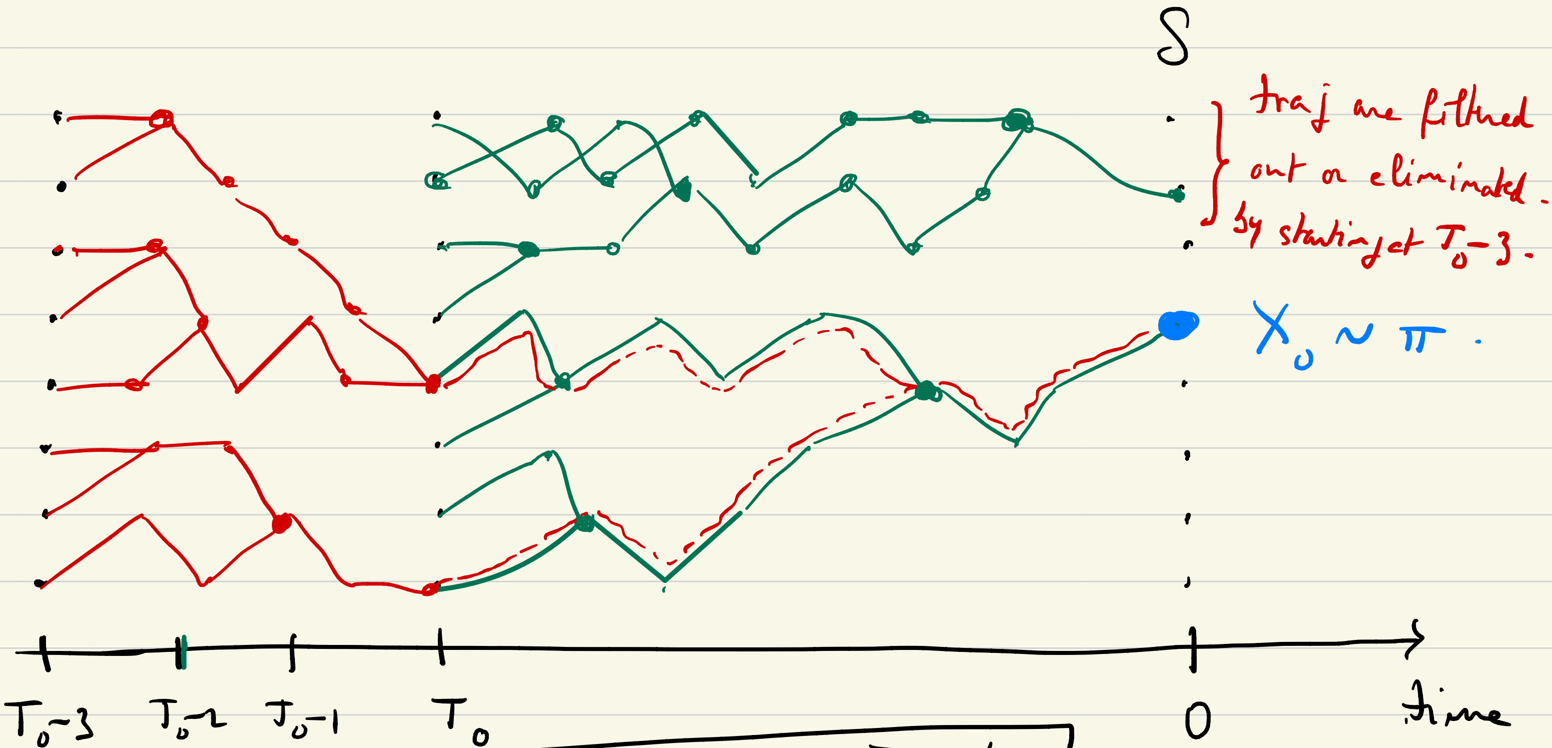
Discussion of lecture 13 (exact simulation.)

1) "Filtering" of bad trajectories in Propp Wilson algo.



$\boxed{\text{Trajectories move according to } U_{-n}\text{'s}}$

- No total coalescence of green trajectories.

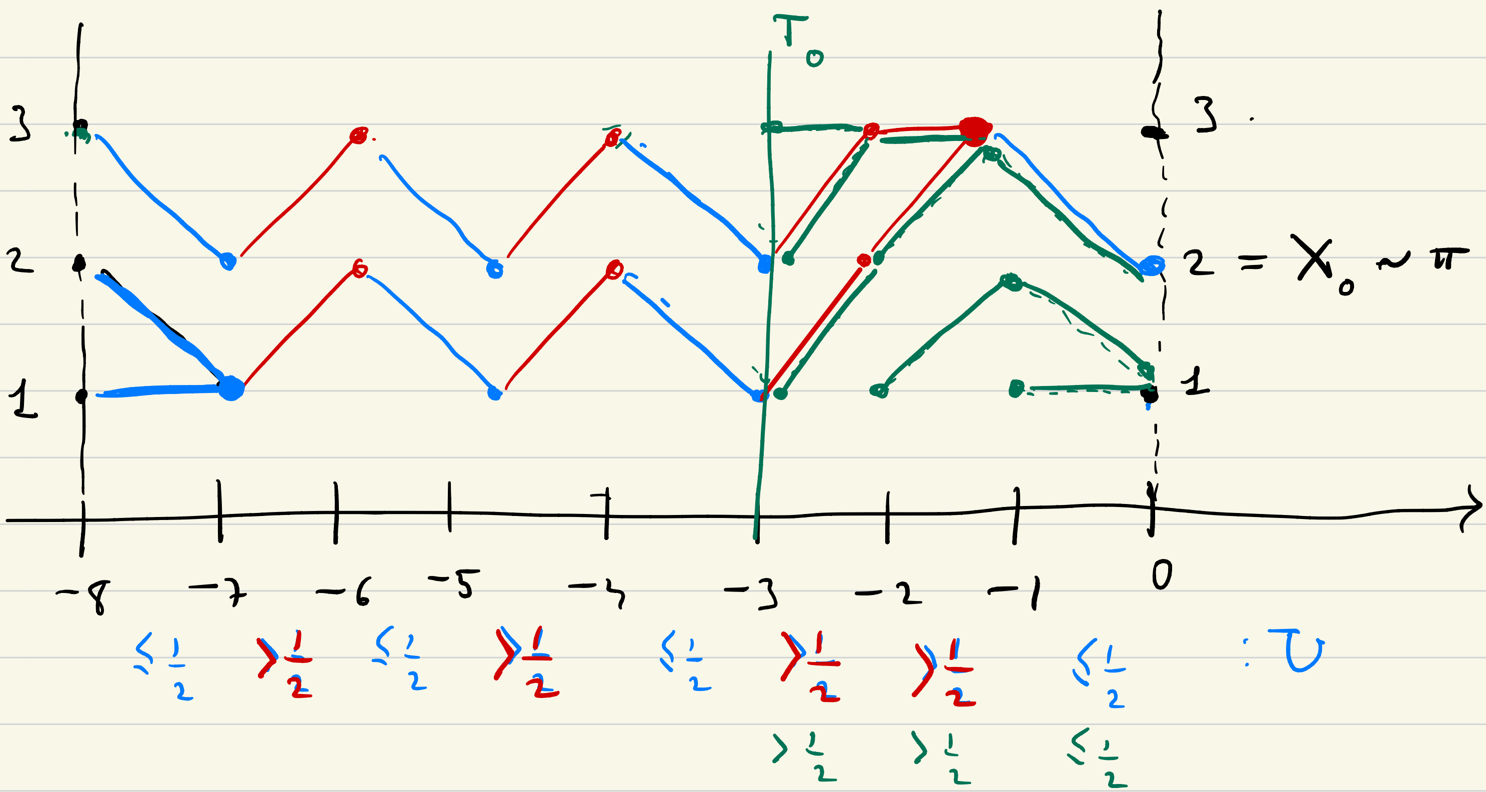
  $\Rightarrow$ shift back $T_0 \leftarrow T_0 - 1$ ect...

- Start further in the past will generate some coalescence events in the past and eventually filter out part of trajectories.

- Since $\mathbb{P}$(coalescence in finite time) $= 1$ eventually we end up with a set of traj where at time 0 there is a single sample. $X_0 \sim \pi$

2) Easy example of monotone coupling.

Sym random walk on $S = \{1, 2, 3\}$.



RM repr: $0 < u \leq \frac{1}{2}$ $\quad \phi(0, u) = 0 \quad \phi(i, u) = i - 1$ down

$\frac{1}{2} < u \leq 1 \quad \phi(N, u) = N \quad \phi(i, u) = i + 1$. up

- Coalescence happens only at states 1 or 3.
- We just have to test traj issued from 1 and 3 because the coupling is monotone: $i \leq j \Rightarrow \underbrace{\phi(i, u)}_{i \pm 1} \leq \underbrace{\phi(j, u)}_{j \pm 1}$.

3) Is MH chain monotone for ferromagnetic model?

Select v. Make $\underline{\sigma} \to \underline{\sigma}'$ with $\sigma'_w = \sigma_w$, $w \neq v$

and $\sigma'_v = -\sigma_v$ with prob $\min(1, e^{-\beta \Delta E})$

where $\Delta E = H(\underline{\sigma}') - H(\underline{\sigma}) = 2\sigma_v \sum_w J_{vw} \sigma_w$

RM repr: $0 \leq u \leq \min(1, e^{-\beta \Delta E})$ do $\sigma'_v = -\sigma_v$.

$\min(1, e^{-\beta \Delta E}) \leq u \leq 1$ do $\sigma'_v = \sigma_v$.

This is Not Monotone!

$(\sigma_1 \dots \sigma_N) \preceq (\tau_1 \dots \tau_N) \iff \sigma_i \leq \tau_i$

✓ we have that $\sigma_w \leq \tau_w \Rightarrow \sigma'_w \leq \tau'_w$ for $w \neq v$.

✓ for $\min(1, e^{-\beta \Delta E}) \leq u \leq 1$ we have that

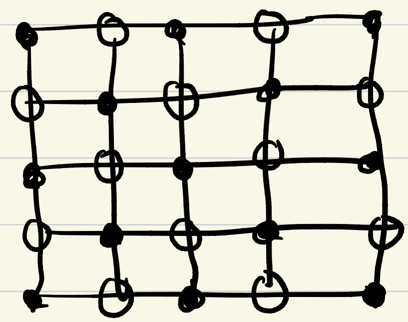
$$\sigma_v \leq \tau_v \Rightarrow \underbrace{\sigma'_v}_{\sigma_v} \leq \underbrace{\tau'_v}_{\tau_v}$$

But for $0 \leq u \leq \min(1, e^{-\beta \Delta E})$ we have that

$\sigma_v \leq \tau_v$ and $\underbrace{\sigma'_v}_{-\sigma_v} \geq \underbrace{\tau'_v}_{-\tau_v}$ $\boxed{\text{SO NOT MONOTONE}}$

4) Antiferromagnetic Model on square lattice.

$$H = -\sum_{(v,w)\in E} J_{vw}\,\sigma_v\sigma_w$$



lattice $= A \cup B$ bipartite

$\nearrow$ even $\bullet$ $\qquad$ $\uparrow$ odd. $\circ$

$\boxed{\text{take } J_{vw} < 0.}$

RM of Glauber dynamics: select v. Do $\sigma'_w = \sigma_w$ for $w \neq v$

$$0 \le u \le \frac{1}{2}\Big(1 + \tanh \sum_w J_{vw}\sigma_w\Big) \quad \text{do} \quad \sigma'_v = +1$$

$$\frac{1}{2}\Big(1 + \tanh \sum_w J_{vw}\sigma_w\Big) \le u \le 1 \quad \text{do} \quad \sigma'_v = -1.$$

This is Monotone wrt the partial order:

$$(\sigma_1 \cdots \sigma_N) \preceq (\tau_1 \cdots \tau_N) \iff \begin{array}{ll} \sigma_i \le \tau_i & i \in A \\ \sigma_i \ge \tau_i & i \in B \end{array}$$

Proof: as in class for Ferro case and usual partial order $(\sigma_1 \cdots \sigma_N) \preceq (\tau_1 \cdots \tau_N) \iff \sigma_i \le \tau_i$ all i