Question 1: Exchange option

You hold an exchange option, with payout $max(S_1(T) - S_2(T), 0)$ at the expiration of the option T. Explain how you would build an (instantaneously) riskless portfolio containing this option. Derive the PDE for the exchange option and its solution (hint: do a change of variable $y \equiv S_1/S_2$ and $X(t, S_1/S_2) = S_2 f(t, y)$ where X is the option price).

Question 2: Jump-Diffusion model

In the Merton jump-diffusion model,

$$\frac{dS_t}{S_t} = (r - q + \lambda^Q \gamma)dt + \sigma dW_t - \gamma dN_t \tag{1}$$

with constants r, q, λ^Q, γ, σ, find a formula to price a contract paying $(S_T)^n$ at time T for an arbitrary power n.

Question 3: Breeden-Litzenberger formula

Show that the cumulative distribution function (cdf) of a stock price at a future time T can be written in terms of the partial derivative of a call option with expiration T with respect to strike.

Show that the distribution function (pdf) of a stock price at a future time T can be written in terms of the second partial derivative of a call option with expiration T with respect to strike.

Explain how you use this formula to price an arbitrary option whose payout depends only on the stock price at expiration, if you have option prices of all strikes for that expiration.

Question 4: Explicit and implicit method for numerical PDE solution

Write the Black-Scholes PDE. Show how you discretize this equation if you want to solve it numerically with an explicit or an implicit method. What are the advantages and disadvantages of each method?

Question 5: Vasicek model

Derive the stochastic process (under the Q measure) for the bond price $P(t, T)$ in the Vasicek model. Under the Q measure, the short rate follows the process

$$dr_t = k(\theta - r_t)dt + \sigma dW_t \tag{2}$$

You can use the fact that $P(t, T) = A(t, T)e^{-B(t,T)r_t}$ (You don't have to derive an explicit formula for $A(t, T)$ and $B(t, T)$).