LECTURE 10

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Before we initiate the study of differential k-forms on manifolds, we shall state some basic properties of alternating tensors and the wedge product. Let V be an n-dimensional vector space. Recall that $\otimes^k V^*$ is the n^k-dimensional real vector space of covariant k-tensors. Each element of this vector space is a multilinear map $V \times ... \times V \to \mathbf{R}$, where the domain is a k-fold product. We denote the $\binom{n}{k}$-dimensional subspace of alternating k-tensors by $\Lambda^k V^*$. Given a multi-index $I \subseteq \{1, ..., n\}$, we define

$$\varepsilon^I \in \Lambda^k V^* \qquad \varepsilon^I = \sum_{\sigma \in S_k} sgn(\sigma) \varepsilon^{i_{\sigma(1)}} \otimes ... \otimes \varepsilon^{i_{\sigma(k)}} \qquad I = (i_1, ..., i_k)$$

Given $\omega \in \Lambda^k V^*, \nu \in \Lambda^l V^*$, we define the wedge product $\omega \wedge \nu \in \Lambda^{k+l} V^*$ as the unique multilinear map that satisfies

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^K \qquad K = (i_1, ..., i_k, j_1, ..., j_l), I = (i_1, ..., i_k), J = (j_1, ..., j_l)$$

The following are some elementary features of the wedge product.

Proposition 0.1. (1) *(Bilinearity) For $a, b \in \mathbf{R}$*

$$(a\omega + b\omega') \wedge \nu = a(\omega \wedge \nu) + b(\omega' \wedge \nu)$$

(2) *(Associativity)*

$$\omega \wedge (\nu \wedge \psi) = (\omega \wedge \nu) \wedge \psi$$

(3) *(Anticommutativity) For $\omega \in \Lambda^k V^*, \nu \in \Lambda^l V^*$ we have*

$$\omega \wedge \nu = (-1)^{kl} \nu \wedge \omega$$

(4) *If $\varepsilon^1, ..., \varepsilon^n$ is a basis for V^* and $I = (i_1, ..., i_k)$ is a multiindex, then*

$$\varepsilon^I = \varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge ... \wedge \varepsilon^{i_k}$$

(5) *For any covectors $\omega^1, ..., \omega^k$ and vectors $v_1, ..., v_k$*

$$\omega^1 \wedge ... \wedge \omega^k (v_1, ..., v_k) = Det(\omega^j(v_i))$$

1. Differential k-forms on manifolds

Let M be a smooth manifold of dimension n. We shall apply the definitions from the previous lecture to the case where $V = T_pM$. For each $p \in M$, we obtain the $\binom{n}{k}$-dimensional vector space of alternating k-tensors, which is denoted as $\Lambda^k T_p^* M$. If $(x^1, ..., x^n)$ are local coordinates on a neighbourhood of p then a basis of $\Lambda^k(T_p^* M)$ is

$$dx_p^I = dx_p^{i_1} \wedge ... \wedge dx_p^{i_k} \qquad I = (i_1, ..., i_k), 1 \leq i_1 < ... < i_k \leq n$$

Evaluated on a k-tuple of coordinate vectors

$$\frac{\partial}{\partial x^{j_1}} |_p, ..., \frac{\partial}{\partial x^{j_k}} |_p \qquad J = (j_1, ..., j_k)$$

we obtain

$$dx_p^I(\frac{\partial}{\partial x^{j_1}} |_p, ..., \frac{\partial}{\partial x^{j_k}} |_p) = sgn(\sigma)$$

where σ is the permutation such that $J = \sigma(I)$. The union

$$\Lambda^k(T^* M) = \bigcup_{p \in M} \Lambda^k(T_p^* M)$$

is a smooth vector bundle of rank $\binom{n}{k}$ in a natural way as follows.

Given a smooth chart $(U, (x^1, ..., x^n))$ around $p \in M$, a local trivialisation for $\Lambda^k T^* M$ is given by

$$\bigcup_{p \in U} \Lambda^k T_p^* M \to U \times \mathbf{R}^{\binom{n}{k}} \qquad (p, \sum_{I \text{ an increasing multi-index}} C_I dx_p^I) \mapsto (p, (C_I)_{I \text{ an increasing multi-index}})$$

The maps $p \to dx^I_p$ provide smooth sections and a smooth local frame for this is

$$p \to \{dx^I_p \mid I = (i_1, ..., i_k), 1 \leq i_1 < ... < i_k \leq n\}$$

Example 1.1. The vector bundle $\Lambda^2(T^*\mathbf{R}^3)$ has a global frame given by $dx \wedge dy, dy \wedge dz, dx \wedge dz$.

Definition 1.2. A smooth section of $\Lambda^k T^* M$ is called a differential k-form. The set of differential k-forms on M is denoted as $\Omega^k(M)$. Note that for $k = 0$ a differential 0-form is just a smooth function, so $\Omega^0(M) = C^\infty(M)$.

We shall fix the following notational convention. Sums over increasing multiindices of the form

$$\sum_{I \subset \{1,...,n\}, I \text{ is an increasing multiindex}} C_I dx^I_p$$

shall be denoted as

$$\sum_I{}' C_I dx^I_p$$

Definition 1.3. For $\omega \in \Omega^k(M)$, we denote $\omega_p = \omega(p)$. In local coordinates for a chart (U, ϕ), we denote

$$\omega_p = \sum_I{}' \omega_I(p) dx^I_p$$

for some smooth functions $\omega_I : U \to \mathbf{R}$ which are called the *component functions* of ω in the given chart. Note that the smoothness is equivalent to requiring that for any list of smooth vector fields $X^1, ..., X^k$ on U, the function

$$U \to \mathbf{R} \qquad p \mapsto \omega_p(X^1_p, ..., X^k_p)$$

is smooth.

We now generalise the notions of wedge product and pullback to the bundle $\Omega^k(M)$

Definition 1.4. (Wedge product) Let $\omega \in \Omega^k(M), \nu \in \Omega^l(M)$. Then the wedge product $\omega \wedge \nu \in \Omega^{k+l}(M)$ is defined as

$$(\omega \wedge \nu)_p = \omega_p \wedge \nu_p$$

(Pullback) Let $F : M \to N$ be a smooth map. Let $\omega \in \Omega^k(N)$. Then we define a linear map

$$F^* : \Lambda^k(T_{F(p)}N) \to \Lambda^k(T_p M) \qquad F^*(\omega)(X_1, ..., X_k) = \omega(F_*(X_1), ..., F_*(X_k))$$

for

$$X_1, ..., X_k \in T_p(M), \omega \in \Lambda^k(T_{F(p)}N)$$

Specialising to differential k-forms, we obtain:

$$F^* : \Omega^k(N) \to \Omega^k(M) \qquad F^*(\omega)_p = F^*(\omega_{F(p)})$$

For a 0-form $f \in C^\infty(N)$, we set $F^* f = f \circ F$.

In the special case where we have an inclusion map $i : U \to M$, we denote $i^*\omega \in \Omega^k(U)$ for $\omega \in \Omega^k(M)$ as the *restriction of* ω. Indeed for $p \in U$, $i^*\omega_p = \omega_p$.

Lemma 1.5. *For $F : M \to N$ a smooth map, $\omega \in \Omega^k(N), \nu \in \Omega^l(N)$, we have:*

1. $F^*(\omega \wedge \nu) = F^*(\omega) \wedge F^*(\nu)$.
2. *In any coordinate chart on N,*

$$F^*(\sum_I{}' \omega_I dy^I) = \sum_I{}' (\omega_I \circ F) d(y^{i_1} \circ F) \wedge ... \wedge d(y^{i_k} \circ F)$$

Exercise 1.6. *Prove the above Lemma. Hint: For part (2), combine part (1) with Lemma 1.9 from Lecture notes 8.*

Example 1.7. Let $F : \mathbf{R}^2 \to \mathbf{R}^2$ be the map

$$F(r, \theta) = (r\cos(\theta), r\sin(\theta))$$

The pullback of

$$dx \wedge dy \in \Omega^2(F(\mathbf{R}^2)$$

is

$$F^*(dx \wedge dy) = (F^*dx) \wedge F^*(dy) = d(x \circ F) \wedge d(y \circ F)$$

$$= (\frac{\partial}{\partial r} r\cos(\theta)dr + \frac{\partial}{\partial \theta} r\cos(\theta)d\theta) \wedge (\frac{\partial}{\partial r} r\sin(\theta)dr + \frac{\partial}{\partial \theta} r\sin(\theta)d\theta)$$
$$= (\cos(\theta)dr - r\sin(\theta)d\theta) \wedge (\sin(\theta)dr + r\cos(\theta)d\theta)$$
$$= r\cos^2(\theta)(dr \wedge d\theta) + r\sin^2(\theta)(dr \wedge d\theta) = rdr \wedge d\theta$$

In general the pullback of an n-form for a smooth map between two n-dimensional manifolds is given by the following formula. (Note that the value n is the same throughout).

Proposition 1.8. *Let $F : M \to N$ be a smooth map. Let (U, x^i) be a chart on M and (V, y^i) be a chart on N. Let $u \in C^\infty(M)$. Then*

$$F^*(udy^1 \wedge ... \wedge dy^n) = (u \circ F)(Det(\frac{\partial F^j}{\partial x^i}))dx^1 \wedge ... \wedge dx^n \qquad F^j = y^j \circ F$$

Proof. Since $\Lambda^n(T^*_pM), \Lambda^n(T^*_{F(p)}N)$ are 1-dimensional, it suffices to show that the evaluation of both sides of the equality on $(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})$ are equal. From the Lemma above, the left hand side equals

$$(u \circ F)dF^1 \wedge ...dF^n \qquad F^i = y^i \circ F$$

The result then follows from

$$dF^1 \wedge ... \wedge dF^n(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}) = Det(dF^i(\frac{\partial}{\partial x^j})) = Det(\frac{\partial F^i}{\partial x^j})$$

□

Note that specialising the above Proposition to the case when $M = N$, $F = id_M, u = 1_M$, and $(U, x^i), (V, y^i)$ just two different charts on M, we obtain the following change of coordinates formula:

$$dy^1 \wedge ... \wedge dy^n = Det(\frac{\partial y^j}{\partial x^i})dx^1 \wedge ... \wedge dx^n$$

2. The exterior derivative

The differential of a smooth function in $C^\infty(M)$ is a 1-form on M. Since we view $C^\infty(M)$ as $\Omega^0(M)$, this provides a map

$$d : \Omega^0(M) \to \Omega^1(M)$$

The generalisation of this is the following operation on k-forms which is first defined locally as:

$$(*) \qquad d : \Omega^k(U) \to \Omega^{k+1}(U) \qquad U \subseteq M, (U, x^i) \text{ a chart on } M$$

$$\omega = \sum_I{}' \omega_I dx^I \mapsto d_U\omega = \sum_I{}' d\omega_I \wedge dx^I$$

Here $d\omega_I \in \Omega^1(U)$ is the differential, viewing $\omega_I \in C^\infty(U) = \Omega^0(U)$. Note that this is a local definition. The global definition is provided by the following theorem.

First we state the following Lemma shall be needed in the proof of the theorem.

Lemma 2.1. *Let $f, g \in C^\infty(M)$. Then $d(fg)_p = f(p)dg_p + g(p)df_p$.*

Proof. We choose a chart (U, x^i) centered at p.

$$d(fg)_p = \sum_{1\le i\le n} \frac{\partial fg}{\partial x^i} |_p \, dx^i_p = f(p) \sum_{1\le i\le n} \frac{\partial g}{\partial x^i} |_p \, dx^i_p + g(p) \sum_{1\le i\le n} \frac{\partial f}{\partial x^i} |_p \, dx^i_p$$
$$= f(p)dg_p + g(p)df_p$$

□

Theorem 2.2. *Let M be a smooth n-manifold. For each $k \in \mathbf{N}$, there is a unique map*

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

satisfying the following:

1. *If $f \in C^\infty(M) = \Omega^0(M)$ then df is the usual differential, i.e. $df(X) = Xf$.*
2. *For $\omega \in \Omega^k(M), \nu \in \Omega^l(M)$, we have*

$$d(\omega \wedge \nu) = d\omega \wedge \nu + (-1)^k\omega \wedge d\nu$$

3. $d^2 = 0$.

Moreover, for each individual chart, in local coordinates d is given by the formula (∗) above. We call d the **exterior derivative**.

Proof. First we will show that the three conditions hold for the case when (U, x^i) is a global chart on M, i.e. when M is diffeomorphic to $\mathbf{R}^n$. Part 1 is immediate from the definition. We first show part 2. Let

$$\omega = \sum_I{}' \omega_I dx^I \qquad \nu = \sum_I{}' \nu_I dx^I$$

By multilinearity, it suffices to show that

$$d(\omega_I dx^I \wedge \nu_J dx^J) = (d\omega_I \wedge dx^I) \wedge (\nu_J dx^J) + (-1^k)(\omega_I dx^I) \wedge (d\nu_J \wedge dx^J)$$

We have

$$\begin{aligned} d(\omega_I dx^I \wedge \nu_J dx^J) &= d(\omega_I \nu_J dx^I \wedge dx^J) \\ &= (\nu_J d\omega_I + \omega_I d\nu_J) \wedge dx^I \wedge dx^J \\ &= (\nu_J d\omega_I) \wedge dx^I \wedge dx^J + (\omega_I d\nu_J) \wedge dx^I \wedge dx^J \\ &= (d\omega_I \wedge dx^I) \wedge (\nu_J dx^J) + (d\nu_J) \wedge (\omega_I dx^I) \wedge dx^J \\ &= (d\omega_I \wedge dx^I) \wedge (\nu_J dx^J) + (-1)^k (\omega_I dx^I) \wedge (d\nu_J \wedge dx^J) \end{aligned}$$

Note that the $(-1)^k$ appears from the anticommutativity law

$$d\psi_1 \wedge d\psi_2 = (-1)^{kl} d\psi_2 \wedge d\psi_1 \qquad \psi_1 \in \Omega^k(M), \psi_2 \in \Omega^l(M)$$

and since $\omega_I dx^I \in \Omega^k(M), d\nu_J \in \Omega^1(M)$.

Now we show part 3. First we show this for a 0-form $\omega \in C^\infty(M) = \Omega^0(M)$. Then

$$\begin{aligned} d(df) &= d(\sum_{1 \le i \le n} \frac{\partial f}{\partial x^i} dx^i) = \sum_{1 \le i,j \le n} \frac{\partial f}{\partial x^i \partial x^j} dx^i \wedge dx^j \\ &= \sum_{1 \le i < j \le n} (\frac{\partial f}{\partial x^i \partial x^j} - \frac{\partial f}{\partial x^j \partial x^i}) dx^i \wedge dx^j = 0 \end{aligned}$$

We now prove the general case for $\nu \in \Omega^k(M), \nu = \sum_I' \nu_I dx^I$.

$$d(d\nu) = d(\sum_I{}' d\nu_I \wedge dx^{i_1} \wedge ... \wedge dx^{i_k})$$

$$= \sum_I{}' d(du) \wedge dx^{i_1} \wedge ... \wedge dx^{i_k} + \sum_I{}' \sum_{1 \le j \le k} du \wedge ... \wedge d(dx^{i_j}) \wedge ... \wedge dx^{i_k}$$

We now prove the existence and uniqueness of the exterior derivative for the general case, i.e. for an arbitrary smooth manifold. Let $\omega \in \Omega^k(M)$. We wish to define $d\omega \in \Omega^{k+1}(M)$ by means of local coordinate charts, locally, and then prove the global existence and independence of the choice of coordinates. Let (U, ϕ) be a chart for M. We set

$$d\omega = \phi^* d((\phi^{-1})^* \omega)$$

Given any other chart (V, η), we obtain

$$\begin{aligned} \eta^* d((\eta^{-1})^* \omega) &= \phi^* (\phi^{-1})^* \eta^* d(\eta^{-1})^* \omega) \\ &= \phi^* (\eta \circ \phi^{-1})^* d(\eta^{-1})^* \omega) = \phi^* d((\eta \circ \phi^{-1})^* \eta^{-1})^* \omega) \\ &= \phi^* d((\eta^{-1} \circ \eta \circ \phi^{-1})^* \omega) = \phi^* d((\phi^{-1})^* \omega) \end{aligned}$$

Therefore, the definition is well defined for $U \cap V$.

□

Exercise 2.3. *Use the existence of smooth bump functions to prove the uniqueness part of the previous theorem.*