LECTURE 11

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1. de Rham Cohomology

Definition 1.1. For a smooth n-dimensional manifold M we call the sequence of maps and spaces

$$0 \to \Omega^0(M) \to \Omega^1(M) \to ... \to \Omega^n(M) \to 0$$

the *de Rham complex* of M. We say that $\omega \in \Omega^k(M)$ is *closed* if $d\omega = 0$. We say that $\omega \in \Omega^k(M)$ is *exact* if there is a $\nu \in \Omega^{k-1}(M)$ such that $\omega = d\nu$. (We fix the convention that $\Omega^{-1}(M) = 0$.)

The set of exact k-forms is denoted as $\mathcal{B}^k(M)$ and the set of closed k-forms is denoted as $\mathcal{Z}^k(M)$. It follows from $d^2 = 0$ that $\mathcal{B}^k(M) \subseteq \mathcal{Z}^k(M)$, and both are linear subspaces of $\Omega^k(M)$.

We define the vector space $H^k_{dR}(M) = \mathcal{Z}^k(M)/\mathcal{B}^k(M)$ as the k-th *de Rham cohomology* group.

Theorem 1.2. *The de Rham cohomology groups are topological invariants, i.e. the groups for homeomorphic manifolds are isomorphic. In particular, the invariants do not depend on the smooth structure, just on the topology.*

Exercise 1.3. *Show that $H^0_{dR}(M) = \mathbf{R}^m$ where m is the number of connected components of M.*

Example 1.4.

$$H^1_{dR}(\mathbf{R}^2 \setminus \{0\}) \cong \mathbf{R}$$

$$H^k_{dR}(\mathbf{T}^n) = \mathbf{R}^{\binom{n}{k}}$$

$$H^k_{dR}(\mathbf{S}^n) = \mathbf{R} \text{ if } k \in \{0, n\} \qquad H^k_{dR}(\mathbf{S}^n) = 0 \text{ otherwise}$$

2. Integration of differential forms

Let M be a smooth manifold and $\omega \in \Omega^k(M)$. We define

$$supp(\omega) = \overline{\{p \in M \mid \omega_p \neq 0\}}$$

We denote by $\Omega^k_c(M)$ as the set of k-forms with compact support. Note that $\Omega^0_c(M) = C^\infty_c(M)$, where the latter is the set of compactly supported smooth functions on M..

Definition 2.1. Let $\omega \in \Omega^n_c(\mathbf{R}^n)$. Then $\omega = f dx^1 \wedge ... \wedge dx^n$ for some $f \in C^\infty_c(\mathbf{R}^n)$. We define

$$\int_{\mathbf{R}^n} \omega = \int_{\mathbf{R}^n} f dx^1 \wedge ... \wedge dx^n = \int_{\mathbf{R}^n} f dx^1 ... dx^n$$

where the last expression is the standard Riemann integral. Note that this is defined since the function is compactly supported.

2.1. Orientation.

In order to generalise the above definition of integration to smooth manifolds, we shall need the following notion of orientation on smooth manifolds. First we recall the vector space notion.

Definition 2.2. We say that two bases $E_1, ..., E_n$ and $E'_1, ..., E'_n$ for a vector space V have the same orientation if the transition matrix

$$M = (a_{i,j})_{1 \leq i,j \leq n} \qquad E_i = \sum_{1 \leq j \leq n} (a_{i,j}) E_j$$

has positive determinant. We define an equivalence relation on the set of all bases of V by declaring two bases to be equivalent if they have the same orientation. Recall that there are precisely two equivalence classes. An *orientation for* V is a choice of equivalence class for V. A basis that belongs to the class of the chosen orientation is said to have *positive orientation* and a basis that belongs to the other class is said to have *negative orientation*. The equivalence class of the basis $E_1, ..., E_n$ is denoted as $[E_1, ..., E_n]$.

Let $U \subset \mathbf{R}^n$ be open. A local diffeomorphism $F : U \to \mathbf{R}^n$ (i.e. a diffeomorphism onto its image) is said to be *orientation preserving*, if $detDF(x) > 0$ for each $x \in U$, where $DF(x)$ is the jacobian matrix at x.

Definition 2.3. Let M be a smooth manifold and let $(U, \phi_1), (V, \phi_2)$ be two charts on it. They are said to be *consistently oriented*, if $\phi_2 \circ \phi_1^{-1}$ is orientation preserving.

A smooth atlas on a manifold is said to be *oriented*, if every pair of charts is consistently oriented. We say that a smooth manifold is *orientable* if it admits an oriented atlas. An *orientation* on M is a maximal oriented atlas.

Example 2.4. $\mathbf{S}^n$ is orientable. Check that the two charts given by the stereographic projection maps are consistently oriented.

Definition 2.5. Let M be a smooth manifold. A pointwise orientation on M is an assignment of an orientation $\mathcal{O}_p$ to T_pM for each $p \in M$.

For instance, given an orientable manifold $(M, \mathcal{A})$, there is an induced pointwise orientation given as follows. For any $p \in M$ we pick a chart $(U, \phi) \in \mathcal{A}$ containing p. Then

$$\mathcal{O}_p = [\frac{\partial}{\partial x^1} \mid_p, ..., \frac{\partial}{\partial x^n} \mid_p]$$

A pointwise orientation is said to be *continuous*, if around every point $p \in M$ there is an open set U and an n-tuple of smooth vector fields $(X_1, ..., X_n)$ such that for all $p \in U$ the vectors $(X_1)_p, ..., (X_n)_p$ form a positively oriented basis, i.e.

$$\mathcal{O}_p = [(X_1)_p, ..., (X_n)_p]$$

Proposition 2.6. *A smooth manifold is orientable if and only if it admits a continuous pointwise orientation.*

Proposition 2.7. *Let M be a smooth n-manifold. Then M is orientable if and only if there is a nowhere vanishing n-form on M. (Nowhere vanishing means that the value at each point is nonzero.)*

Proof. We will show that if given an oriented atlas on M, we can construct a nowhere vanishing n-form on M. The converse shall be an exercise. Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ be the smooth oriented atlas. Let

$$\omega_\alpha = d\phi_\alpha^1 \wedge ... \wedge d\phi_\alpha^n \in \Omega^n(U_\alpha)$$

We shall glue these together using partitions of unity. Let $\{\chi_\alpha \mid \alpha \in I\}$ be a partitions of unit subordinate to the atlas. We define

$$\omega = \sum_{\alpha \in I} \chi_\alpha \omega_\alpha$$

We claim that ω is nonvanishing. (Note that the sum is locally finite so it is well defined.) Let $p \in M$. Let $\beta \in I$ be such that $\chi_\beta(p) > 0$ Then

$$\omega_p(\frac{\partial}{\partial \phi_\beta^1} \mid_p, ..., \frac{\partial}{\partial \phi_\beta^n} \mid_p) = \sum_{\alpha \in I} \chi_\alpha \omega_\alpha(\frac{\partial}{\partial \phi_\beta^1} \mid_p, ..., \frac{\partial}{\partial \phi_\beta^n} \mid_p) > 0$$

since for each $\alpha \in I$ we have that

$$(\omega_\alpha)_p(\frac{\partial}{\partial \phi_\beta^1} \mid_p, ..., \frac{\partial}{\partial \phi_\beta^n} \mid_p) = Det(D(\phi_\alpha \circ \phi_\beta^{-1})) > 0$$

since the charts are consistently oriented.

□

2.2. Back to integration of forms.

Definition 2.8. (Integration of compact forms supported in a single chart) Let M be an oriented smooth manifold, (U, ϕ) be a smooth chart in the oriented atlas, and let $\omega \in \Omega_c^n(M)$. Suppose that ω is supported in the single chart (U, ϕ). Then we define

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega$$

We show that the above definition is independent of the choice of chart.

Proposition 2.9. *Let M be an oriented smooth manifold, $(U, \phi), (V, \chi)$ be smooths chart in the oriented atlas that are consistently oriented, and let $\omega \in \Omega_c^n(M)$. Suppose that ω is supported in $U \cap V$. Then*

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\chi(V)} (\chi^{-1})^* \omega$$

Proof. Let $\omega = f dx^1 \wedge ... \wedge dx^n$ in the coordinates provided by (U, ϕ) and $\omega = g dy^1 \wedge ... \wedge dy^n$ in the coordinates provided by (V, χ). First observe that by Proposition 1.8 from Lecture notes 10 it follows that

$$\begin{aligned} &g dy^1 \wedge ... \wedge dy^n \\ &= (\phi \circ \chi^{-1})^*(f dx^1 \wedge ... \wedge dx^n) = (f \circ \phi \circ \chi^{-1})|Det(D(\phi \circ \chi^{-1}))| dy^1 \wedge ... \wedge dy^n \\ &= (f \circ \phi \circ \chi^{-1}) Det(D(\phi \circ \chi^{-1})) dy^1 \wedge ... \wedge dy^n \qquad \text{since } Det(D(\phi \circ \chi^{-1})) > 0 \end{aligned}$$

For simplicity of notation, we denote

$$G = \phi \circ \chi^{-1} \qquad G : \chi(U \cap V) \to \phi(U \cap V)$$

Note that $Det(G) > 0$ since the charts are consistently oriented.

It follows that

$$\int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\phi(U)} f dx^1 \wedge ... \wedge dx^n$$

Using the change of variables formula for Riemann integrals (recall this from calculus) applied to the map G above we get

$$\int_{\phi(U)} f dx^1 \wedge ... \wedge dx^n = \int_{\chi(V)} Det(G)(f \circ G) dy^1 \wedge ... \wedge dy^n$$

Again, note that $Det(G) > 0$ since the charts are consistently oriented so we don't need the absolute value sign. Finally, by the above we have

$$\int_{\chi(V)} Det(G)(f \circ G) dy^1 \wedge ... \wedge dy^n = \int_{\chi(V)} g dy^1 \wedge ... \wedge dy^n = \int_{\chi(V)} (\chi^{-1})^* \omega$$

It follows that

$$\int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\chi(V)} (\chi^{-1})^* \omega$$

□

Definition 2.10. Let $(M, \mathcal{A})$ be a smooth manifold with an oriented atlas $\mathcal{A}$. Let $\omega \in \Omega^n_c(M)$. We define the integral $\int_M \omega$ as follows. Take any finite collection $\{(U_i, \phi_i)\}_{i \in I}$ of oriented charts such that

$$supp(\omega) \subset \bigcup_{i \in I} U_i$$

Let χ_i be a partition of unity subordinate to $\{U_i \mid i \in I\}$. Then we define

$$\int_M \omega = \sum_{i \in I} \int_M \chi_i \omega = \sum_{i \in I} \int_{\phi_i(U_i)} (\phi_i^{-1})^*(\chi_i \omega)$$

Proposition 2.11. *The above definition is independent of the choice of the charts and of the partition of unity subordinate to them.*

Proof. Let $\{(V_j, \psi_j)\}_{j \in J}$ be another collection of charts and let $\{\zeta_j\}_{j \in J}$ be a partition of unity subordinate to $\{V_j \mid j \in J\}$. Then we have

$$\int_M \omega = \sum_{j \in J} \int_M \zeta_j \omega = \sum_{j \in J} \int_M \sum_{i \in I} \chi_i \zeta_j \omega = \sum_{i \in I, j \in J} \int_M \chi_i \zeta_j \omega$$

Note that $\chi_i \zeta_j \omega$ is supported on a single chart, since its support lies in $U_i \cap V_j$. Since the charts are consistently oriented and by Proposition 2.9, $\int_M \chi_i \zeta_j \omega$ can be evaluated using either (U_i, ϕ_i) or (V_j, ψ_j) and the output is the same. It follows that

$$\sum_{i \in I, j \in J} \int_M \chi_i \zeta_j \omega = \sum_{i \in I} \int_M \sum_{j \in J} \zeta_j \chi_i \omega = \sum_{i \in I} \int_M \chi_i \omega$$

□