LECTURE 9

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1. Tensors and Differential k-forms

Definition 1.1. Let V be an n-dimensional real vector space and $k \geq 1$. A covariant k-tensor T on V is a multilinear function

$$T : V^k = V \times ... \times V \to \mathbf{R}$$

where multilinear means that it is linear on each factor of V, i.e.

$$T(v_1, ..., av_i + bv_i', ..., v_n) = aT(v_1, ..., v_i, ..., v_n) + bT(v_1, ..., v_i', ..., v_n)$$

We denote the set of covariant k-tensors as $T^k(V)$. This is a real vector space with scalar multiplication and pointwise addition. Covariant 1-tensors are just linear maps to $\mathbf{R}$. We extend the definition to $k = 0$ by declaring that a covariant 0-tensor is just a constant.

Exercise 1.2. *Show that covariant 2-tensors (also called bilinear maps) $V \times V \to \mathbf{R}$ are in bijective correspondence with $n \times n$ matrices. Show that the determinant map viewed as a function on an $n \times n$ matrix is a covariant n-tensor on $\mathbf{R}^n$, with input the n row vectors of the matrix.*

Definition 1.3. Let $S \in T^k(V), T \in T^l(V)$. We define the *tensor product* as

$$S \otimes T : V^{k+l} \to \mathbf{R} \qquad S \otimes T(v_1, ..., v_{k+l}) = S(v_1, ..., v_k)T(v_{k+1}, ..., v_{k+l})$$

Note that the definition allows us to write the tensor product of three or more tensors unambiguously without parenthesis.

Lemma 1.4. *Let $e_1, ..., e_n$ be a basis for V, and let $\varepsilon^1, ..., \varepsilon^n$ be a dual basis (i.e. the bais for $V*$). Then a basis for $T^k(V)$ consists of elements of the form*

$$\varepsilon^{i_1} \otimes ... \otimes \varepsilon^{i_k} \qquad 1 \leq i_1, ..., i_k \leq n$$

In particular, $dim T^k(V) = n^k$.

Proof. First we show that this collection is linearly independent. Assume that

$$\sum_{1 \leq i_1, ..., i_k \leq n} T_{i_1, ..., i_k} \varepsilon^{i_1} \otimes ... \otimes \varepsilon^{i_k} = 0$$

Then evaluating the above on $(e_{i_1}, ..., e_{i_k})$ we obtain that $T_{i_1,...,i_k} = 0$, proving our claim.

Now we show that these vectors span $T^k(V)$. Let $T \in T^k(V)$. Let $T_{i_1,...,i_k} = T(e_{i_1}, ..., e_{i_k})$ for each $1 \leq i_1, ..., i_k \leq n$. Then it follows that

$$T = \sum_{1 \leq i_1, ..., i_k \leq n} T_{i_1, ..., i_k} \varepsilon^{i_1} \otimes ... \otimes \varepsilon^{i_k}$$

by multilinearity since the evaluation of the above on the basis vectors $(e_{i_1}, ..., e_{i_k})$ for $1 \leq i_1, ..., i_k \leq n$ of $V \times ... \times V$ (k-times) is equal. □

We remark that it is also common to use the notation $\otimes^k V^*$ for $T^k(V)$. The previous Lemma makes it clear why this is natural, and we shall use it from now on.

1.1. Alternating tensors.

Definition 1.5. We say that a tensor $T \in \otimes^k V^*$ is *alternating*, if

$$T(x_1, ..., x_i, ..., x_j, ..., x_k) = -T(x_1, ..., x_j, ..., x_i, ..., x_k) \qquad \forall x_1, ..., x_k \in V$$

The set of alternating covariant k-tensors is denoted as $\Lambda^k V^*$ and is a subspace of $\otimes^k V^*$.

Exercise 1.6. *$T \in \otimes^k V^*$ is alternating if and only if it is of the form*

$$T = \sum_{1 \leq i_1,...,i_k \leq n} T_{i_1,...,i_k} \varepsilon^{i_1} \otimes ... \otimes \varepsilon^{i_k}$$

with the property that $T_{i_1,...,i_k} = sgn(\sigma) T_{i_{\sigma(1)},...,i_{\sigma(k)}}$ for each $\sigma \in S_k$. Here S_k is the group of permutations on k elements and $sgn(\sigma)$ is the sign of the permutation. (Recall this from your group theory or algebra course).

Let $e_1, ..., e_n$ be a basis for V and let $\varepsilon^1, ..., \varepsilon^n$ be the corresponding dual basis for V^* as usual. For a multi-index $I = (i_1, ..., i_k) \in \{1, ..., n\}^k$, we define the following alternating covariant tensor

$$\varepsilon^I = \sum_{\sigma \in S_k} sgn(\sigma) \varepsilon^{i_{\sigma(1)}} \otimes ... \otimes \varepsilon^{i_{\sigma(n)}}$$

These are called *elementary alternating tensors.*

When evaluating ε^I on $(X_1, ..., X_k) \in V \times ... \times V$, we obtain

$$(*) \qquad \varepsilon^I(X_1, ..., X_k) = Det \begin{pmatrix} X_1^{i_1} & ... & X_k^{i_1} \\ . & & . \\ . & & . \\ . & & . \\ X_1^{i_k} & ... & X_k^{i_k} \end{pmatrix}$$

Exercise 1.7. *Demonstrate the last formula. Hint: Use induction on k.*

Lemma 1.8. *The elementary tensors satisfy the following:*

(1) *$\varepsilon^I = 0$ if I has a repeated index.*
(2) *If $J = \sigma(I)$ for some permutation σ, then $\varepsilon^J = sgn(\sigma)\varepsilon^I$.*
(3) *For $J = (j_1, ..., j_k)$ a multi-index, we have*

$$\varepsilon^I(e_{j_1}, ..., e_{j_k}) = 0 \qquad \text{if } I \text{ or } J \text{ have a repeated index or are not permutations of each other}$$

$$\varepsilon^I(e_{j_1}, ..., e_{j_k}) = sgn(\sigma) \qquad \text{if } J = \sigma(I) \text{ and } J \text{ has no repeated indices}$$

We call a multi-index $I = (i_1, ..., i_k)$ *increasing*, if $i_1 < ... < i_k$. We show that the elementary alternating tensors given by the increasing multi-indices provide a basis for the space $\Lambda^k V^*$.

Proposition 1.9. *The set*

$$\{\varepsilon^I \mid I = (i_1, ..., i_k), 1 \leq i_1 < ... < i_k \leq n\}$$

is a basis for $\Lambda^k V^*$.

Exercise 1.10. *Prove the above proposition.*

The above Proposition implies that $dim \Lambda^k V^* = \binom{n}{k}$. For $k = n$ this is just a 1-dimensional space, spanned by $\varepsilon^{(1,...,n)}$, which is just the determinant function as given by the formula $(*)$ above.

Definition 1.11. We define the wedge product

$$\wedge : \Lambda^k V^* \times \Lambda^l V^* \to \Lambda^{k+l} V^*$$

as the unique bilinear map that satisfies

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^K$$

where

$$I = \{i_1, ..., i_k\} \qquad J = \{j_1, ..., j_l\} \qquad K = \{i_1, ..., i_k, j_1, ..., j_l\}$$

Note that if we have $\omega \in \Lambda^k V^*, \nu \in \Lambda^l V^*$ then

$$\omega \wedge \nu = (-1)^{kl} \nu \wedge \omega$$

This motivates the notation: the space $\Lambda^k V^*$ is a linear combination of k-fold wedge product of covariant 1-tensors, which are just covectors.

Note that any 0-tensor is alternating, and that $\Lambda^0 V^* = \otimes^0 V^* = \mathbf{R}$. The wedge product of a 0-tensor $\lambda \in \Lambda^0 V^* = c \in \mathbf{R}$ with $\nu \in \Lambda^k V^*$ is $\lambda \wedge \nu = c\nu$.

2. Differential k-forms on manifolds

Let M be a smooth manifold of dimension n. We shall apply the definitions from the previous lecture to the case where $V = T_pM$. For each $p \in M$, we obtain the $\binom{n}{k}$-dimensional vector space of alternating k-tensors, which is denoted as $\Lambda^k T_p^* M$. If $(x^1, ..., x^n)$ are local coordinates on a neighbourhood of p then a basis of $\Lambda^k(T_p^* M)$ is

$$dx_p^I = dx_p^{i_1} \wedge ... \wedge dx_p^{i_k} \qquad I = (i_1, ..., i_k), 1 \leq i_1 < ... < i_k \leq n$$

Evaluated on a k-tuple of coordinate vectors

$$\frac{\partial}{\partial x^{j_1}} |_p, ..., \frac{\partial}{\partial x^{j_k}} |_p \qquad J = (j_1, ..., j_k)$$

we obtain

$$dx_p^I(\frac{\partial}{\partial x^{j_1}} |_p, ..., \frac{\partial}{\partial x^{j_k}} |_p) = sgn(\sigma)$$

where σ is the permutation such that $J = \sigma(I)$. The union

$$\Lambda^k(T^*M) = \bigcup_{p \in M} \Lambda^k(T_p^* M)$$

is a smooth vector bundle of rank $\binom{n}{k}$ in a natural way as follows.

Given a smooth chart $(U, (x^1, ..., x^n))$ around $p \in M$, a local trivialisation for $\Lambda^k T^* M$ is given by

$$\bigcup_{p \in U} \Lambda^k T_p^* M \to U \times \mathbf{R}^{\binom{n}{k}} \qquad (p, \sum_{I \text{ an increasing multi-index}} C_I dx_p^I) \mapsto (p, (C_I)_{I \text{ an increasing multi-index}})$$

The maps $p \to dx_p^I$ provide smooth sections and a smooth local frame for this is

$$p \to \{dx_p^I \mid I = (i_1, ..., i_k), 1 \leq i_1 < ... < i_k \leq n\}$$

Example 2.1. The vector bundle $\Lambda^2(T^*\mathbf{R}^3)$ has a global frame given by $dx \wedge dy, dy \wedge dz, dx \wedge dz$.

Definition 2.2. A smooth section of $\Lambda^k T^* M$ is called a differential k-form. The set of differential k-forms on M is denoted as $\Omega^k(M)$. Note that for $k = 0$ a differential 0-form is just a smooth function, so $\Omega^0(M) = C^\infty(M)$.