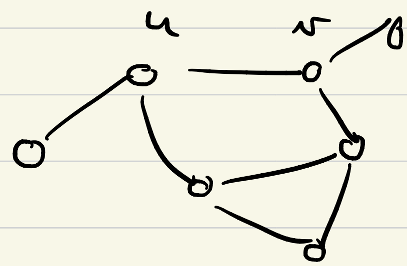
Coloring Problem: Analysis of the Metropolis chain.

Recap the problem.

$G = (V, E)$



sample from the space of proper colorings.

i.e s.t two Neighbouring vertices $(u, v) \in E$

dont have same color.

Notation: $\{1, 2, 3 \ldots q\}$ = set of colors.

Δ = max degree of a vertex $v \in V$.

$\underline{x} = (x_1, x_2, \ldots, x_N)$ $\quad N = |V|$.

and x_v color assigned to $v \in V$.

$$\pi(\underline{x}) = \frac{\mathbb{1}(\underline{x} \text{ is proper})}{Z}$$

$Z \leftarrow$ counts total # of proper cols

$N = |V|$

Recap the proposed algorithm.

- Start from an initial proper coloring.
- Select $v \in V$ uniformly at random
- Select a color $c \in \{1, 2, \dots q\}$ unif at random
- Recolor vertex v iff c is an allowed color.
  (otherwise we do nothing).

Recap the theorem.

Theorem: Assume $q > 3\Delta$ then for any initial proper coloring $\underline{x}$

$$\| P_{\underline{x}}^{n} - \pi \|_{TV} \leq N e^{-\frac{n}{N}\left(1 - \frac{3\Delta}{q}\right)}.$$

distr with initial cond $\underline{x}$ after n iterations.

and the mixing time $T_\epsilon = \inf\left\{ n \geq 1 : \max_{\underline{x} \text{ prop col}} \| P_{\underline{x}}^{n} - \pi \| \leq \epsilon \right\}$

satisfies

$$T_\epsilon \leq \left(1 - \frac{3\Delta}{q}\right)^{-1} \left\{ N \log N + \log \frac{1}{\epsilon} \right\}.$$

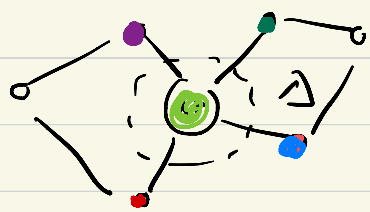
Remark: The theorem with a similar but more advanced proof holds for $q > 2\Delta$.

q smaller: out of our scope here and the problem becomes much harder.

(q too small: thm breaks down.).

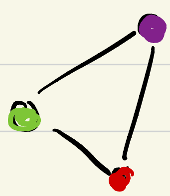
Remark:

- $q = \Delta + 1 \Rightarrow$ certainly we can color the graph.

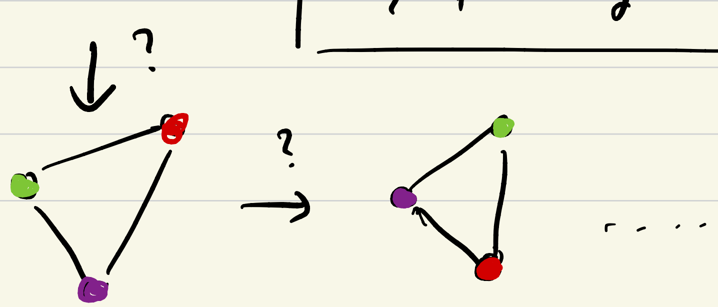


- but the chain of the algo might not be irreducible.
  For example

$\Delta = 2$ and $q = 3$



$\boxed{\text{Can you move out of this configuration by following the algo steps?}}$



• It is a fact that for $q > 3\Delta$ the chain is irreducible.

But also for $q \geq \Delta + 2$ as one can show.

↑

follow up this during the quiz session for a little proof.

Remark: The proof of the thm. will use this fact that for $q > 3\Delta$ the chain is irreducible.

The proof will proceed by a coupling argument:

$$\underbrace{X_n}_{\text{of interest}} \quad \text{and} \quad \underbrace{Y_n}_{\text{another chain}} \quad \leftarrow \text{coupled chains.}$$

Recap the property: $\| P_x^n - Q_y^n \|_{TV} = \inf_{\{\text{couplings}\}} \mathbb{P}(X_n \neq Y_n)$

Proof of theorem.

Coupled chains.

- $(X_n, n \geq 0)$ the chain starting at $X_0 = \underline{x}$ the proper initial coloring and follows the steps of the algo: select $v \in V$ at random, select $c \in \{1, \dots, q\}$ at random, recolor if c is allowed.

- $(Y_n, n \geq 0)$ starts at $Y_0 \sim \pi$ and then evolves with the same steps than $(X_n, n \geq 0)$ with the same $v \in V$ and $c \in \{1 \dots q\}$ at each time step.

so the chains are coupled.

Hamming distance between chains at each time step.

$$d(\underline{x}_n, \underline{y}_n) \equiv \sum_{v \in V} \mathbb{1}\left(x_v^{(n)} \neq y_v^{(n)}\right).$$

Since this is $\underline{\underline{\text{same}}}$ coupling between chain:

$$\| P_{\underline{x}}^{n} - \pi \| \leq \underbrace{\mathbb{P}(X_n \neq Y_n)}$$

at all times

distr of chain $(Y_n, n \geq 0)$ is π because π is stat distr.

$$\mathbb{P}\big(d(X_n, Y_n) \geq 1\big).$$

Markov inequality: $\leq$

$$\mathbb{E}\big(d(X_n, Y_n)\big)$$

We have a nice inequality to start with:

$$\boxed{\| P_x^n - \pi \|_{TV} \leq \mathbb{E}\big(d(X_n, Y_n)\big).}$$

we are going to bound this expectation now. to prove

$$\leq N \exp\left(-\frac{n}{N}\left(1 - \frac{3\Delta}{q}\right)\right).$$

Proceed by induction:

- First we assume that $d(X_0, Y_0) = 1$ prove the bound.

- Then we generalize to $d(X_0, Y_0) = r \geq 1$.

- Finally we conclude.

#.

Assume at time at $n=0$ $d(X_0, Y_0) = 1$.

$\exists\, v \in V$ s.t $x_v \neq y_v$ (at time zero)

and $x_w = y_w$ $\forall\, w \neq v$.

$$\Rightarrow d(X_1, Y_1) \in \{0, 1, 2\}.$$

$$\Rightarrow \mathbb{E}(d(X_1, Y_1)) = 0 \cdot \mathbb{P}(d(X_1, Y_1) = 0)$$

$1 - \mathbb{P}(d=0) - \mathbb{P}(d=2)$

$$+ 1 \cdot \mathbb{P}(d(X_1, Y_1) = 1)$$

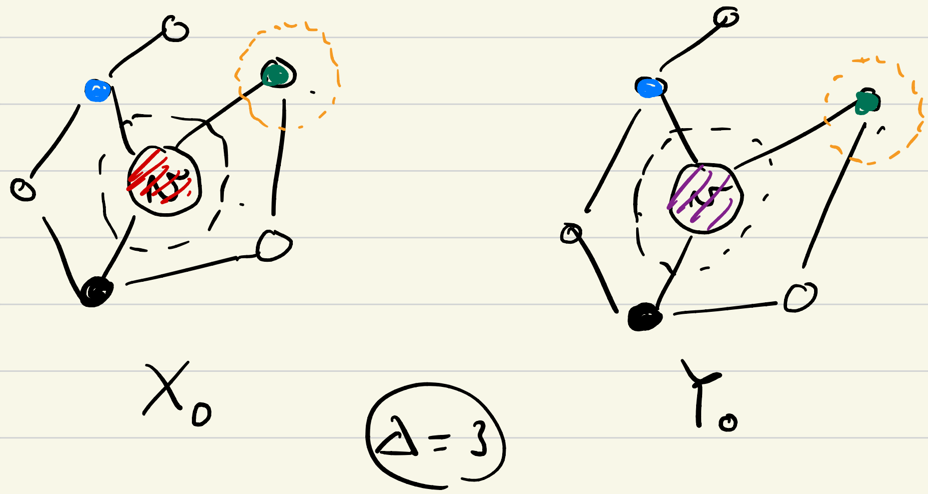
$$+ 2 \cdot \mathbb{P}(d(X_1, Y_1) = 2)$$

$$= \{1 - \mathbb{P}(d(X_1, Y_1) = 0)\} + \mathbb{P}(d(X_1, Y_1) = 2).$$

$\boxed{\text{at time } t_0 \text{ if } d(X_0, Y_0) = 1}$

$$\mathbb{E}(d(X_1, Y_1)) = 1 - \mathbb{P}(d(X_1, Y_1) = 0) + \mathbb{P}(d(X_1, Y_1) = 2).$$

$$\mathbb{P}(d(X_1, Y_1) = 0) = \frac{1}{N} \frac{\#\text{ allowed color at } v}{q} \geq \frac{1}{N} \frac{q - \Delta}{q}.$$



$$\mathbb{P}(d(X_1, Y_1) = 2) \leq \frac{\Delta}{N} \frac{2}{q}.$$

- if the selected vertex w is not a neighbord of v you will do the same recoloring or non-recoloring of w in both chains and $d(X_1, Y_1)$ remain $= 1$.

- The event $d(X_1, Y_1) = 2$ happens only if w is a Neighbord $\to$ It should be that you recolor w is X and Not in Y OR recolor w in Y and Not in X.

$$\boxed{\mathbb{E}(d(X_1, Y_1)) \leq 1 - \frac{1}{N}\left(1 - \frac{3\Delta}{q}\right)} \quad (*)$$

$\boxed{\text{Generalize to case } d(X_0, Y_0) = r}$:

Claim by irreducibility (for $q > 3\Delta$ or in fact for $q \geq \Delta + 2$):

$\exists$ path between assignments :

$$X_0 \to Z_0^{(0)} \to Z_0^{(1)} \to \cdots Z_0^{(k)} \to \cdots Z_0^{(r-1)} \to Y_0 \ . \ \checkmark$$

s.t. $d(Z_0^{(0)}, Z_0^{(1)}) = 1$. all dist between assignments are equal to 1.

Let evolve the chains by one time unit :

$$X_1 \to Z_1^{(0)} \to Z_1^{(1)} \to \cdots Z_1^{(k)} \to \cdots Z_1^{(r-1)} \to Y_1 .$$

By triangle inequality

$$d(X_1, Y_1) \leq d(X_1, Z_1^{(0)}) + d(Z_1^{(0)}, Z_1^{(1)}) + \cdots + d(Z_1^{(r-1)}, Y_1).$$

Take the expectation, use linearity, use result $(*)$ under $\checkmark$

$\Longrightarrow$

$$\boxed{\mathbb{E}(d(X_1, Y_1)) \leq r\left\{1 - \frac{1}{N}\left(1 - \frac{3\Delta}{q}\right)\right\}} \leftarrow$$

Conclusion of proof:

Remark by homogeneity of the Markov chain:

$$\mathbb{E}\left(d(X_{m+1}, Y_{m+1}) \mid d(X_m, Y_m) = r\right)$$

$$\leq r\left(1 - \frac{1}{N}\left(1 - \frac{3\Delta}{q}\right)\right).$$

Average over r:

$$\sum_r \mathbb{E}\left(d(X_{m+1}, Y_{m+1}) \mid d(X_m, Y_m) = r\right) \mathbb{P}(d(X_m, Y_m) = r)$$

$$= \mathbb{E}(d(X_{m+1}, Y_{m+1})).$$

$$\leq \left(1 - \frac{1}{N}\left(1 - \frac{3\Delta}{q}\right)\right) \underbrace{\sum_r r\, \mathbb{P}(d(X_m, Y_m) = r)}_{\mathbb{E}(d(X_m, Y_m))}.$$

$\Rightarrow$D.

We found:

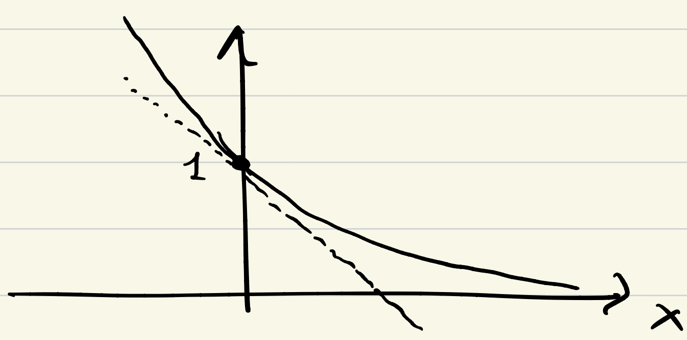
$$\mathbb{E}(d(X_{m+1}, Y_{m+1})) \leq \mathbb{E}(d(X_m, Y_m)) \left\{1 - \frac{1}{N}\left(1 - \frac{3\lambda}{9}\right)\right\}.$$

$$\vdots$$

$$\leq \underbrace{\mathbb{E}(d(X_0, Y_0))}_{\substack{\leq N \\ \text{total \# of vertex.}}} \left(1 - \frac{1}{N}\left(1 - \frac{3\lambda}{9}\right)\right)^m$$

$$\Rightarrow \quad \mathbb{E}(d(X_{m+1}, Y_{m+1})) \leq N \left(1 - \frac{1}{N}\left(1 - \frac{3\lambda}{9}\right)\right)^m.$$

use $\quad 1 - x \leq e^{-x}$



QED. ∎