STOKES THEOREM

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1. The Stokes theorem

Theorem 1.1. *Let $(M, \mathcal{A})$ be an oriented n-manifold with boundary. (Here $\mathcal{A}$ is the orientation on M. As usual, we let $\tilde{\mathcal{A}}$ be the induced orientation on ∂M.) Let $\omega \in \Omega_c^{n-1}(M)$, and given the inclusion map $i : \partial M \to M$, we denote $i^*\omega \in \Omega^{n-1}(\partial M)$ as ω. Then*

$$\int_M d\omega = \int_{\partial M} \omega$$

Proof. We first consider the special case of $M = \mathbf{H}^n$. Let $\omega \in \Omega_c^{(n-1)}(\mathbf{H}^n)$. Since ω is supported in a compact set, there is a "rectangle" of the form

$$A = [-r, r] \times ... \times [-r, r] \times [0, s] \subset \mathbf{H}^n \qquad r > 0$$

such that $\omega(x) = 0$ for each $x \in \mathbf{H}^n \setminus A$. Note that in particular, this means that $\omega(x) = 0$ whenever $x \in Int(\mathbf{H}^n) \cap \partial A$, however the restriction of ω to $\partial\mathbf{H}^n \cap \partial A$ may be nontrivial (this is indeed exactly the case when the theorem is interesting!).

We write

$$\omega = \sum_{1 \le i \le n} \omega_i dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n$$

where the $\widehat{dx^i}$ denotes that the dx^i term has been omitted. We compute

$$d\omega = \sum_{1 \le i \le n} d\omega_i \wedge dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n$$

$$= \sum_{1 \le i,j \le n} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n$$

$$= \sum_{1 \le i \le n} \frac{\partial \omega_i}{\partial x^i} dx^i \wedge dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n$$

$$= \sum_{1 \le i \le n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge ... \wedge dx^n$$

We compute

$$\int_{\mathbf{H}^n} d\omega = \int_{\mathbf{H}^n} \sum_{1 \le i \le n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge ... \wedge dx^n$$

$$= \sum_{1 \le i \le n} (-1)^{i-1} \int_{-r}^{r} ... \int_{-r}^{r} \int_0^r \frac{\partial \omega_i}{\partial x^i} dx^1 ... dx^n$$

Since the order of the integrals can be changed without changing the integral (by Fubini's theorem), we obtain that the above equals:

$$= \sum_{1 \le i < n} (-1)^{i-1} \int_0^r \int_{-r}^{r} ... \int_{-r}^{r} \frac{\partial \omega_i}{\partial x^i} dx^i dx^1 ... dx^{i-1} dx^{i+1} ... dx^n + (-1)^{n-1} \int_{-r}^{r} ... \int_{-r}^{r} \int_0^r \frac{\partial \omega_n}{\partial x^n} dx^n dx^1 ... dx^{n-1} \qquad (*)$$

We analyse the first term to obtain

$$\sum_{1 \le i < n} (-1)^{i-1} \int_0^r \int_{-r}^{r} ... \int_{-r}^{r} \frac{\partial \omega_i}{\partial x^i} dx^i dx^1 ... dx^{i-1} dx^{i+1} ... dx^n$$

$$= \sum_{1 \le i < n} (-1)^{i-1} \int_0^r \int_{-r}^{r} ... \int_{-r}^{r} [\omega_i(x)]_{x_i=-r}^{x_i=r} dx^1 ... dx^{i-1} dx^{i+1} ... dx^n = 0$$

Since $\omega(x) = 0$ whenever $x^i \in \{-r, +r\}$ for $i \neq n$. So in the sum $(*)$ above, only the second term remains:

$$(-1)^{n-1}\int_{-r}^{r}...\int_{-r}^{r}\int_{0}^{r}\frac{\partial\omega_n}{\partial x^n}dx^n dx^1...dx^{n-1}$$

$$= (-1)^{n-1}\int_{-r}^{r}...\int_{-r}^{r}(\omega_n(x^1,...,x^{n-1},r) - \omega_n(x^1,...,x^{n-1},0))dx^1...dx^{n-1}$$

$$= (-1)^{n-1}\int_{-r}^{r}...\int_{-r}^{r}(0 - \omega_n(x^1,...,x^{n-1},0))dx^1...dx^{n-1}$$

$$= (-1)^{n}\int_{-r}^{r}...\int_{-r}^{r}\omega_n(x^1,...,x^{n-1},0)dx^1...dx^{n-1} \qquad (**)$$

Now we analyse

$$\int_{\partial\mathbf{H}^n}\omega = \int_{\partial\mathbf{H}^n}\sum_{1\leq i\leq n}\omega_i dx^1\wedge...\wedge\widehat{dx^i}\wedge...\wedge dx^n$$

$$\sum_{1\leq i\leq n}\int_{\partial\mathbf{H}^n}\omega_i dx^1\wedge...\wedge\widehat{dx^i}\wedge...\wedge dx^n$$

Since $dx^n = 0$ on $\partial\mathbf{H}^n$, the only nonzero term in the above sum is

$$\int_{\partial\mathbf{H}^n}\omega_n dx^1\wedge...\wedge dx^{n-1}$$

This equals

$$\int_{A\cap\partial\mathbf{H}^n}\omega_n(x^1,...,x^n,0)dx^1\wedge...\wedge dx^{n-1} \qquad (***)$$

Taking into account the orientation for $\partial\mathbf{H}^n$: when n is even, it is positively oriented and when n is odd, it is negatively oriented, we obtain that $(***) = (**)$, proving the Theorem for the case of $\mathbf{H}^n$.

Note that in the special case of $M = \mathbf{R}^n$, where there is no boundary, the same computation as above goes through with the additional fact that the $i = n$ term vanishes as well. So both sides of the equation are equal to 0.

Now let M be an arbitrary smooth manifold with boundary. Consider a form $\omega \in \Omega_c^{(n-1)}(M)$ such that $supp(\omega) \subset U$ for a smooth positively oriented chart (U, ϕ). Then we have

$$\int_M d\omega = \int_{\mathbf{H}^n}(\phi^{-1})^* d\omega = \int_{\mathbf{H}^n}d((\phi^{-1})^*\omega) = \int_{\partial\mathbf{H}^n}(\phi^{-1})^*\omega$$

where $\partial\mathbf{H}^n$ is given the induced orientation. Note that $\phi \restriction U \cap \partial M$ is an orientation preserving diffeomorphism onto $\phi(U) \cap \partial\mathbf{H}^n$. It follows that

$$\int_{\partial\mathbf{H}^n}(\phi^{-1})^*\omega = \int_{\partial M}\omega$$

This proves the theorem in this case.

Finally, let $\omega \in \Omega_c^{n-1}(M)$ be an arbitrary compactly supported smooth $n-1$ form. Let $\{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ be a finite collection of charts that covers $supp(\omega)$. Let $\{\psi_\alpha \mid \alpha \in I\}$ be a partition of unity subordinate to this collection. Then we have

$$\int_{\partial M}\omega = \sum_{\alpha\in I}\int_{\partial M}\psi_\alpha\omega = \sum_{\alpha\in I}\int_M d(\psi_\alpha\omega)$$

(To understand the second equality above, note that since each $\psi_\alpha\omega$ is an $n-1$ form whose support is contained in a single chart, we can apply the theorem for this special case).

$$\sum_{\alpha\in I}\int_M d(\psi_\alpha\omega) = \sum_{\alpha\in I}\int_M(d\psi_\alpha\wedge\omega + \psi_\alpha d\omega)$$

$$= \int_M\sum_{\alpha\in I}d\psi_\alpha\wedge\omega + \sum_{\alpha\in I}\int_M\psi_\alpha d\omega$$

$$= \int_M d(\sum_{\alpha\in I}\psi_\alpha)\wedge\omega + \sum_{\alpha\in I}\int_M\psi_\alpha d\omega$$

$$= \int_M d(1_M)\wedge\omega + \sum_{\alpha\in I}\int_M\psi_\alpha d\omega$$

$$= 0 + \sum_{\alpha\in I}\int_M\psi_\alpha d\omega = \int_M d\omega$$

□

Remark 1.2. (Fundamental theorem of calculus) The stokes theorem generalised the fundamental theorem of calculus in the following manner. Given

$$\omega = f \in \Omega^0([a,b]) = C^\infty([a,b])$$

we have that

$$\int_{[a,b]} d\omega = \int_{[a,b]} f'(x)dx = \int_{\partial[a,b]} f = f(b) - f(a)$$

Note that here we have the standard "left to right" orientation on $[0,1]$, and in the induced orientation on the boundary, the point 1 is assigned $+1$ and the point 0 is assigned -1.

Remark 1.3. (Fundamental theorem of line integrals) Let M be a smooth manifold and let $\gamma : [a,b] \to M$ be a smooth embedding, so that $S = \gamma([a,b])$ is an embedded 1-submanifold with boundary in M. If we give S the orientation such that γ is orientation preserving, then for any smooth function $f \in C^\infty(M)$, the Stokes theorem says that

$$\int_\gamma df = \int_{[a,b]} \gamma^* df = \int_S df = \int_{\partial S} f = f(\gamma(b)) - f(\gamma(a))$$

Here are a few important corollaries of the theorem.

Corollary 1.4. *(Integrals of closed forms over boundaries) Suppose that M is a compact oriented smooth n-manifold with boundary. If $\omega \in \Omega^n(M)$ is a closed form, then*

$$\int_{\partial M} \omega = \int_M d\omega = 0$$

Corollary 1.5. *(Integrals of exact forms) If M is a compact oriented smooth manifold without boundary, then the integral of every exact form over M is zero:*

$$\int_M d\omega = \int_{\partial M} \omega = 0$$

Corollary 1.6. *(Submanifolds) Let M be a smooth n-manifold (with or without boundary). Let $S \subseteq M$ be an oriented compact smooth k-submanifold (without boundary). Let $\omega \in \Omega_c^k(M)$ be closed. If $\int_S \omega \neq 0$, then the following holds:*

1. *ω is not exact on M.*
2. *S is not the boundary of an oriented compact smooth submanifold with boundary in M.*

Definition 1.7. In a manifold M, a regular compact domain is a set $D \subset M$, which is compact and has the property that for each $p \in \partial D$, there is a chart (U, ϕ) of M containing p such that

$$\phi(U \cap \partial D) \subset \partial \mathbf{H}^n \qquad \phi(U \cap D) \subset \mathbf{H}^n$$

Example 1.8. The closed 1-form

$$\omega = \frac{(xdy - ydx)}{x^2 + y^2}$$

has a nonzero integral over $\mathbf{S}^1$. Moreover, it is not exact on $\mathbf{R}^2 \setminus \{0\}$. The preceding corollary also tells us that $\mathbf{S}^1$ is not the boundary of a compact regular domain in $\mathbf{R}^2 \setminus \{0\}$.

Theorem 1.9. *(Green's theorem) Suppose that D is a compact regular domain in $\mathbf{R}^2$, and P, Q are smooth real valued functions on D. Then*

$$\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy = \int_{\partial D} Pdx - Qdy$$

Proof. We apply the Stokes theorem to the form

$$Pdx + Qdy \in \Omega^1(D)$$

□