EPFL - Autumn 2019	
Introduction to differentiable manifolds	**Exercises**
Solutions 12	10 December 2019

Exercise 1.

(1) Let $\mathcal{A}$ by any smooth atlas on M, wlog we assume the charts in $\mathcal{A}$ all have connected domains. For each chart (U, x^i) in $\mathcal{A}$, a nowhere vanishing n-form $\omega \in \Omega^n(M)$ can be expressed as

$$\omega = f \; dx^1 \wedge \dots, \wedge dx^n \quad \text{in } U$$

Since $f \in C^\infty(U)$ and U is connected, we know f can't change sign in U. For each coordinates chart, if $f < 0$ in U, we replace x^1 by $-x^1$ and denote the new atlas $\mathcal{A}'$. We show that $\mathcal{A}'$ is consistently oriented atlas. Let (U_α, x^i_α) and (U_β, x^i_β) be two charts in $\mathcal{A}'$ where $U_\alpha \cap U_\beta \neq \emptyset$. Then we have

$$\begin{aligned} \omega &= f \; dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha \\ &= g \; dx^1_\beta \wedge \cdots \wedge dx^n_\beta \end{aligned}$$

where $f, g > 0$ on $U_\alpha \cap U_\beta$. By the change of coordinates formula, we have

$$0 < g = \det(d\phi_{\alpha\beta}) \; f$$

where $\phi_{\alpha\beta} := x_\beta \circ x_\alpha^{-1}$ is the transition function of the two charts. Hence $\det(d\phi_{\alpha\beta}) > 0$ for all α, β. Thus M is orientable.

(2) For an orientable manifold M, we have an orientated atlas $\mathcal{A} := \{(U_\alpha, \phi_\alpha) | \; \alpha \in I\}$. Then for each local coordinate chart $(U_\alpha, \phi_\alpha) \in \mathcal{A}$, it determines a corresponding pointwise orientation $O^\alpha_p := \left[\left(\frac{\partial}{\partial \phi^1_\alpha}\Big|_p, \dots, \frac{\partial}{\partial \phi^n_\alpha}\Big|_p\right)\right]$ for $p \in U_\alpha$. Then considering two charts (U_α, ϕ_α) and (U_β, ϕ_β) where $U_\alpha \cap U_\beta \neq \emptyset$, by the change of coordinate formula, we have

$$\left(\frac{\partial}{\partial \phi^1_\beta}, \dots, \frac{\partial}{\partial \phi^n_\beta}\right) = J_{d\phi_{\alpha\beta}} \cdot \; \left(\frac{\partial}{\partial \phi^1_\alpha}, \dots, \frac{\partial}{\partial \phi^n_\alpha}\right)$$

where $\phi_{\alpha\beta} := \phi_\beta \circ \phi_\alpha$ is the transition function and $J_{d\phi_{\alpha\beta}}$ denotes the Jacobian of the transition function.

Since $\mathcal{A}$ is consistently orientable, we have $\det(J_{d\phi_{\alpha\beta}}) > 0$. Thus $O^\alpha_p = O^\beta_p$ for $p \in U_\alpha \cap U_\beta$. Then we can define an orientation $O := \{O_p\}_{p \in M}$ s.t. $O_p := O^\alpha_p$ whence $p \in U_\alpha$. It is a well defined pointwise orientation on M. And it's continuos by the definition.

Conversely, suppose M has a continuos pointwise orientation. We can use the same method as in (1) to construct a new atlas $\mathcal{A}'$ from $\mathcal{A}$. If a chart $(U_\alpha, \phi_\alpha) \in \mathcal{A}$ is negatively oriented, i.e. $\left[\left(\frac{\partial}{\partial \phi^1_\alpha}\Big|_p, \dots, \frac{\partial}{\partial \phi^n_\alpha}\Big|_p\right)\right] \notin O_p$, then we change ϕ^1_α to $-\phi^1_\alpha$ to construct a new chart in $\mathcal{A}'$. Thereafter, by the change of coordinate formula (2), we can conclude that $\det(J_{d\phi_{\alpha\beta}}) > 0$ for all $\alpha, \beta \in I'$. Thus $\mathcal{A}'$ is a consistently oriented atlas. So M is orientable.

Exercise 2. For a parallelizable manifold M, its tangent bundle TM is trivial. From Exercise 1 of Sheet 5, we know that TM has a global frame. We can just pick a smooth vector field $(X_1, \dots, X_n)$ on M as a global frame for TM. Let's define the pointwise orientation $O_p = [(X_1(p), \dots, X_n(p))]$. Then we know pointwise $O := \{O_p\}_{p \in M}$ is a continuos pointwise orientation by the definition. From Exercise 1, we can conclude that any parallelizable manifold is orientable.

For a torus $\mathbb{T}^n := S^1 \times \dots S^1$, we know $T\mathbb{T}^n \cong TS^1 \times \dots \times TS^1$. Since TS^1 is trivial as we proved in Exercise 1 of Sheet 5. We can conclude that $T\mathbb{T}^n \cong TS^1 \times \dots \times TS^1 \cong \mathbb{T}^n \times \mathbb{R}^n$ is trivial as well. Thus $\mathbb{T}^n$ is parallelizable, so it is also orientable.

Exercise 3. Assume that the real projective plane $\mathbb{P}^2 := \mathbb{R}^3 \setminus \{0\}/\sim$, where $x \sim y$ iff $x = \lambda y$ for $\lambda \in \mathbb{R}$ as defined in Exercise 5 of Sheet 1 is orientable, then we can find a nowehre vanishing 2-form $\omega \in \Omega^2(\mathbb{P}^2)$.

We use the standard atlas on $\mathbb{P}^2$, where $U_i := \{[x] \in \mathbb{P}^2 | x_i \neq 0\}$ and $\phi_i : U_i \to \mathbb{R}^2$ for $i = 1, 2, 3$. For $p := [(x_1, x_2, x_3)] \in U_1 \cap U_3$, we can express ω in term of the two charts as

$$\omega = f d\phi_1^1 \wedge d\phi_1^2 = g d\phi_3^1 \wedge d\phi_3^2,$$

where $f, g \in C^\infty(\mathbb{P}^2)$ and f, g will not change sign on U_1 and U_3 correspondingly. Since U_3 is a connected domain, w.l.o.g, we can assume that $g > 0$ on U_3. Then we apply the change of coordinate formula, we have

$$0 < g = \det(J_{d\phi_{1,3}})\ f$$

where $\phi_{1,3} = \phi_3 \circ \phi_1^{-1} : (x_2, x_3) \to \left(\frac{1}{x_3}, \frac{x_2}{x_3}\right)$ is the transition function of the two charts. Notice that $\det(J_{d\phi_{1,3}}\big|_{(x_2,x_3)}) = -\frac{1}{(x_2)^3}$, thus $\det(J_{d\phi_{1,3}})$ will change sign on $\phi_1(U_1 \cap U_3)$. Therefore, f will change sign on U_3 which will contradict the assumption that f doesn't change sign on U_1. So $\mathbb{P}^2$ is not orientable.

Exercise 4. Suppose that $\partial M \cap \mathrm{Int} M \neq \emptyset$, then we can find $p \in M$ and smooth charts $\phi : U \to \mathbb{H}^n$ and $\psi : U \to \mathbb{H}^n$ s.t.

$$\begin{cases} \phi(p) \in \partial\mathbb{H}^n \\ \psi(p) \in \mathrm{Int}\mathbb{H}^n \end{cases} \tag{1}$$

Then let's denote $V := \phi(U)$ and $\tilde{V} := \psi(U)$. We can suppose $\tilde{V}$ is open in $\mathbb{R}^n$. Then $\phi \circ \psi^{-1} : \tilde{V} \to V$ is a smooth map in the standard sense and $J_{(\phi\circ\psi^{-1})}(\psi(p))$ is regular. In particular

$$\frac{\partial(\phi^n \circ \psi^{-1})}{\partial(x^1, \dots, x^n)}\ (\psi(p)) \neq (0, \dots, 0)$$

Therefore, any neighborhood of $\psi(p)$ in $\tilde{V}$ has a point such that $\phi^n \circ \psi^{-1}$ (the n^{th} coordinate) is negative which is not possible, since ϕ maps into $\mathbb{H}^n$ (i.e. $\phi^n(x) \geq 0$ for all $x \in U$).