12.1. We have

$$\frac{d^2}{dt^2}\|Z_t\|^2 = 2\frac{d}{dt}\langle \nabla_t Z, Z\rangle = 2\langle (\nabla_t)^2 Z, Z\rangle + 2\|\nabla_t Z\|^2$$
$$= -2\langle R(Z,\dot\gamma)\dot\gamma, Z\rangle + 2\|\nabla_t Z\|^2.$$

Since the manifold has nonpositive sectional curvature, the above quantity is nonnegative.

Using the identity $\frac{d^2}{dt^2}\sqrt{f} = \frac{ff''-\frac{1}{2}f'^2}{2f^{3/2}}$ with $f(t) = \|Z_t\|^2$, we obtain

$$\frac{d^2}{dt^2}\|Z_t\| = \frac{-\|Z\|^2\langle R(Z,\dot\gamma)\dot\gamma, Z\rangle + \|Z\|^2\|\nabla_t Z\|^2 - \langle \nabla_t Z, Z\rangle^2}{\|Z\|^3}.$$

Now the Schwarz's inequality $\|Z\|^2\|\nabla_t Z\|^2 \geq \langle \nabla_t Z, Z\rangle^2$ implies that $\frac{d^2}{dt^2}\|Z_t\| \geq 0$ since $\langle R(Z,\dot\gamma)\dot\gamma, Z\rangle \leq 0$.

12.2. This is a direct calculation.

12.3. La solution se trouve dans le Do Carmo p.107

12.4. First we shall prove the following:

Theorem (Weinstein) Let f be an isometry of a compact, oriented Riemannian manifold M with positive sectional curvature. Suppose further that M is even dimensional and f preserves the orientation. Then f has a fixed point.

An important step in the proof of the above is the following:

Lemma Let A be an orthogonal linear transformation of $\mathbb{R}^{n-1}$ such that $det(A) = (-1)^n$. Then A fixes some non zero vector.

Proof:

If n is even, then the associated polynomial $det(A - \lambda I)$ has odd degree. Therefore A has a real eigenvalue. Since A is orthogonal, this eigenvalue is either -1 or 1. The product of the complex eigenvalues of A is non negative, and the determinant of A is 1. Hence at least one of the real eigenvalues of A must equal 1. The case when n is odd is analogous.

Proof of Weinstein's theorem

Assume by way of contradiction that there is no fixed point. Let $p \in M$ be such that $d(p, f(p)) = l$ realises the infimum of the quantity $d(q, f(q))$ for $q \in M$. Since M is complete, we can find a geodesic $\gamma : [0, l] \to M$ such that $\gamma(0) = p, \gamma(l) = f(p)$.

Let P be the parallel transport along γ from $f(p)$ to p. Let A be the matrix corresponding to $P \circ df_p$. Note that A is an isometry.

Next, consider the geodesic $f(\gamma)$ that joins $f(p)$ to $f^2(p)$. Observe that the concatenation of $\gamma, f(\gamma)$ is also a geodesic. (Use the triangle inequality and the assumption that $d(p, f(p)) = l$ is the infimum as stated above). Conclude that A leaves $\gamma'(0)$ fixed.

Applying the Lemma to the orthogonal complement of $\gamma'(0)$ in $T_p(M)$, conclude that A fixes a unit vector v in this subspace. Note that here we use the fact that f, P are isometries that preserves the orientation and so $det(A) = 1$.

Let $e_1(t)$ be a parallel vector field along γ such that for each t, $e_1(t)$ belongs to the orthogonal complement of $\gamma'(t)$ and $A(e_1(0)) = e_1(0)$. Consider the variation given by $h(s,t) = exp_{\gamma(t)}(se_1(t))$. Use the second variation formula, and the assumption on curvature, to show that there is a curve in the variation that contradicts the assumption that

$$d(p, f(p)) = l = inf\{d(q, f(q)) \mid q \in M\}$$

Proof of the main result

Let $\pi : \tilde{M} \to M$ be the universal cover of M. Introduce the covering metric on $\tilde{M}$, and an orientation so that π is orientation preserving. Because M is compact and has positive sectional curvature, this is bounded below by some positive δ. Since π is a local isometry this also holds for $\tilde{M}$. Since $\tilde{M}$ is complete, it follows from the assumption on curvature that it is also compact. Now using Weinstein's theorem, we conclude that every covering transformation of $\tilde{M}$ has a fixed point, and hence is trivial. This means that M is simply connected.

12.5. As in the previous problem, show that there is a vector v in the orthogonal complement of $\gamma'(0)$ which is left invariant under parallel transport around the curve. Use this parallel vector field to build a variation. Use the second variation formula and the assumption on curvature to prove that there is a curve in the variation that is of smaller length. Clearly, this curve is homotopic to γ.