LECTURE 12

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1. Structures on manifolds with boundaries

Recall that a manifold with boundary is an n-dimensional Hausdorff, second countable space in which every point has a neighbourhood homeomorphic to an open set of the upper half space

$$\mathbf{H}^n = \{(x_1, ..., x_n) \mid x_n \geq 0\}$$

An open set U of a manifold with boundary, M, together with a homeomorphism ϕ from U to an open subset of $\mathbf{H}^n$ is called a *generalised chart.*

Let

$$\partial\mathbf{H}^n = \{(x_1, ..., x_n) \mid x_n = 0\} \qquad Int\mathbf{H}^n = (x_1, ..., x_n) \mid x_n > 0\}$$

A point in M that is the inverse image of a point in $\partial\mathbf{H}^n$ of a generalised chart is called a *boundary point.* The set of such points is called ∂M. A point in M that is the inverse image of a point in $Int\mathbf{H}^n$ of a generalised chart is called an *interior point.* The set of such points is called $IntM$.

Exercise 1.1. *Let M be an n-dimensional topological manifold with boundary. Then $\partial M \cap IntM = \emptyset$.*

We use the convention that a map $F : A \to \mathbf{R}^n$ on an arbitrary subset $A \subset M$ is smooth if it has a smooth extension to an open set of M containing A.

Definition 1.2. (Smooth manifold with boundary) Let M be an n-dimensional topological manifold with boundary. We define an *atlas* for M as a collection of generalised charts whose domains cover M. Two such generalised charts $(U, \phi), (V, \chi)$ are *smoothly compatible* if

$$\chi \circ \phi^{-1} : \phi(U \cap V) \to \chi(U \cap V)$$

is a diffeomorphism. A *smooth atlas* on M is a collection of generalised charts on M that are pairwise smoothly compatible. A *smooth structure* on M is a maximal smooth atlas. An n-dimensional smooth manifold with boundary is an n-dimensional topological manifold with boundary endowed with a smooth structure.

Note that given a map $F : M \to N$ between manifolds with boundary, we define smoothness just as usual, i.e. we demand that the coordinate representation of the map is smooth with respect to any smooth chart.

Proposition 1.3. *Let M be a smooth n-dimensional manifold with boundary. Endowed with the subspace topology, $IntM, \partial M$ can be further endowed with smooth structures with the property that the inclusion maps are smooth.*

Proof. We first show this for $IntM$. For each generalised smooth chart (U, ϕ) for M, we define a smooth chart $(U \cap IntM, \phi \restriction_{IntM})$. The set of such charts clearly covers the space $IntM$ are they are pairwise smoothly compatible.

Now we show this for ∂M. Let (U, ϕ) be a smooth chart for M. Then we define the chart $(\tilde{U}, \tilde{\phi})$ for $\tilde{U} = U \cap \partial M$ where

$$\tilde{\phi} : U \cap \partial M \to \mathbf{R}^{n-1} \qquad \tilde{\phi} = \pi \circ \phi \text{ where } \pi : \mathbf{R}^n \to \mathbf{R}^{n-1} \text{ is the projection map}$$

Let $(U, \phi), (V, \chi)$ be two different charts on M, and let $(\tilde{U}, \tilde{\phi}), (\tilde{V}, \tilde{\chi})$ be the corresponding charts on ∂M. Then we have

$$\tilde{\phi} \circ \tilde{\chi}^{-1} = \pi \circ \phi \circ \chi^{-1} \circ j \text{ where } j : \mathbf{R}^{n-1} \to \mathbf{R}^n \text{ is the inclusion map}$$

(Check that $(\pi \circ \chi)^{-1} = \chi^{-1} \circ j$.) This is clearly smooth.

□

For the rest of the course, we shall fix the smooth structure that emerge from the previous proposition as the natural smooth structures on $\partial M, IntM$. So when we refer to either of $\partial M, IntM$, we shall implicitly refer to these smooth structures.

1.1. **The tangent space of a manifold with boundary.**

Definition 1.4. Let M be a manifold with boundary and $p \in M$. The tangent space at p, denoted by $T_p(M)$, is the vector space of derivations of $C^\infty(M)$ at p. That is, linear maps $w : C^\infty(M) \to \mathbf{R}$ that satisfy the product rule $w(fg) = f(p)w(g) + g(p)w(f)$.

We define the pushforward $F_* : T_pM \to T_{F(p)}N$ of a smooth map $F : M \to N$ between manifolds with boundary in the same way as before.

Proposition 1.5. *Let $i : \mathbf{H}^n \to \mathbf{R}^n$ be the inclusion map. Then*

$$i_* : T_p\mathbf{H}^n \to T_p\mathbf{R}^n$$

is an isomorphism with inverse given as follows. For $X \in T_p\mathbf{R}^n$ and $f \in C^\infty(\mathbf{H}^n)$, we consider any smooth extension $\tilde{f} : \mathbf{R}^n \to \mathbf{R}$ of f and set

$$Y(f) = X(\tilde{f}) \qquad (i_*)^{-1}(Y) = X$$

It follows that applying $(i_*)^{-1}$ to the coordinate bases $\frac{\partial}{\partial x^1} \mid_p, ... \frac{\partial}{\partial x^n} \mid_p$, we obtain a basis for $T_p(\mathbf{H}^n)$. We abuse notation and also refer to this basis as $\frac{\partial}{\partial x^1} \mid_p, ... \frac{\partial}{\partial x^n} \mid_p$.

Now let M be a smooth n-manifold with boundary. Let $p \in M$ and let (U, ϕ) be a smooth generalised chart such that $p \in U$. The pushforward

$$\phi_* : T_p(M) \to T_{\phi(p)}(\mathbf{H}^n)$$

induces an isomorphism. Hence applying ϕ_*^{-1} to the coordinate vectors $\frac{\partial}{\partial x^1} \mid_{\phi(p)}, ..., \frac{\partial}{\partial x^n} \mid_{\phi(p)}$, we obtain the basis for $T_p(M)$ which we denote as $\frac{\partial}{\partial \phi^1} \mid_p, ..., \frac{\partial}{\partial \phi^n} \mid_p$.

To compute the derivation $\frac{\partial}{\partial \phi^i} \mid_p$ to $f \in C^\infty(M)$, we do the following. Let $\widetilde{f \circ \phi^{-1}}$ be a smooth extension of $f \circ \phi^{-1}$ from $\phi(U) \subset \mathbf{H}^n$ to $\mathbf{R}^n$. Then we obtain

$$\frac{\partial}{\partial \phi^i} \mid_p f = \frac{\partial}{\partial x^i} \mid_{\phi(p)} \widetilde{f \circ \phi^{-1}}$$

Definition 1.6. Let M be a smooth n-manifold with boundary. The definitions of $\Lambda^k T_p^*(M)$ and $\Omega^k T_p^*(M)$ are analogous to the definitions as in the case of manifold without boundary.

1.2. **Orientations on smooth manifolds with boundary.**

Definition 1.7. Let M be a smooth n-manifold with boundary. Two generalised charts $(U, \phi), (V, \chi)$ are said to be *consistently oriented* if the transition function $\phi \circ \chi^{-1}$ is orientation preserving. An orientation on a smooth manifold with boundary is an atlas consisting of charts which are pairwise oriented. M is called *orientable* if it admits an orientation.

Equivalently, M is orientable if and only if it admits a continuous pointwise orientation, and if and only if it admits a nowhere vanishing n-form.

Note that if $IntM$ is oriented, then we can define a consistently oriented atlas on M by taking all the generalised charts ϕ such that $\phi \restriction IntM$ is positively oriented. Therefore, an orientation on $IntM$ induces an orientation on M.

1.3. **Integration on a manifold with boundary.** Let $f dx^1 \wedge ... \wedge dx^n$ be an n-form on $\mathbf{H}^n$. Then we define

$$\int_{\mathbf{H}^n} f dx^1 \wedge ... \wedge dx^n = \int_{\mathbf{H}^n} f dx^1 ... dx^n$$

Let M be an n-manifold with boundary. Given an n-form $\omega \in \Omega^n(M)$ that is supported in a chart (U, ϕ), we define

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega$$

The integral of a compactly supported n-form is define using partitions of unity, just like the case of a manifold without boundary.

1.4. **The induced orientation on the boundary.** Given an oriented atlas $\mathcal{A}$ on an n-manifold (possibly with a boundary), we define the oriented atlas $-\mathcal{A}$ as follows. For each chart (U, ϕ) in $\mathcal{A}$, we define a chart $(U, \phi') \in -\mathcal{A}$ where $\phi' \circ \phi^{-1}$ is orientation reversing. Note that $-\mathcal{A}$ is also an orientation (prove this!).

Let $(M, \mathcal{A})$ be a smooth oriented n-manifold with boundary. Let $\tilde{A}$ be the induced smooth atlas on ∂M. Recall that this is given as follows: Let (U, ϕ) be a smooth chart for M. Then we define the chart $(\tilde{U}, \tilde{\phi}) \in \tilde{A}$ for $\tilde{U} = U \cap \partial M$ where

$$\tilde{\phi} : U \cap \partial M \to \mathbf{R}^{n-1} \qquad \tilde{\phi} = \pi \circ \phi \text{ where } \pi : \mathbf{R}^n \to \mathbf{R}^{n-1} \text{ is the projection map}$$

Definition 1.8. If n is even, then we endow ∂M with the orientation $\tilde{\mathcal{A}}$. If n is odd, then we endow ∂M with the orientation $-\tilde{\mathcal{A}}$.

The above definition/convention will be justified in the proof of the Stokes theorem!

Remark 1.9. The above definition has the following feature. If $(X_1, ..., X_n)$ is a positively oriented bases for $T_p(\partial M), p \in \partial M$, then

$$(N, X_1, ..., X_n)$$

is a positively oriented basis for $T_p(M)$, where N is the "outward pointing" vector at $p \in M$. Outward pointing means that in any generalised chart (U, ϕ) containing p, the vector

$$N = \sum_{1 \le i \le n} c^i \frac{\partial}{\partial \phi^i} |_p$$

satisfies that $c^n < 0$. Think of it as "outward pointing" from $\mathbf{H}^n$ to its complement in $\mathbf{R}^n$.

Also, note that in the special case of a 1-dimensional manifold with boundary (for example, $[0, 1]$), we assign numerical quantities $+1$ or -1 to each of the boundary points as the "induced orientation". In the standard "left to right" orientation on $[0, 1]$, the point 1 is assigned $+1$ and the point 0 is assigned -1.

2. The Stokes theorem

Theorem 2.1. *Let $(M, \mathcal{A})$ be an oriented n-manifold with boundary. (Here $\mathcal{A}$ is the orientation on M. As usual, we let $\tilde{\mathcal{A}}$ be the induced orientation on ∂M.) Let $\omega \in \Omega_c^{n-1}(M)$, and given the inclusion map $i : \partial M \to M$, we denote $i^*\omega \in \Omega^{n-1}(\partial M)$ as ω. Then*

$$\int_M d\omega = \int_{\partial M} \omega$$

Remark 2.2. (Fundamental theorem of calculus) The stokes theorem generalised the fundamental theorem of calculus in the following manner. Given

$$\omega = f \in \Omega^0([a, b]) = C^\infty([a, b])$$

we have that

$$\int_{[a,b]} d\omega = \int_{[a,b]} f'(x)dx = \int_{\partial[a,b]} f = f(b) - f(a)$$

Note that here we have the standard "left to right" orientation on $[0, 1]$, and in the induced orientation on the boundary, the point 1 is assigned $+1$ and the point 0 is assigned -1.

Remark 2.3. (Fundamental theorem of line integrals) Let M be a smooth manifold and let $\gamma : [a, b] \to M$ be a smooth embedding, so that $S = \gamma([a, b])$ is an embedded 1-submanifold with boundary in M. If we give S the orientation such that γ is orientation preserving, then for any smooth function $f \in C^\infty(M)$, the Stokes theorem says that

$$\int_\gamma df = \int_{[a,b]} \gamma^* df = \int_S df = \int_{\partial S} f = f(\gamma(b)) - f(\gamma(a))$$

Here are a few important corollaries of the theorem.

Corollary 2.4. *(Integrals of closed forms over boundaries) Suppose that M is a compact oriented smooth n-manifold with boundary. If $\omega \in \Omega^n(M)$ is a closed form, then*

$$\int_{\partial M} \omega = \int_M d\omega = 0$$

Corollary 2.5. *(Integrals of exact forms) If M is a compact oriented smooth manifold without boundary, then the integral of every exact form over M is zero:*

$$\int_M d\omega = \int_{\partial M} \omega = 0$$

Corollary 2.6. *(Submanifolds) Let M be a smooth n-manifold (with or without boundary). Let $S \subseteq M$ be an oriented compact smooth k-submanifold (without boundary). Let $\omega \in \Omega^k_c(M)$ be closed. If $\int_S \omega \neq 0$, then the following holds:*

(1) *ω is not exact on M.*
(2) *S is not the boundary of an oriented compact smooth submanifold with boundary in M.*

Definition 2.7. In a manifold M, a regular compact domain is a set $D \subset M$, which is compact and has the property that for each $p \in \partial D$, there is a chart (U, ϕ) of M containing p such that

$$\phi(U \cap \partial D) \subset \partial \mathbf{H}^n \qquad \phi(U \cap D) \subset \mathbf{H}^n$$

Example 2.8. The closed 1-form

$$\omega = \frac{(xdy - ydx)}{x^2 + y^2}$$

has a nonzero integral over $\mathbf{S}^1$. Moreover, it is not exact on $\mathbf{R}^2 \setminus \{0\}$. The preceding corollary also tells us that $\mathbf{S}^1$ is not the boundary of a compact regular domain in $\mathbf{R}^2 \setminus \{0\}$.

Theorem 2.9. *(Green's theorem) Suppose that D is a compact regular domain in $\mathbf{R}^2$, and P, Q are smooth real valued functions on D. Then*

$$\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy = \int_{\partial D} Pdx - Qdy$$

Proof. We apply the Stokes theorem to the form

$$Pdx + Qdy \in \Omega^1(D)$$

□