Definition :

$$\alpha_i(t) = P(w_1, ..., w_i, T_i = t)$$

To be proven :

$$\alpha_{i+1}(t') = P(w_{i+1}|T_{i+1} = t') \cdot \sum_{t \in \mathscr{T}} \big(\alpha_i(t) \cdot P(T_{i+1} = t|T_i = t)\big)$$

First notice, going back to "HMM main formula" (an denoting t by t_i) :

$$\begin{aligned}\alpha_i(t) &= \sum_{t_1^{i-1}} P(w_1, ..., w_i, t_1^{i-1}, T_i = t) \\ &= \sum_{t_1^{i-1}} P(T_1 = t_1) \prod_{k=2}^{i} P(T_k = t_k|T_{k-1} = t_{k-1}) \prod_{k=1}^{i} P(w_k|T_k = t_k)\end{aligned}$$

Similarly,

$$\begin{aligned}\alpha_{i+1}(t') &= \sum_{t_1^{i-1}} \sum_{t} P(w_1, ..., w_i, w_{i+1}, t_1^{i-1}, T_i = t, T_{i+1} = t') \\ &= \sum_{t} \sum_{t_1^{i-1}} P(T_1 = t_1) \prod_{k=2}^{i} P(T_k = t_k|T_{k-1} = t_{k-1}) \prod_{k=1}^{i} P(w_k|T_k = t_k) \cdot P(T_{k+1} = t'|T_k = t) \cdot P(w_{k+1}|T_{k+1} = t') \\ &= \sum_{t} \alpha_i(t) \cdot P(T_{k+1} = t'|T_k = t) \cdot P(w_{k+1}|T_{k+1} = t')\end{aligned}$$