EPFL - Autumn 2019	
Introduction to differentiable manifolds	**Exercises**
Solutions 7	5 November 2019

Exercise 1. We need to show that the injective immersion F is a homeomorphism onto its image. Since F is smooth, it is continuous. Let us denote $\tilde{F}$ the map where we restrict the codomain of F to the set $F(M)$, i.e. $\tilde{F} : N \to F(M) : x \to F(x)$. Note that $\tilde{F}$ is a continuous bijection (where the image has the subspace topology of M). We know that a continuous bijection from a compact topological space to a Hausdorff topological space is open. Thus we can conclude that $\tilde{F}$ is open in the sense of subspace topology of $\tilde{F}(M)$. Hence $\tilde{F}^{-1}$ is continuous. Therefore F is a homeomorphism onto its image.

Exercise 2.
(i) First notice that since each S^1 is embedded in $\mathbb{R}^2$ then $\mathbb{T}^2$ may be embedded in $\mathbb{R}^4$. However below we prove that we can think of the torus $\mathbb{T}^2$ as the surface of a "doughnut" in $\mathbb{R}^3$. We use the smooth atlas on S^1 given by graph coordinates, i.e. we have 4 local charts: $\{(U_{i,\pm}, \varphi_{i,\pm})\}_{i=1,2}$, where

$$U_{i,\pm} = \Big\{(u_1, u_2) \in \mathbb{R}^2 : u_i \in (-1,1),\ u_j = \pm\sqrt{1-u_i^2}\Big\}, \quad \text{and} \quad \varphi_{i,\pm}(u_1,u_2) = u_i$$

Recall that the inverse of the charts are $(\varphi_{1,\pm})^{-1}(u) = \big(u, \pm\sqrt{1-u^2}\big)$, and $(\varphi_{2,\pm})^{-1}(u) = \big(\pm\sqrt{1-u^2}, u\big)$. Let us use the variables u_i for the first circle and the variables v_i for the second circle, then the natural smooth structure on $\mathbb{T}^2$ is composed of 8 local charts: $\{(U_{i,\pm} \times V_{j,\pm}, \varphi_{i,\pm} \times \varphi_{j,\pm})\}_{i,j=1,2}$. Consider the function F as a function from $\mathbb{R}^4$ to $\mathbb{R}^3$:

$$F(u_1,u_2,v_1,v_2) = ((2+u_1)v_1, (2+u_1)v_2, u_2)$$

The coordinates representation of F w.r.t the first chart is

$$\hat{F} := F \circ \big((\varphi_{1,+})^{-1} \times (\varphi_{1,+})^{-1}\big) : (-1,1)^2 \to \mathbb{R}^3$$
$$\hat{F}(u,v) = ((2+u)v, (2+u)\sqrt{1-v^2}, \sqrt{1-u^2})$$

which has Jacobian matric

$$J_{\hat{F}}(u,v) = \begin{pmatrix} v & \sqrt{1-v^2} & -\frac{u}{\sqrt{1-v^2}} \\ 2+u & -(2+u)\frac{v}{\sqrt{1-v^2}} & 0 \end{pmatrix}$$

The rank of this matrix is maximal: $\mathrm{rank} J_{\hat{F}}(u,v) = 2$ for every $(u,v) \in (-1,1)^2$. Analogously one can write the coordinates representation of F w.r.t the other charts and show that the rank of the Jacobian is 2, hence F is an immersion. Moreover F is injective and since $S^1 \times S^1$ is compact, from Exercise 1 we obtain the F is an embedding. Then $F(\mathbb{T}^2)$ is an embedded submanifold of $\mathbb{R}^3$ since it is the image of an embedding.

(ii) From the previous point we know that we can think of $S = F(\mathbb{T}^2)$ as the surface in $\mathbb{R}^3$ parametrized by

$$\begin{cases} x = (2+\cos\varphi)\cos\theta \\ y = (2+\cos\varphi)\sin\theta \\ z = \sin\varphi \end{cases}$$

Then we obtain $f(x,y,z) = (2-(2+\cos\varphi))^2 + \sin^2\varphi = 1$, hence $S \subset f^{-1}(1)$. To show that the level surface does not contain any points other than those in S set $u = \sqrt{x^2+y^2} - 2$. Then the level surface is given by $u^2 + z^2 = 1$ which is the equation of a circle of radius 1 in (u,z). Hence, we can find φ such that $u = \cos(\varphi)$ and $z = \sin\varphi$. Then we also have $(u+2)^2 = x^2+y^2$ which is the equation of a circle of radius $u+2$ in (x,y) and so we can find θ such that $x = (u+2)\cos\theta$ and $y = (u+2)\sin\theta$. Therefore $S = f^{-1}(1)$

We have to show that f has rank 1 at all points of S, that is that $J_f = \nabla f \neq 0$ on S. However

$$\nabla f(x,y,z) = \Big(\frac{x\sqrt{x^2+y^2}-2x}{\sqrt{x^2+y^2}}, \frac{y\sqrt{x^2+y^2}-2y}{\sqrt{x^2+y^2}}, 2z\Big)$$

is not well defined along the z-axis. Thus let us consider $\tilde{f} = f\big|_{\mathbb{R}^3\setminus\{z\text{-axis}\}}$, then we still have that $S = \tilde{f}^{-1}(1)$ and moreover $\tilde{f}$ is smooth and has constant rank 1 on $\mathbb{R}^3 \setminus \{z\text{-axis}\}$. From the Constant Rank Level Set Theorem we deduce that S is an embedded submanifold of $\mathbb{R}^3 \setminus \{z\text{-axis}\}$. This implies that S is an embedded submanifold of $\mathbb{R}^3$.

Exercise 3. As in Exercise 1, we denote $\tilde{\sigma} : N \to \sigma(N) : x \to \sigma(x)$ be the map where we restrict the codomain of σ to the set $\sigma(N)$. Since σ is a smooth embedding, then $\tilde{\sigma}$ is a smooth map as well. Recall that the natural smooth structure on $\sigma(N)$ is the one obtained by slice charts, i.e. if $\mathcal{A}'$ is the smooth structure of M then the smooth structure on $\sigma(N)$ [1] is

$$\mathcal{A} = \{(V,\psi) :\ V = U\cap\sigma(N),\ \psi = \pi\circ\varphi\big|_V,\ \text{where } (U,\varphi)\in\mathcal{A}' \text{ is a slice chart}\}$$

Let $p \in N$. We want to look at the coordinate representation of $\tilde{\sigma}$ in a neighborhood of p with respect to appropriate charts. Let (W,θ) be a smooth chart for N, $p \in N$. Let (U,φ) be a slice chart for $\sigma(N)$ in M, $\sigma(p) \in U$, and let (V,ψ) be the corresponding chart on $\sigma(N)$ as defined above, i.e. $V = U \cap \sigma(N)$, $\psi = \pi \circ \varphi\big|_V$. Then the coordinate representations for σ and $\tilde{\sigma}$ respectively are

$$\begin{aligned} \varphi\circ\sigma\circ\theta^{-1} : \theta(W\cap\sigma^{-1}(U)) \subset \mathbb{R}^k &\ \to\ \varphi(U\cap\sigma(W)) \subset \mathbb{R}^m \\ (x^1,\dots,x^k) &\ \mapsto\ (y^1,\dots,y^k,0,\dots,0) \end{aligned}$$

and

$$\begin{aligned} \psi\circ\tilde{\sigma}\circ\theta^{-1} : \theta(W\cap\sigma^{-1}(U)) \subset \mathbb{R}^k &\ \to\ \varphi(U\cap\sigma(W)) \subset \mathbb{R}^k \\ (x^1,\dots,x^k) &\ \mapsto\ (y^1,\dots,y^k) \end{aligned}$$

for some functions y^i, which are smooth, because σ is smooth by assumption. Here $k = \dim N = \dim\sigma(N)$, and $m = \dim M$. Note that we have implicitly used here that $\sigma^{-1}(U) = \sigma^{-1}(U\cap\sigma(N)) = \tilde{\sigma}^{-1}(V)$. In particular, from the above we see that the coordinate representation of $\tilde{\sigma}$ is smooth. Moreover, the rank of the Jacobian

$$D(\varphi\circ\sigma\circ\theta^{-1})(x)$$

is maximal, i.e. equal to k, for all points $x \in W\cap\sigma^{-1}(U)$ (actually we only need $x = \theta(p)$ here), because σ is an immersion. From the above coordinate representation we see that the last $m-k$ rows of this matrix are 0 and the upper $k\times k$ submatrix is precisely the Jacobian of the coordinate representation of $\tilde{\sigma}$, hence

$$\operatorname{rank} D(\psi\circ\tilde{\sigma}\circ\theta^{-1})(x) = k,$$

i.e. it is a regular matrix. In particular this holds for $x = \theta(p)$. Therefore, by the implicit function theorem $\psi\circ\tilde{\sigma}\circ\theta^{-1}$ is a local diffeomorphism at p and since p was arbitrary, $\tilde{\sigma}$ is a local diffeomorphism on N. Since $\tilde{\sigma}$ is a bijection, $\tilde{\sigma}$ is a diffeomorphism. So σ is diffeomorphism onto its image $\sigma(N)$.

[1] $\pi : \mathbb{R}^n \to \mathbb{R}^k$ is the projection onto first k-components

Exercise 4.

(i) The gradient of f,
$$\nabla f(x,y) = (3x^2, 3y^2),$$
vanishes precisely at the origin $(x,y) = (0,0)$. Thus $f_* : T_p\mathbb{R}^2 \to T_{f(p)}\mathbb{R}$ has rank 0 if and only if $p = (x,y) = (0,0)$. Thus every point $c \in \mathbb{R}$ is a regular value except $c = 1$.

By the constant rank theorem, each level set $f^{-1}(\{c\})$ with $c \neq 1$ is a smooth embedded submanifold in $\mathbb{R}^2$. As for the level set $f^{-1}(\{1\})$ we have to argue differently. The theorem does not say that $f^{-1}(\{1\})$ is not a smooth submanifold. We have to study this case separately. Observe that in this case one has
$$f^{-1}(\{1\}) = \{x^3 + y^3 = 0\} = \{x = -y\}$$
i.e., $f^{-1}(\{1\})$ is a line going through the origin. Thus, also $f^{-1}(\{1\})$ is a smooth submanifold of $\mathbb{R}^2$. Summing up, all level sets of this function are smooth submanifolds.

(ii) Let us apply Proposition 3.1.15, which holds if c is a regular value. Thus suppose $c \neq 1$, then we obtain $T_pS = \ker f_*\big|_p$. Therefore if $X = (X_1, X_2) \in \mathbb{R}^2 \cong T_p\mathbb{R}^2$, then $f_*\big|_p X = 3p_x^2 X_1 + 3p_y^2 X_2$, where $p = (p_x, p_y)$. Hence
$$\ker f_*\big|_p = \{(X_1, X_2) : p_x^2 X_1 + p_y^2 X_2 = 0\}$$
where $p_x^3 + p_y^3 - 1 = c$. When $c = 1$ we already notice that $S = \{x = -y\}$, thus $T_pS = S$.