MICRO-461
Low-power Radio Design for the IoT

10. Oscillators
10.1. Basic Oscillators

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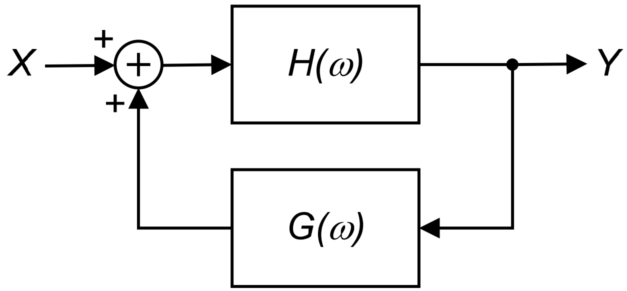
Outline

- General considerations
- The 3-points oscillator
- The cross-coupled pair oscillator

The Barkhausen Criteria



$$Y = \frac{H(\omega)}{1 - H(\omega)G(\omega)} X$$

- Most oscillators can be viewed as **positive feedback** systems with $H(\omega)$ being the feed forward gain and $G(\omega)$ the transfer function of the feedback circuit which is usually a frequency selective network (resonator)
- Oscillations occur at ω_0 if the loop gain $H(\omega_0)G(\omega_0)$ is **exactly** equal to unity, leading to the **Barkhausen** criteria

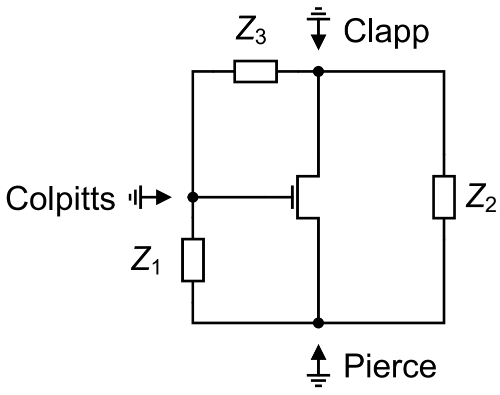
$$|H(\omega_0)G(\omega_0)| = 1 \quad \text{and} \quad \arg\left(H(\omega_0)G(\omega_0)\right) = 0$$

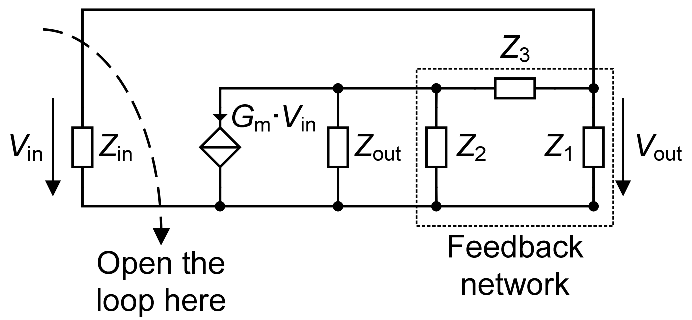
- The feedback network is usually frequency dependent and hence determines the oscillation frequency
- The Barkhausen criteria allows to derive the oscillation frequency, but does not say anything about the oscillation amplitude
- The latter is determined by the circuit nonlinearities

Outline

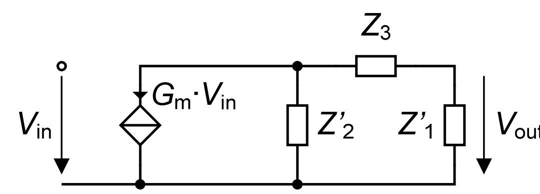
- General considerations
- The 3-points oscillator
- The cross-coupled pair oscillator

The 3-Points Oscillator – Barkhausen Criteria





Z_{in}: transistor input impedance



$$Z_1' = Z_1 // Z_{in}$$

$$Z_2' = Z_2 // Z_{out}$$

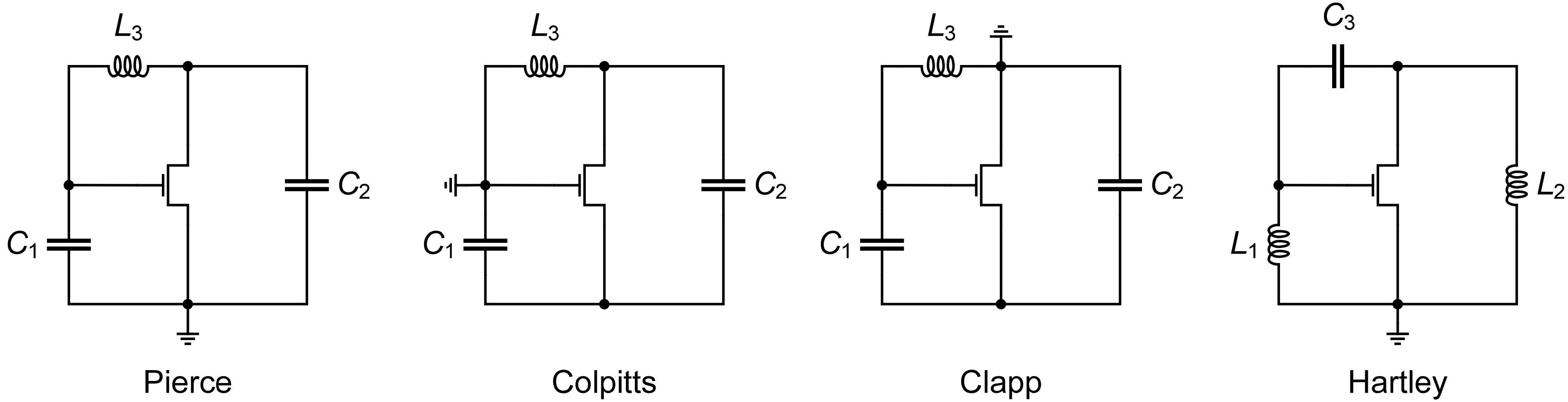
- Many basic (single transistor) oscillators can be described by the generic 3-points oscillator
- The transistor parasitic can be embedded into the impedances Z_k (like for example the transistor input and output impedances are included in Z_1 and Z_2 defining Z_1' and Z_2')
- Opening the loop at the gate allows to calculate the loop gain

$$G \cdot H = \frac{V_{out}}{V_{in}} = \frac{-G_m Z'_1 Z_2'}{Z'_1 + Z_2' + Z_3} = \frac{-G_m}{Y'_1 \left(1 + Y_2' Z_3\right) + Y_2'}$$

- The loop gain has to be equal to unity to satisfy the Barkhausen criteria

$$G_m Z_1' Z_2' + Z_1' + Z_2' + Z_3 = 0 \quad \text{or} \quad G_m + Y_1'\left(1 + Y_2' Z_3\right) + Y_2' = 0$$

The 3-Points Oscillator – Basic Oscillators



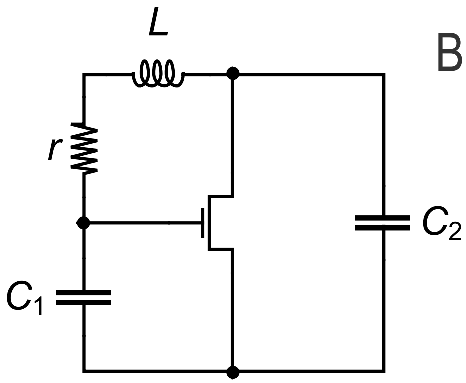
- In the case all the components of the feedback network are reactive $Z_k = jX_k$ $(k = 1,2,3)$, neglecting the input impedance Z_{in} but accounting for the output impedance $Z_{out} = 1/G_{ds}$

$$X_1 + X_2 + X_3 + j\left[\left(G_m + G_{ds}\right)X_1X_2 + G_{ds}X_2X_3\right] = 0$$

$$X_2 = A_{dc} \cdot X_1 \quad \text{and} \quad X_3 = -\left(A_{dc} + 1\right) \cdot X_1 \quad \text{with } A_{dc} = \frac{G_m}{G_{ds}}$$

- Since $A_{dc} > 0$, Z_2 should be of the same type of reactance than Z_1, whereas Z_3 should be of opposite sign leading to the following four basic single transistor oscillators depending on which node is the ground node

The 3-Points Oscillator – Critical Transconductance



Barkhausen criteria: $G_m + Y_1'\left(1 + Y_2' Z_3\right) + Y_2' = 0$

with: $Y_1' = Y_1 = j\omega C_1 \quad Y_2' = G_{ds} + j\omega C_2 \cong j\omega C_2 \quad Z_3 = r + j\omega L$

Leads to:
$$\begin{cases} G_m - \omega^2 r C_1 C_2 = 0 \\ C_1 + C_2 - \omega^2 L C_1 C_2 = 0 \end{cases}$$

- The **resonant frequency** is then given by

$$\omega_0 = \frac{1}{\sqrt{L C_{12}}} \quad \text{with} \quad C_{12} = \frac{C_1 C_2}{C_1 + C_2}$$

- The **critical transconductance** required to maintain the oscillation is given by

$$G_{mcrit} = \omega_0^2 r C_1 C_2 = \frac{\left(C_1 + C_2\right) r}{L} = \frac{\omega_0 \left(C_1 + C_2\right)}{Q_L} \quad \text{with} \quad Q_L = \frac{\omega_0 L}{r}$$

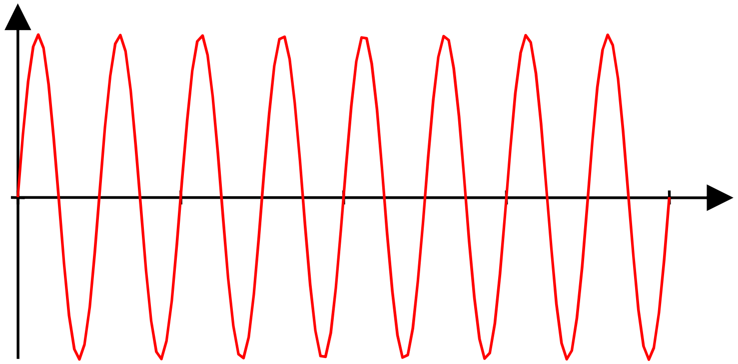
  where Q_L is the unloaded Q of the inductor

- The larger the loss r (the smaller the Q_L), the larger the required G_{mcrit}
- G_{mcrit} also increases with frequency ω_0 and parasitic capacitances C_1 and C_2

The 3-Points Oscillator – Oscillation Conditions

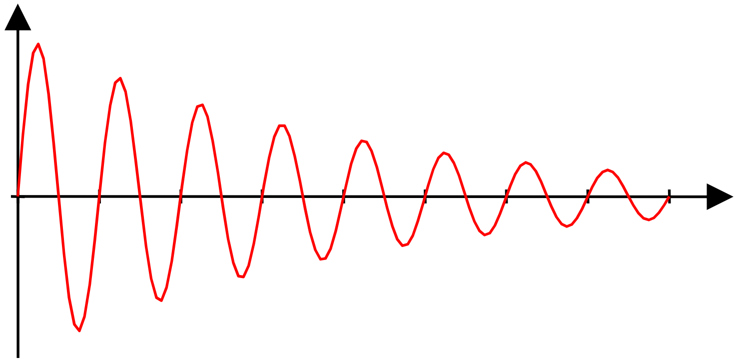
- Oscillations are maintained for

$$G_m = G_{mcrit} = \frac{\omega_0 (C_1 + C_2)}{Q_L}$$



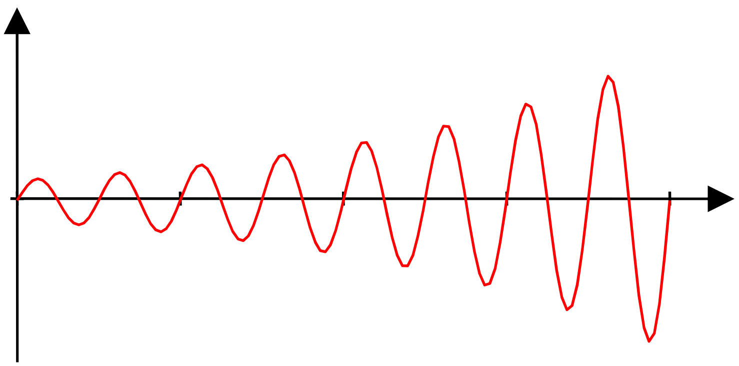
- Oscillations vanish if

$$G_m < G_{mcrit} = \frac{\omega_0 (C_1 + C_2)}{Q_L}$$



- Oscillations amplitude increase if

$$G_m > G_{mcrit} = \frac{\omega_0 (C_1 + C_2)}{Q_L}$$



Accounting for Loss in Output Conductance

- If the output conductance is accounted for, the Barkhausen criteria becomes

$$\begin{cases} G_m + G_{ds} - \omega^2 C_1 \left(G_{ds} L + r C_2 \right) = 0 \\ C_1 + C_2 + C_1 \left(G_{ds} r - \omega^2 L C_2 \right) = 0 \end{cases}$$

- The oscillation frequency is then slightly modified by the presence of the output conductance

$$\omega_0 = \frac{1}{\sqrt{L C_{eq}}} \quad \text{with} \quad C_{eq} \triangleq \frac{C_1 C_2'}{C_1 + C_2'} \quad \text{and} \quad C_2' \triangleq \frac{C_2}{1 + G_{ds} r}$$

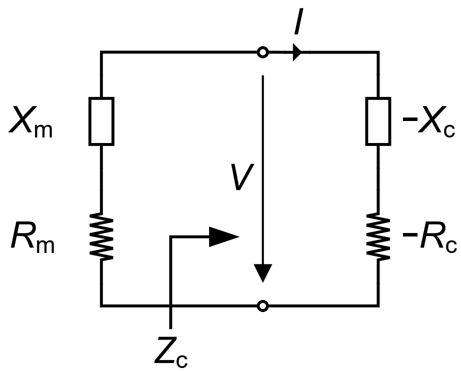
- The critical transconductance is then given by

$$G_{mcrit} = \alpha G_{ds} + (1+\alpha) \frac{\omega_0 C_2}{Q_L} \cong \alpha G_{ds} + \frac{\omega_0 (C_1 + C_2)}{Q_L} \quad \text{with} \quad \alpha = \frac{C_1}{C_2'} = \frac{C_1 (1 + G_{ds} r)}{C_2} \cong \frac{C_1}{C_2}$$

- The critical transconductance has to be larger by $\alpha \cdot G_{ds}$ compared to the case where G_{ds} is negligible

Negative Resistance Analysis Method

- In a **linear analysis**, any oscillator can be viewed as a resonant circuit (X_m and X_c) in series with a negative resistance $-R_c$ that compensates for the loss R_m
- The impedance seen at the input of the circuit Z_c should hence have a negative real part $-R_c$ and a negative imaginary part $-X_c$



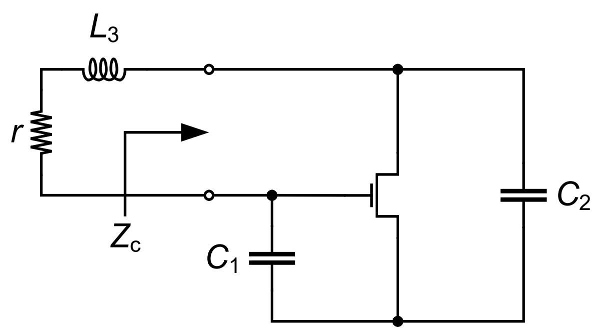
$$Z_m(\omega) = R_m + jX_m$$

$$Z_c(\omega, G_m) = -R_c(\omega, G_m) - jX_c(\omega)$$

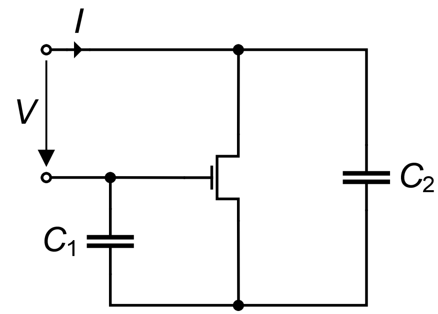
Such that their sum is equal to zero:

$$Z_m(\omega) + Z_c(\omega, G_m) = 0 \quad \rightarrow \quad \begin{cases} -\mathrm{Re}\{Z_c\} = R_c(\omega, G_m) = r \\ -\mathrm{Im}\{Z_c\} = X_c(\omega) = X_m(\omega) \end{cases}$$

- Can be applied to the Pierce oscillator

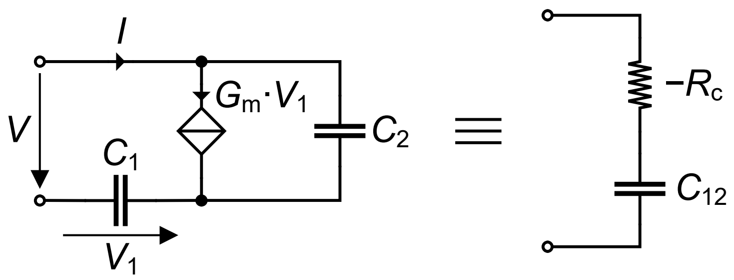


$$Z_c(\omega) = \frac{V}{I}$$



Negative Resistance Analysis Method

- The corresponding small-signal circuit is given by



$$Z_c = \frac{V}{I} = -\frac{G_m + j\omega(C_1 + C_2)}{\omega^2 C_1 C_2} = -\frac{G_m}{\omega^2 C_1 C_2} + \frac{1}{j\omega C_{12}}$$

$$R_c = \frac{G_m}{\omega^2 C_1 C_2}$$

$$X_c = \frac{1}{\omega C_{12}} \quad \text{with} \quad C_{12} = \frac{C_1 C_2}{C_1 + C_2}$$

- The oscillation frequency is then given by the condition on the imaginary part

$$-\operatorname{Im}\{Z_c\} = X_c = X_m \quad \rightarrow \quad \frac{1}{\omega C_{12}} = \omega L \quad \rightarrow \quad \omega_0 = \frac{1}{\sqrt{L C_{12}}}$$

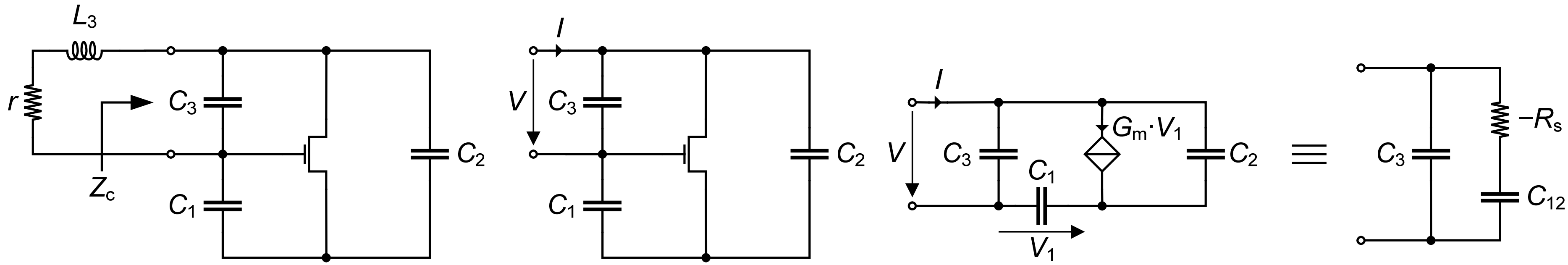
- The critical transconductance to insure oscillation is given by setting $R_c = r$

$$\frac{G_{mcrit}}{\omega^2 C_1 C_2} = r \quad \rightarrow \quad G_{mcrit} = r \cdot \omega^2 C_1 C_2 = \frac{r}{L} \cdot (C_1 + C_2) = \frac{\omega_0 \cdot (C_1 + C_2)}{Q_L}$$

which corresponds to the result obtained earlier using the Barkhausen criteria

Negative Resistance Analysis Method

- The same analysis can be conducted accounting for capacitance C_3 embedding the parasitic capacitances of the inductor and the transistor



$$R_s = \frac{G_m}{\omega^2 C_1 C_2}$$

$$C_{12} = \frac{C_1 C_2}{C_1 + C_2}$$

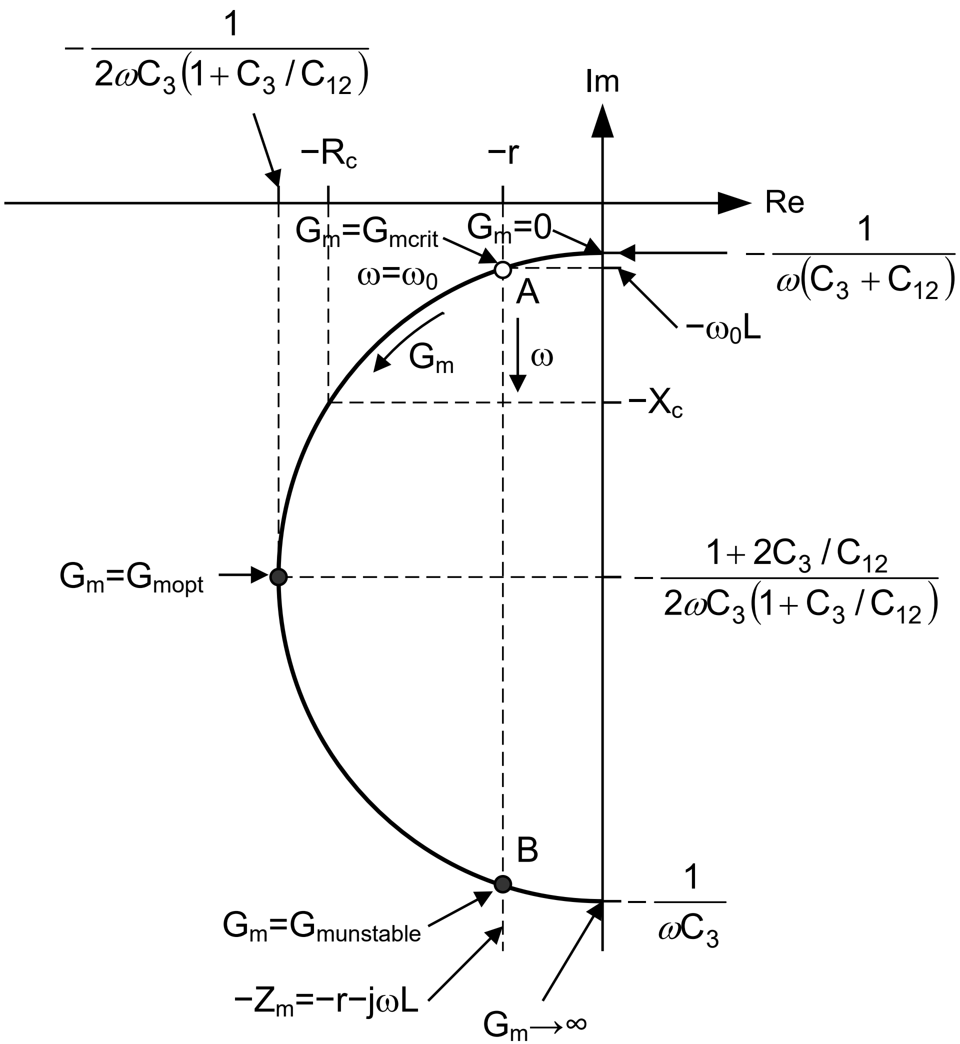
$$Z_c = -\frac{G_m + j\omega(C_1 + C_2)}{\omega^2(C_1C_2 + C_1C_3 + C_2C_3) - j\omega G_m C_3}$$

- Which leads to

$$R_c = \frac{G_m C_1 C_2}{(G_m C_3)^2 + \omega^2 (C_1C_2 + C_1C_3 + C_2C_3)^2}$$

$$X_c = \frac{G_m^2 C_3 + \omega^2 (C_1 + C_2)(C_1C_2 + C_1C_3 + C_2C_3)}{\omega\left[(G_m C_3)^2 + \omega^2 (C_1C_2 + C_1C_3 + C_2C_3)^2\right]}$$

Impedance Locus



- When plotted in the complex plane for a given frequency (usually the resonance frequency ω_0), versus the parameter G_m, the circuit impedance $Z_c(G_m)$ describes a half-circle
- The impedance $-Z_m = -r - j\omega L$ of the lossy inductor can be plotted versus ω and describes a vertical line at $-r$
- The condition $Z_c = -Z_m$ corresponds to the intersections of the circle and the line (points A and B)
- It can be shown that only point A corresponds to a stable point
- By definition, at point A we have:

$$G_m = G_{mcrit} \quad \text{and} \quad \omega = \omega_0$$

Impedance Locus

- G_{mcrit} and ω_0 can be found by solving

$$\begin{cases} -\mathrm{Re}\{Z_c\} = R_c(\omega, G_m) = r \\ -\mathrm{Im}\{Z_c\} = X_c(\omega) = \omega L \end{cases}$$

- R_c reaches a minimum (max in absolute value) given by

$$R_{c,\max} = \frac{1}{2\omega C_3\left(1+\dfrac{C_3}{C_{12}}\right)} \quad \text{for} \quad G_m = G_{mopt} = \omega\left(C_1 + C_2 + \frac{C_1 C_2}{C_3}\right) \quad \text{with} \quad C_{12} = \frac{C_1 C_2}{C_1 + C_2}$$

- If $r > R_{c,max}$ there are no intersections and no oscillations can take place
- The condition for a solution to exist is hence given by

$$r \le R_{c,\max} = \frac{1}{2\omega C_3\left(1+\dfrac{C_3}{C_{12}}\right)}$$

- If C_1 and/or C_2 decrease, point A moves downwards and ω_0 increases
- If $C_3 = 0$ the circle becomes a horizontal line independent of G_m

G_{mcrit} for Given ω_0 and Q_L

- In the case the oscillation frequency ω_0 and the quality factor of the inductor Q_L are set, G_{mcrit} can be found from

$$\frac{X_c(\omega_0, G_{mcrit})}{R_c(\omega_0, G_{mcrit})} = Q_L \quad \Rightarrow \quad \frac{G_{mcrit}^2 C_3 + \omega_0^2 (C_1 + C_2)(C_1C_2 + C_1C_3 + C_2C_3)}{\omega_0 G_{mcrit} C_1 C_2} = Q_L$$

which leads to

$$G_{mcrit} = \omega_0 C_2 \frac{\alpha_1 Q_L}{2\alpha_3}\left[1 - \sqrt{1 - \left(\frac{2\alpha_3}{\alpha_1 Q_L}\right)^2 (\alpha_1 + 1)\left(1 + \alpha_1 + \frac{\alpha_1}{\alpha_3}\right)}\right] \quad \text{where} \quad \alpha_1 = \frac{C_1}{C_2} \quad \alpha_3 = \frac{C_3}{C_2}$$

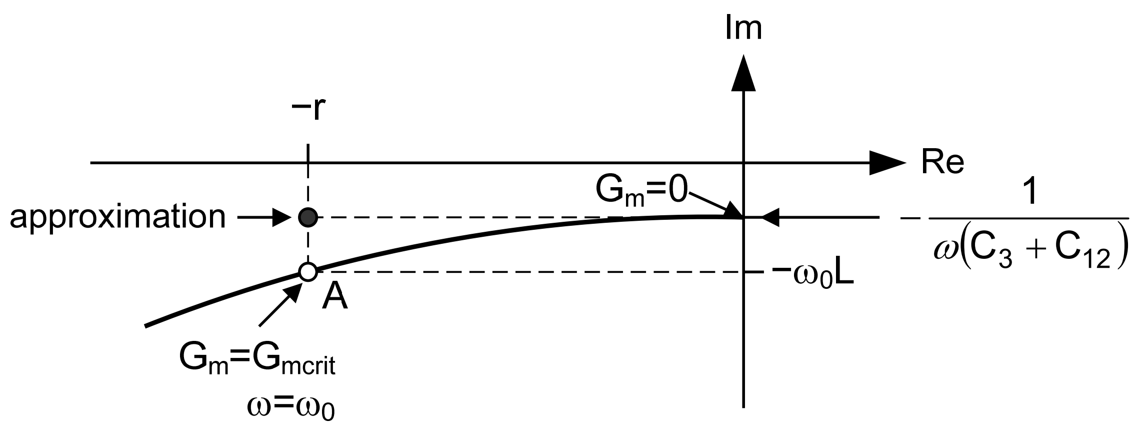
- The solution obviously only exists if

$$Q_L > \frac{2\alpha_3}{\alpha_1} \cdot \sqrt{(\alpha_1 + 1)\left(1 + \alpha_1 + \frac{\alpha_1}{\alpha_3}\right)}$$

- An approximate solution can be found for $Q_L \gg 1$

$$G_{mcrit} \cong \omega_0 C_2 \frac{\alpha_3}{\alpha_1 Q_L}(\alpha_1 + 1)\left(1 + \alpha_1 + \frac{\alpha_1}{\alpha_3}\right) = \frac{\omega_0}{Q_L}(C_1 + C_2)\left(1 + \frac{C_3}{C_{12}}\right) \quad \text{with} \quad C_{12} = \frac{C_1 C_2}{C_1 + C_2}$$

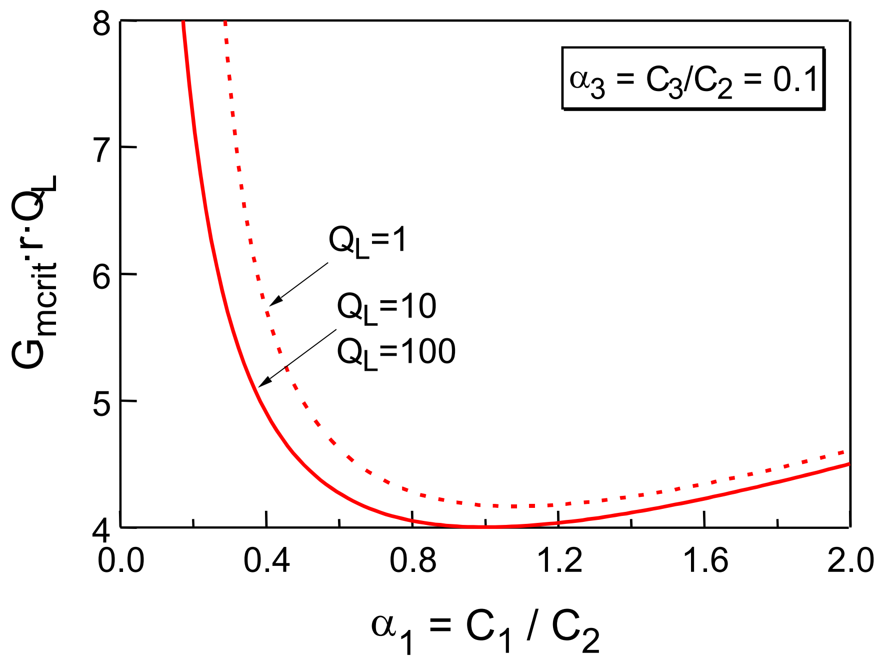
Approximation of G_{mcrit}



- As shown above, the oscillation frequency depends on r and therefore on the quality factor Q_L of the inductor which is not desirable since it may vary significantly
- When losses are small (r small) or Q_L becomes large, the vertical line gets closer to the imaginary axis and the sensitivity of ω_0 to Q_L becomes small
- In this condition, the oscillation frequency can be approximated by setting $G_m = 0$ in $X_c(\omega, G_m) = X_m(\omega)$ and solving for ω leads to

$$\omega_0 \cong \frac{1}{\sqrt{L(C_3 + C_{12})}}$$

Minimum Value of G_{mcrit}



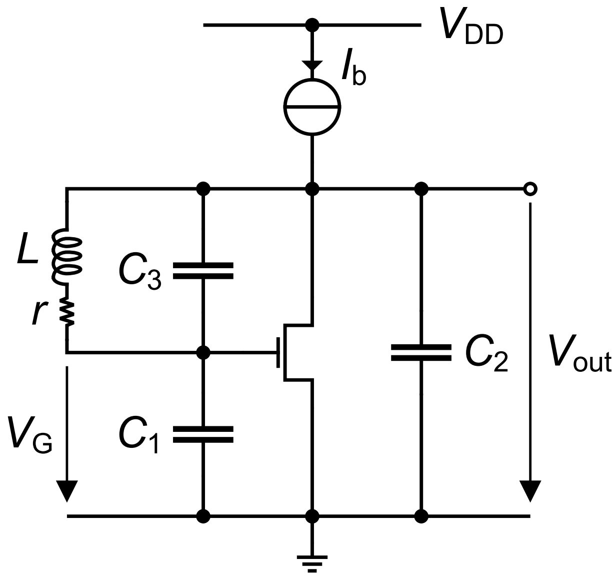
$$G_{mcrit} \cdot r \cdot Q_L \cong \frac{(\alpha_1 + 1)^2}{\alpha_1}$$

- As shown above, G_{mcrit} is minimum for $\alpha_1 = 1$ $(C_1 = C_2)$

$$G_{mcrit,\min} = \frac{1}{r}\left(\frac{2}{Q_L}\right)^2 = \frac{\omega_0}{Q_L} 2(C_1 + 2C_3) \quad \text{for} \quad C_1 = C_2$$

Sinusoidal Control Voltage

- For $G_m > G_{mcrit}$, the oscillation will start and amplitude will grow, generating harmonic components due to the nonlinearity of the active element
- The above analysis was linear assuming small-signal operation. It did not give any information about the oscillation amplitude. This can only be obtained from a nonlinear analysis which is not always possible to achieve in an analytical form

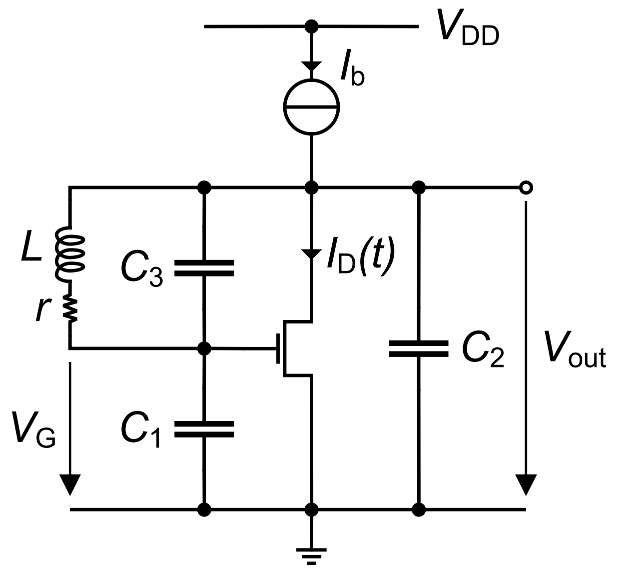


- If the quality factor of the resonator is assumed large (typically $Q_L > 10$), the current going through the LC tank is filtered from its harmonics and generates a voltage at the gate that can be considered as quasi-sinusoidal

$$V_G(t) \cong V_{G0} + A \cdot \cos(\omega_0 t)$$

where V_{G0} is the dc gate voltage when there are no oscillations ($A = 0$)

Nonlinear Analysis of the Pierce Oscillator (weak inv.)



- In the case of the Pierce oscillator the gate voltage can therefore be assumed to be sinusoidal

$$V_G(t) = V_{G0} + A \cdot \cos(\omega_0 t)$$

- If the transistor is biased in weak inversion, the drain current is then given by

$$I_D(t) = I_{D0} \cdot e^{\frac{V_G(t)}{nU_T}} = I_{D0} \cdot e^{\frac{V_{G0}+A\cdot\cos(\omega_0 t)}{nU_T}}$$

$$= I_0 \cdot e^{x\cdot\cos(\omega_0 t)}$$

$$A = \Delta V_G \cong -\Delta V_{out}$$

with $I_0 = I_{D0} \cdot e^{\frac{V_{G0}}{nU_T}} = I_{spec} \cdot e^{\frac{V_{G0}-V_{T0}}{nU_T}}$ and $x \triangleq \dfrac{A}{nU_T}$

- Notice that it is essentially capacitance C_3 that couples harmonic components directly to the gate. Therefore the assumption of the gate voltage being quasi-sinusoidal only holds if C_3 is much smaller than C_{12}

Nonlinear Analysis of the Pierce Oscillator (WI)

- Function $e^{x \cdot cos(\omega_0 t)}$ can be developed in a Fourier series given by

$$e^{x \cdot \cos(\omega_0 t)} = \mathrm{I}_{B0}(x) + 2 \cdot \sum_{n=1}^{+\infty} \mathrm{I}_{Bn}(x) \cdot \cos(n\omega_0 t)$$

where $\mathrm{I}_{B0}(x)$ and $\mathrm{I}_{Bn}(x)$ are the modified Bessel functions of the first kind of order 0 and n

- The drain current is then given by

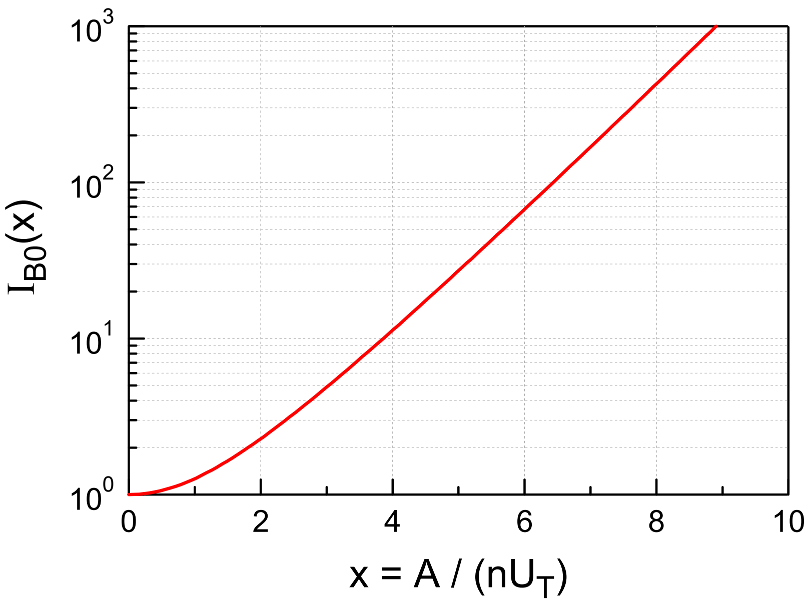
$$I_D(t) = I_0 \cdot e^{x \cdot \cos(\omega_0 t)} = I_{dc} + 2 I_0 \cdot \sum_{n=1}^{+\infty} \mathrm{I}_{Bn}(x) \cdot \cos(n\omega_0 t)$$

where I_{dc} is the average current (dc current) given by

$$I_{dc} = I_0 \cdot \mathrm{I}_{B0}(x)$$

- Notice that in the case of the 3-points oscillators, the dc current I_{dc} is set by a constant bias current I_b, whereas I_0 is the quiescent current defined as the current that flows when there are no oscillations (or their amplitude is zero $x = 0$)

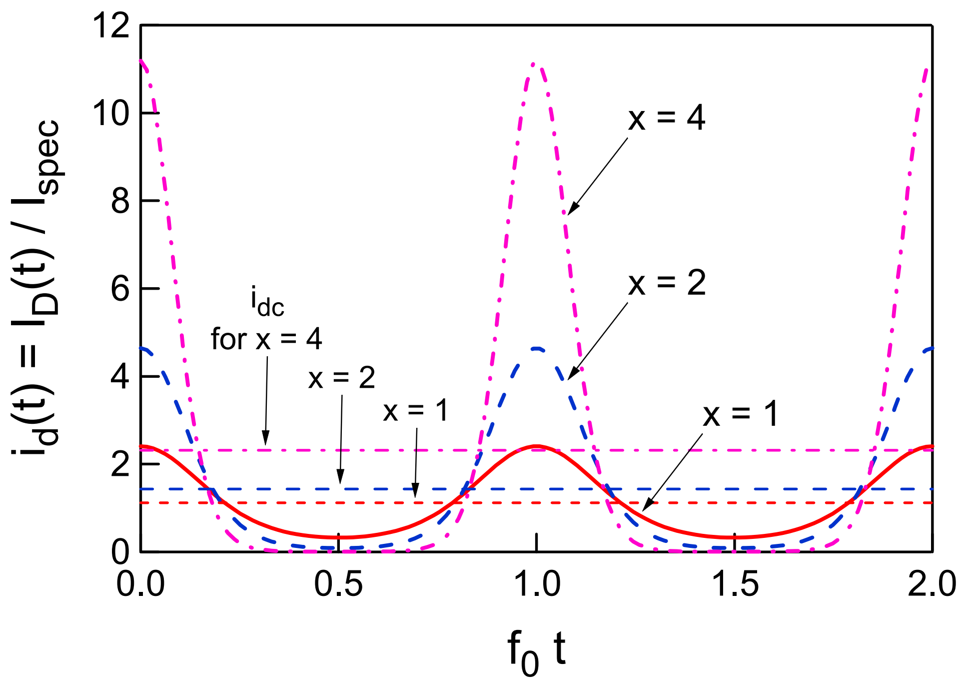
Quiescent Current I_0 and Voltage V_{G0}



$$I_{dc} = I_0 \cdot \mathrm{I}_{B0}(x)$$

- The average of $e^{x \cdot cos(\omega_0 t)}$ is given by $\mathrm{I}_{B0}(x)$ which increases exponentially
- For the 3-points oscillators, the dc current I_{dc} is maintained constant and equal to I_b
- The current I_0 and hence the gate bias voltage V_{G0} need therefore to decrease in order to compensate for the increase in $\mathrm{I}_{B0}(x)$ and maintain the dc current equal to I_b
- There is therefore a relation between the oscillation amplitude and the dc bias which will be derived later

Drain Current Waveforms

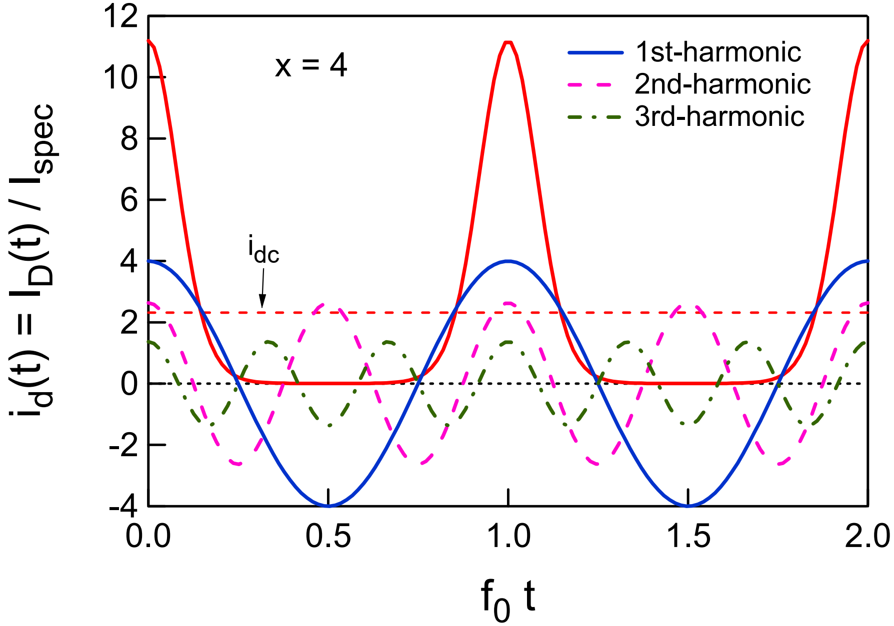


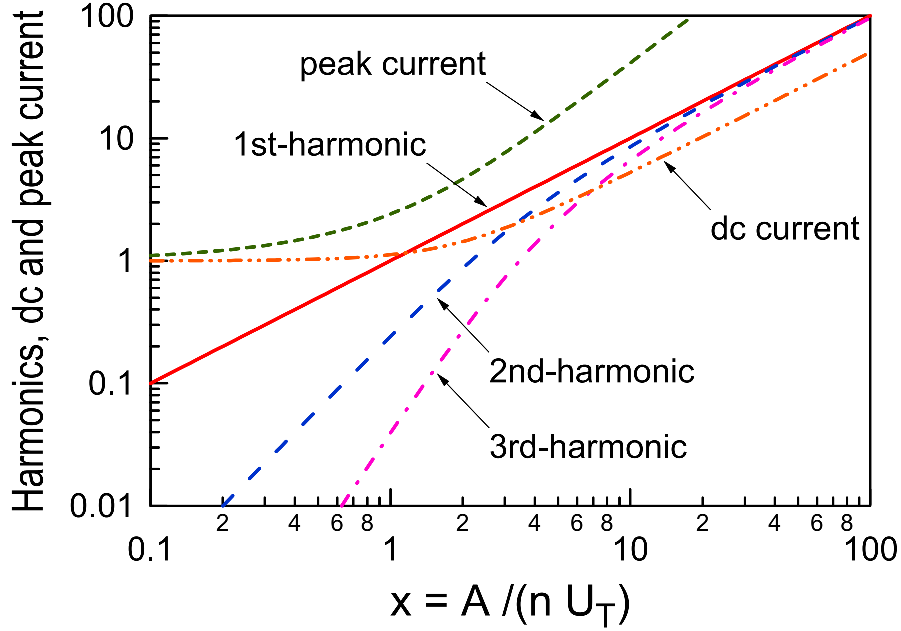
- The above plot shows the drain current normalized to I_{spec} for several oscillation amplitudes and accounting for the dependence of I_0 and I_{dc} (I_b) on x

$$i_d(t) \triangleq \frac{I_D(t)}{I_{spec}} = i_0(x) \cdot e^{x \cdot \cos(\omega_0 t)} = i_{dc}(x) + 2 i_0(x) \cdot \sum_{n=1}^{+\infty} \mathrm{I}_{Bn}(x) \cdot \cos(n \omega_0 t)$$

with $i_0(x) \triangleq \frac{I_0}{I_{spec}} = \frac{x}{2\,\mathrm{I}_{B1}(x)}$ and $i_{dc}(x) \triangleq \frac{I_b}{I_{spec}} = i_0(x) \cdot \mathrm{I}_{B0}(x)$

Drain Current Harmonics

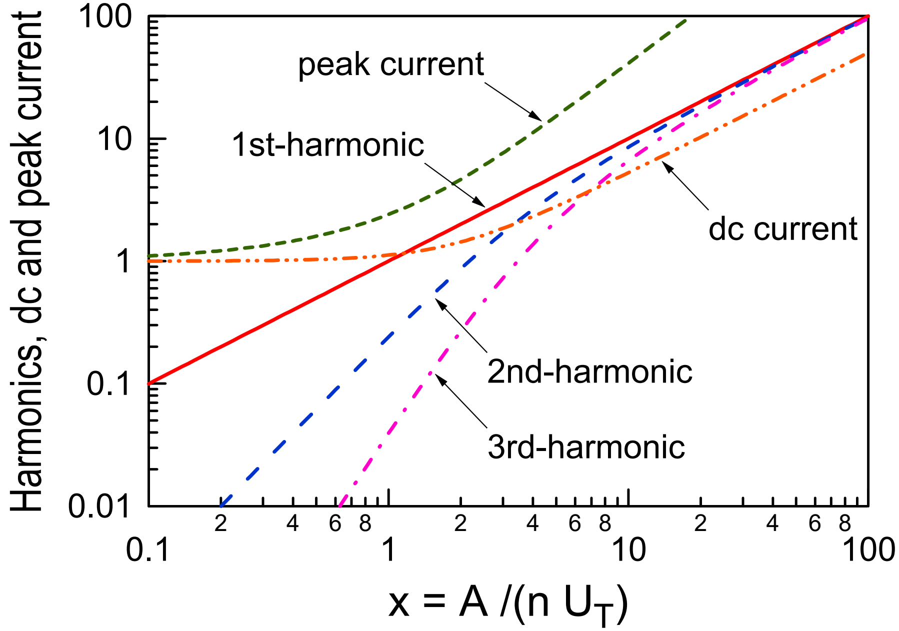


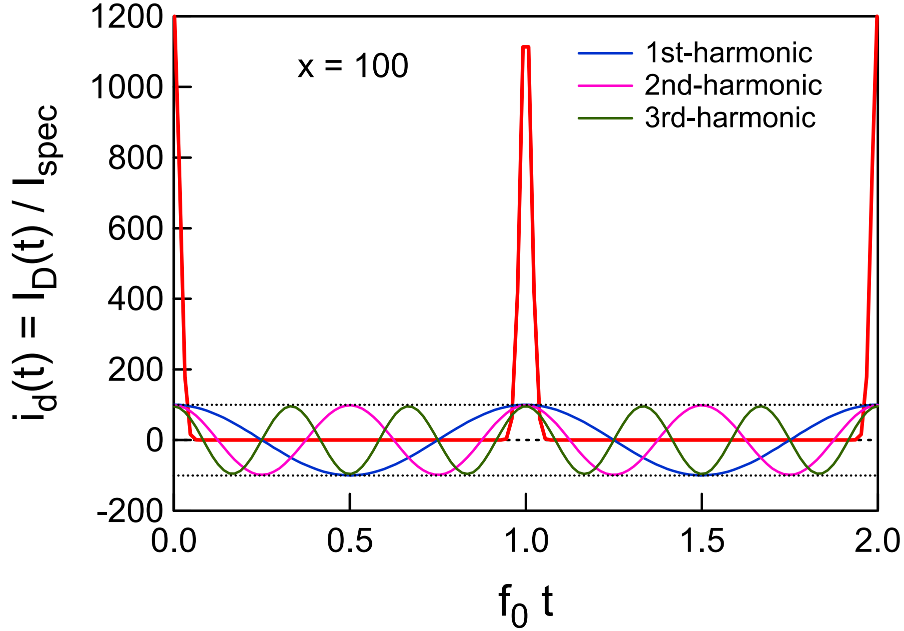


DC current: $$i_{dc}(x) \triangleq \frac{I_b(x)}{I_{spec}} = i_0(x) \cdot \mathrm{I}_{B0}(x) \quad \text{with} \quad i_0(x) \triangleq \frac{I_0}{I_{spec}} = \frac{x}{2\,\mathrm{I}_{B1}(x)}$$

n^{th}-harmonic: $$i_{d(n)} \triangleq \frac{I_{D(n)}}{I_{spec}} = 2 i_0(x) \cdot \mathrm{I}_{Bn}(x) \quad \text{with} \quad i_{d(1)} = \frac{2x}{2\,\mathrm{I}_{B1}(x)} \cdot \mathrm{I}_{B1}(x) = x$$

Harmonics for Large Amplitudes





- It is interesting to note that for large values of x, all harmonics tend to the same value, since

$$\mathrm{I}_{Bn}(x) \cong \frac{e^x}{\sqrt{2\pi x}} \quad \text{for} \quad x \gg 1 \quad \rightarrow \quad i_{d(n)} = 2 i_0(x) \cdot \mathrm{I}_{Bn}(x) = x \cdot \frac{\mathrm{I}_{Bn}(x)}{\mathrm{I}_{B1}(x)} \cong x \quad \text{for} \quad x \gg 1$$

Equivalent Impedance for the Fundamental Component

- The active element is usually nonlinear and generates harmonic components in the drain current
- The latter are filtered out by the resonator even though the current across it can be strongly distorted
- The energy exchange between the active element and the resonator occurs therefore mostly at the fundamental frequency
- The active circuit can therefore be replaced by the impedance for the fundamental defined as

$$Z_{c(1)} = -\frac{V}{I_{(1)}}$$

where $I_{(1)}$ is the complex current at the fundamental frequency which depends on the amplitude of the sinusoidal voltage V

Transconductance for the Fundamental Component

- At low frequency the variation of the fundamental component of the drain current $\Delta I_{D(1)}(t)$ and of the gate voltage $\Delta V_G(t)$ are in-phase
- The small-signal transconductance can be replaced by the transconductance for the fundamental $G_{m(1)}$ given by

$$G_{m(1)} = \frac{\Delta I_{D(1)}}{A} = \frac{2 I_0 \,\mathrm{I}_{B1}(x)}{A} = \frac{I_0}{n U_T} \cdot \frac{2\,\mathrm{I}_{B1}(x)}{x} \quad \text{where} \quad x \triangleq \frac{A}{n U_T}$$

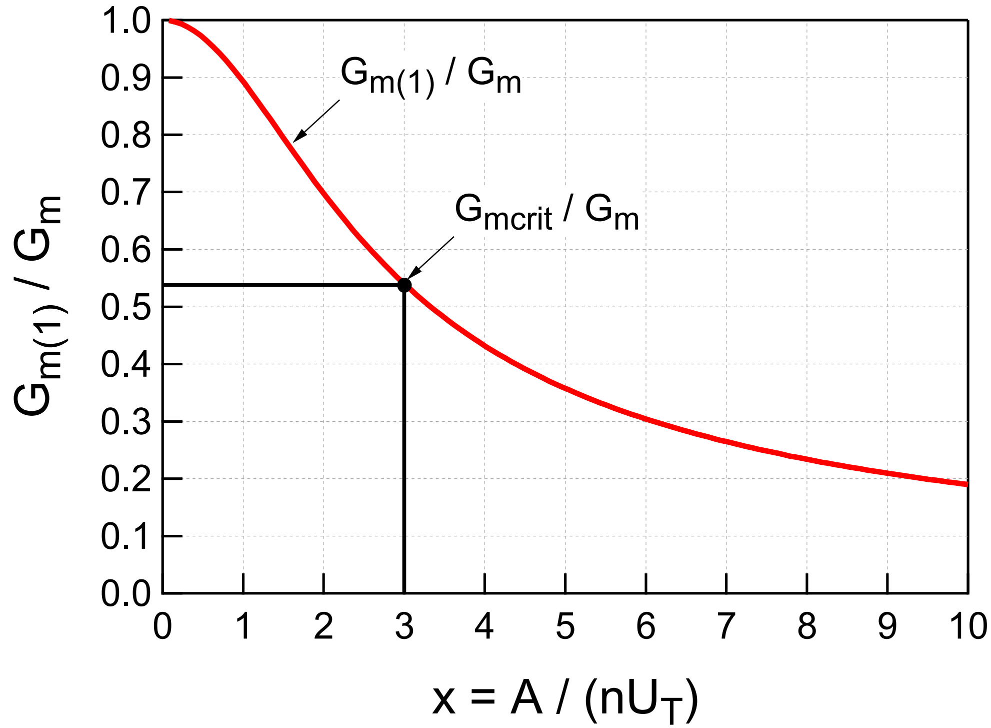
- The transconductance for the fundamental can be rewritten by introducing the dc current I_b

$$I_b = I_0 \cdot \mathrm{I}_{B0}(x) \quad \rightarrow \quad I_0 = \frac{I_b}{\mathrm{I}_{B0}(x)} \quad \rightarrow \quad G_{m(1)} = \frac{I_b}{n U_T} \cdot \frac{2\,\mathrm{I}_{B1}(x)}{x \cdot \mathrm{I}_{B0}(x)} = G_m \cdot \frac{2\,\mathrm{I}_{B1}(x)}{x \cdot \mathrm{I}_{B0}(x)}$$

where $G_m = I_b/(nU_T)$ is the small-signal transconductance set by the bias current I_b

$$G_m = \frac{I_{dc}}{n U_T} = \frac{I_b}{n U_T}$$

Transconductance for the Fundamental Component



$$\frac{G_{m(1)}}{G_m} = \frac{2\,\mathrm{I}_{B1}(x)}{x \cdot \mathrm{I}_{B0}(x)}$$

with $G_m = \dfrac{I_b}{nU_T}$

and $x \triangleq \dfrac{A}{nU_T}$

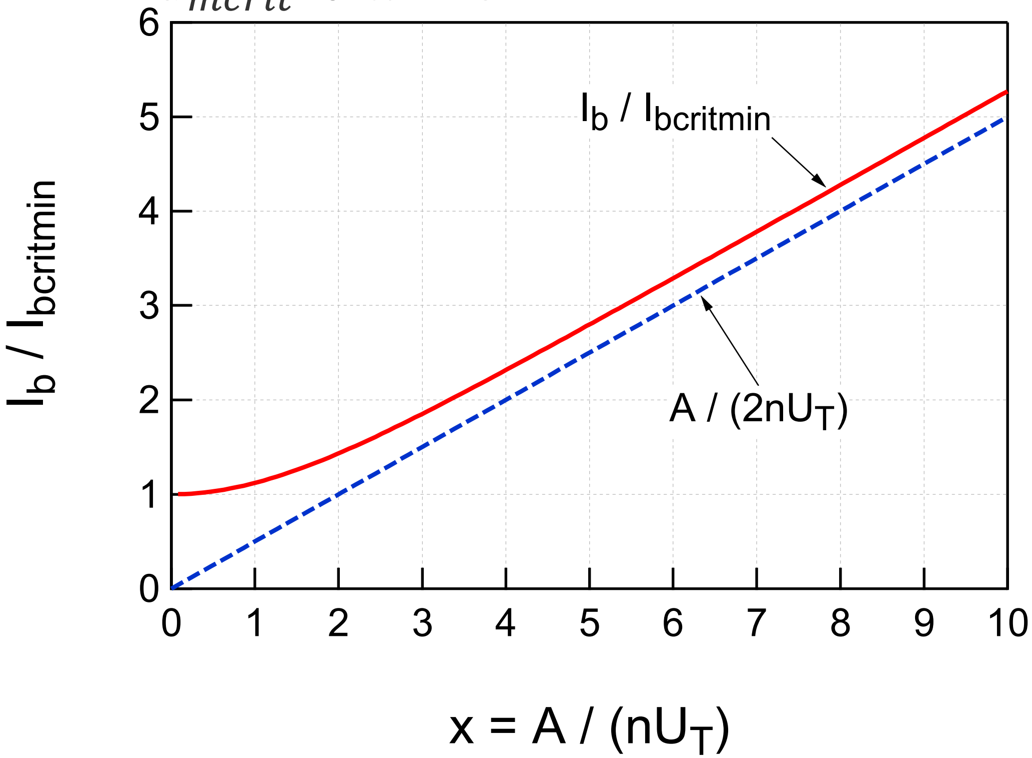
- The above plot shows the transconductance for the fundamental normalized to the small-signal transconductance versus the normalized oscillation amplitude
- The amplitude will stabilize for $G_{m(1)} = G_{mcrit}$ which is the condition that finally determines the oscillation amplitude

Bias Current versus Amplitude

- In weak inversion the condition $G_{m(1)} = G_{mcrit}$ translates into

$$G_{m(1)} = G_{mcrit} \quad \rightarrow \quad \frac{I_b}{nU_T} \cdot \frac{2\,\mathrm{I}_{B1}(x)}{x \cdot \mathrm{I}_{B0}(x)} = \frac{I_{bcritmin}}{nU_T} \quad \rightarrow \quad \frac{I_b}{I_{bcritmin}} = \frac{x \cdot \mathrm{I}_{B0}(x)}{2\,\mathrm{I}_{B1}(x)}$$

- $I_{bcritmin} \triangleq nU_T \cdot G_{mcrit}$ is the minimum current (reached in WI) to achieve G_{mcrit} for $x = 0$



- Since for $x \gg 1$

$$\mathrm{I}_{Bn}(x) \cong \frac{e^x}{\sqrt{2\pi x}} \quad \text{for} \quad x \gg 1$$

- we have

$$\frac{I_b}{I_{bcritmin}} = \frac{x \cdot \mathrm{I}_{B0}(x)}{2\,\mathrm{I}_{B1}(x)} \cong \frac{x}{2} \quad \text{for} \quad x \gg 1$$

- or

$$I_b \cong I_{bcritmin} \cdot \frac{A}{2nU_T} = \frac{G_{mcrit}}{2} \cdot A$$

for $A \gg nU_T$

DC Gate Voltage Bias Shift

- In case the bias current is set to the quiescent current $I_b = I_0$, by definition of I_0, the oscillation amplitude is zero ($x = 0$)

$$x = 0 \quad \rightarrow \quad I_b = I_0 \cdot \mathrm{I}_{B0}(x=0) = I_0 = I_{D0} \cdot e^{\frac{V_{G0}}{nU_T}} = I_{spec} \cdot e^{\frac{V_{G0}-V_{T0}}{nU_T}}$$

- For $I_b > I_0$, oscillations will start to grow until the condition $G_{m(1)} = G_{mcrit}$ is reached, at which the oscillations will stabilize with an amplitude set by $I_b / I_{bcritmin}$
- As shown in the previous plot, the dc drain current would increase wrt x, but it is actually constant and set to I_b by the current source. Since the current cannot grow when the oscillations are growing, the dc gate voltage has to adjust so that $I_{dc} = I_b$
- V_{G0} and I_0 therefore decrease compared to the condition $V_{G0} = V_{Gcrit}$ and $I_0 = I_b = I_{bcritmin}$ for which $x = 0$
- The quiescent voltage V_{G0} and the quiescent current I_0 are therefore indirectly also functions of the oscillation amplitude and hence of the I_b / I_{bcrit} ratio

DC Gate Voltage Bias Shift

- For a given bias current I_b and minimum critical bias current $I_{bcritmin}$, the relation between the quiescent current I_0 and the oscillation amplitude x can be found from the oscillation condition

$$G_{m(1)} = G_{mcrit} \quad \rightarrow \quad \frac{I_b}{I_{bcritmin}} = \frac{I_0 \cdot \mathrm{I}_{B0}(x)}{I_{bcritmin}} = \frac{x \cdot \mathrm{I}_{B0}(x)}{2\,\mathrm{I}_{B1}(x)} \quad \rightarrow \quad \frac{I_0}{I_{bcritmin}} = \frac{x}{2\,\mathrm{I}_{B1}(x)}$$

- Introducing the definition of the quiescent current I_0, we get

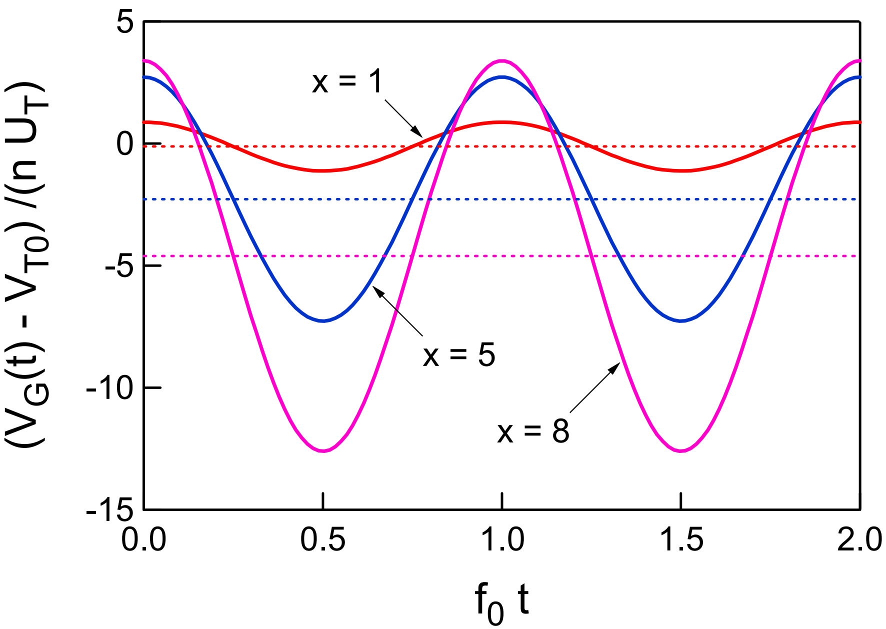
$$\frac{I_0}{I_{bcritmin}} = \frac{I_{spec}}{I_{bcritmin}} \cdot e^{\frac{V_{G0}-V_{T0}}{nU_T}} \quad \rightarrow \quad e^{\frac{V_{G0}-V_{T0}}{nU_T}} = \frac{I_{bcritmin}}{I_{spec}} \cdot \frac{x}{2\,\mathrm{I}_{B1}(x)}$$

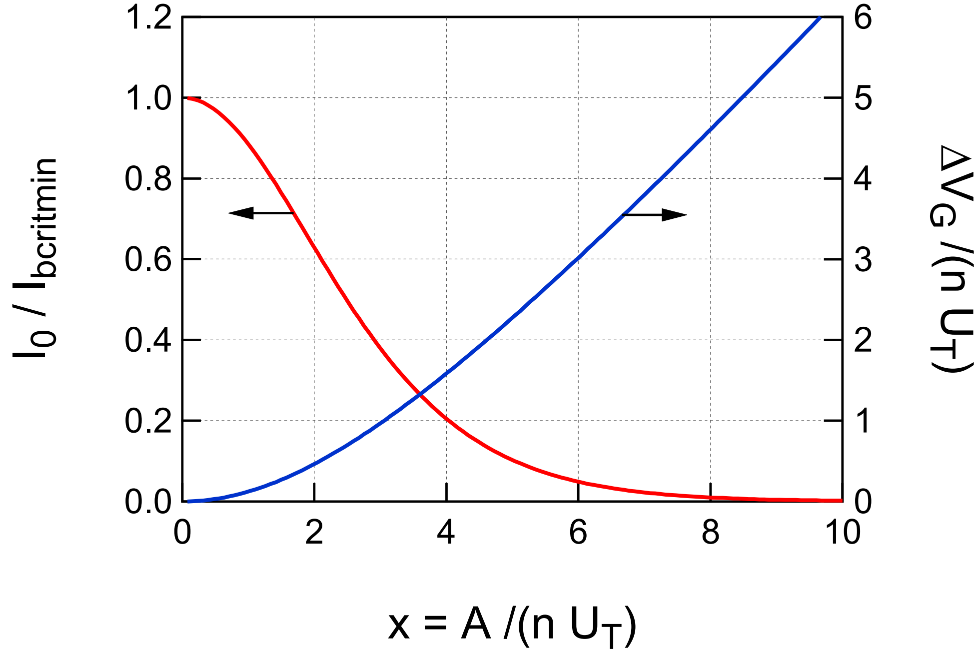
- We see that for a given $I_{bcritmin}$ and bias current I_b, as the amplitude grows, at the same time the overdrive voltage decreases according to

$$\frac{V_{G0}-V_{T0}}{nU_T} = \ln\left(\frac{I_0}{I_{spec}}\right) = \underbrace{\ln\left(\frac{I_{bcritmin}}{I_{spec}}\right)}_{\triangleq \frac{V_{Gcrit}-V_{T0}}{nU_T}} - \underbrace{\ln\left(\frac{2\,\mathrm{I}_{B1}(x)}{x}\right)}_{\triangleq \frac{\Delta V_G(x)}{nU_T}} = \frac{V_{Gcritmin}-V_{T0}}{nU_T} - \frac{\Delta V_G(x)}{nU_T}$$

where $V_{Gcritmin}$ is the gate voltage for a bias current $I_b = I_{bcritmin}$, i.e. $x = 0$

DC Gate Bias Voltage Shift





- As mentioned earlier, the gate bias has to decrease when the oscillations are growing to maintain the dc drain current equal to the bias current

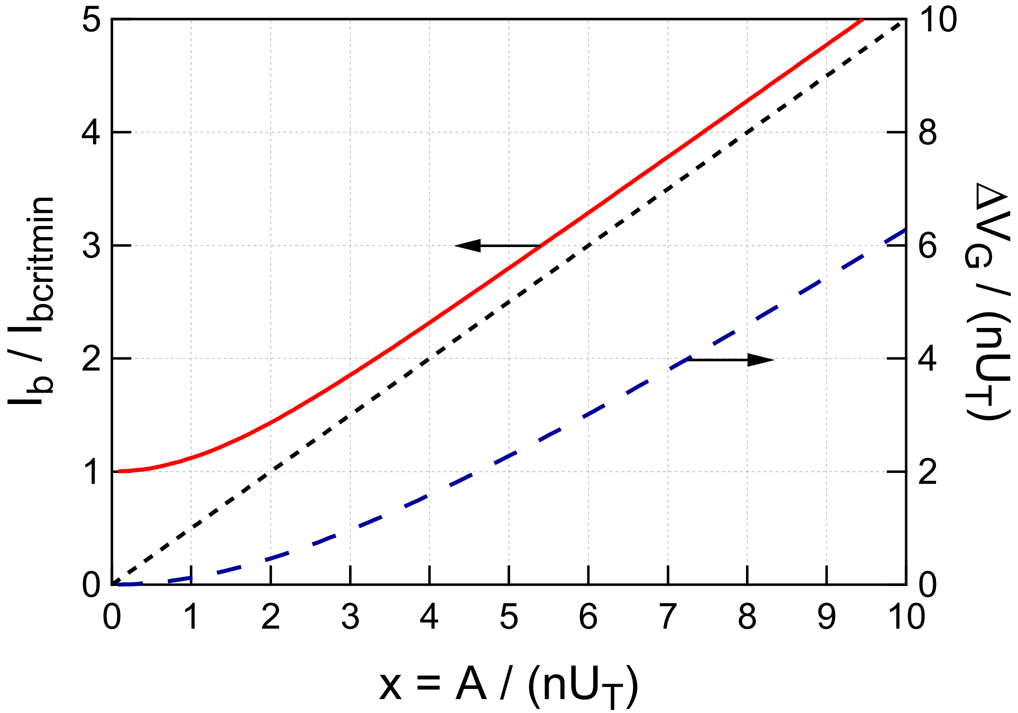
$$\frac{V_G(t)-V_{T0}}{nU_T}=\frac{V_{G0}(x)-V_{T0}}{nU_T}+x\cdot\cos(\omega_0\cdot t)$$

with $$\frac{V_{G0}(x)-V_{T0}}{nU_T}=\ln\left(\frac{I_{bcritmin}}{I_{spec}}\right)-\ln\left(\frac{2\,\mathrm{I}_{B1}(x)}{x}\right)$$

$$\frac{V_{G0}-V_{T0}}{nU_T}=\frac{V_{Gcritmin}-V_{T0}}{nU_T}-\frac{\Delta V_G(x)}{nU_T}$$

with $$\frac{\Delta V_G(x)}{nU_T}\triangleq\ln\left(\frac{2\,\mathrm{I}_{B1}(x)}{x}\right)$$

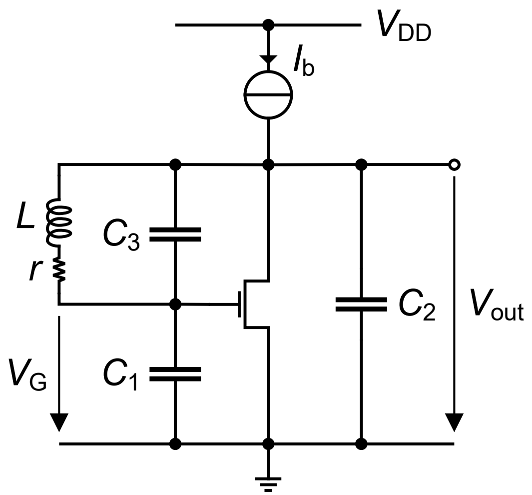
Amplitude and Gate Voltage Bias Shift vs Bias Current



- For a given resonator and hence a given $I_{bcritmin}$, this plot shows the bias current I_b that is required for achieving a given amplitude A and the resulting gate bias shift decrease

$$\frac{I_b}{I_{bcritmin}} = \frac{x \cdot \mathrm{I}_{B0}(x)}{2\,\mathrm{I}_{B1}(x)} \quad \text{and} \quad \frac{\Delta V_G}{nU_T} = \ln\left(\frac{2\,\mathrm{I}_{B1}(x)}{x}\right)$$

Example – The Pierce Oscillator



$$f_0 = 1\,GHz, Q_L = 10, C_1 = C_2 = 1\,pF, C_3 = 1\,pF$$

$$\omega_0 \cong \frac{1}{\sqrt{L(C_3 + C_{12})}} \Rightarrow L \cong \frac{1}{\omega_0^2 (C_3 + C_{12})} = 16.9\,nH$$

$$r = \frac{\omega_0 L}{Q_L} = 10.6\Omega$$

$$G_{mcrit} \cong \frac{\omega_0}{Q_L}(C_1 + C_2)\left(1 + \frac{C_3}{C_{12}}\right) = 3.8\frac{mA}{V}$$

- Since the inductance Q_L is not very high, the above approximation is not very accurate. The exact solution is then given by

$$G_{mcrit} = \omega_0 C_2 \frac{\alpha_1 Q_L}{2\alpha_3}\left[1 - \sqrt{1 - \left(\frac{2\alpha_3}{\alpha_1 Q_L}\right)^2 (\alpha_1 + 1)\left(1 + \alpha_1 + \frac{\alpha_1}{\alpha_3}\right)}\right] = 4\frac{mA}{V}$$

- The inductance value is then found from

$$L = \frac{X_c(\omega_0, G_{mcrit})}{\omega_0}$$

- This leads to $L = 17.256\,nH$ and $r = 10.8\,\Omega$

Pierce Oscillator Example – Bias Current in WI

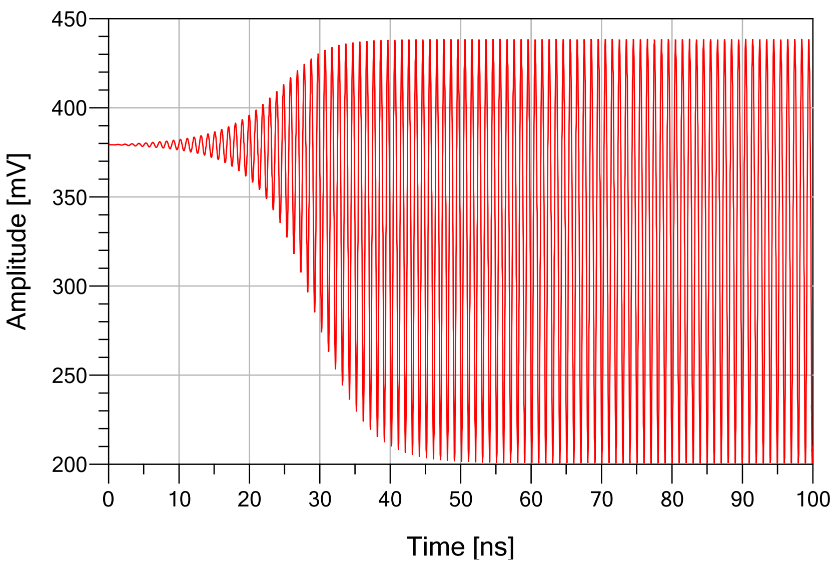
- If we assume that the transistor operates in weak inversion (with $n = 1.3$), the critical current is given by

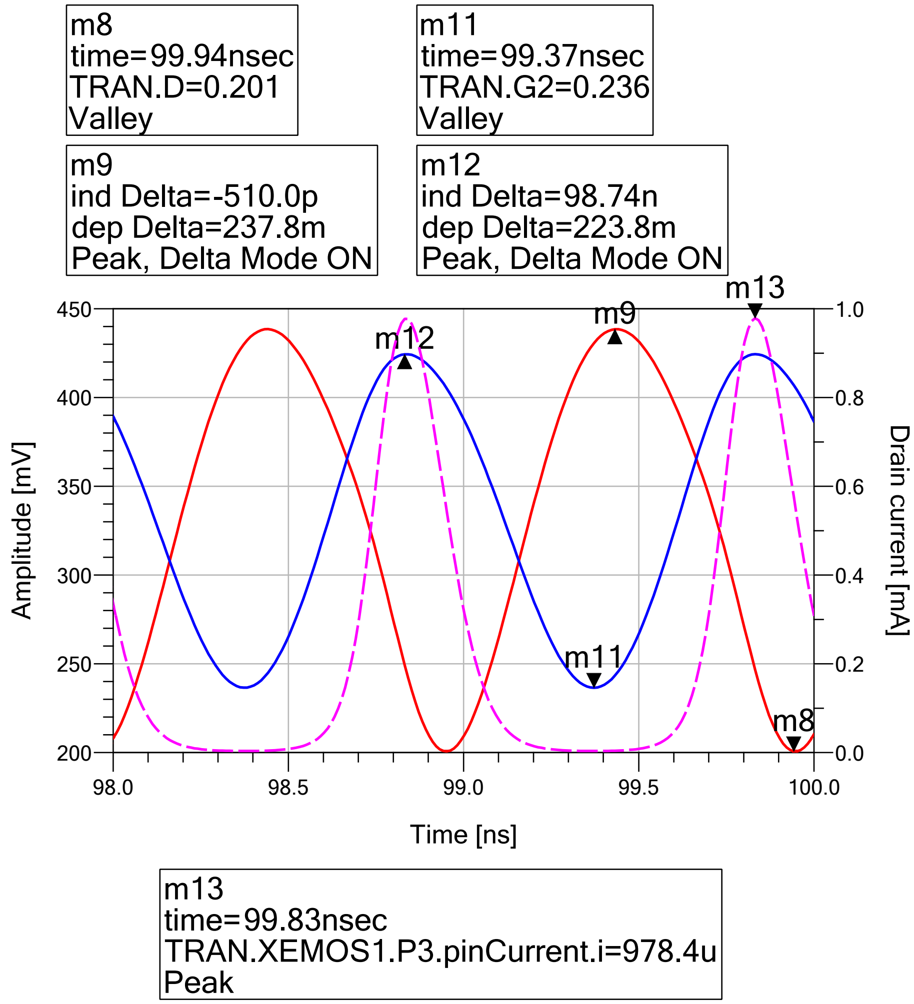
$$I_{bcritmin} = G_{mcrit} \cdot nU_T \cong 132\,\mu A$$

- Setting the oscillation amplitude to $A = 100\ mV$, we get

$$x = \frac{A}{nU_T} = 3 \quad \Rightarrow \quad \frac{x \cdot \mathrm{I}_{B0}(x)}{2\,\mathrm{I}_{B1}(x)} = 1.87 \quad \Rightarrow \quad I_b = I_{bcritmin} \cdot 1.87 = 247.7\,\mu A$$

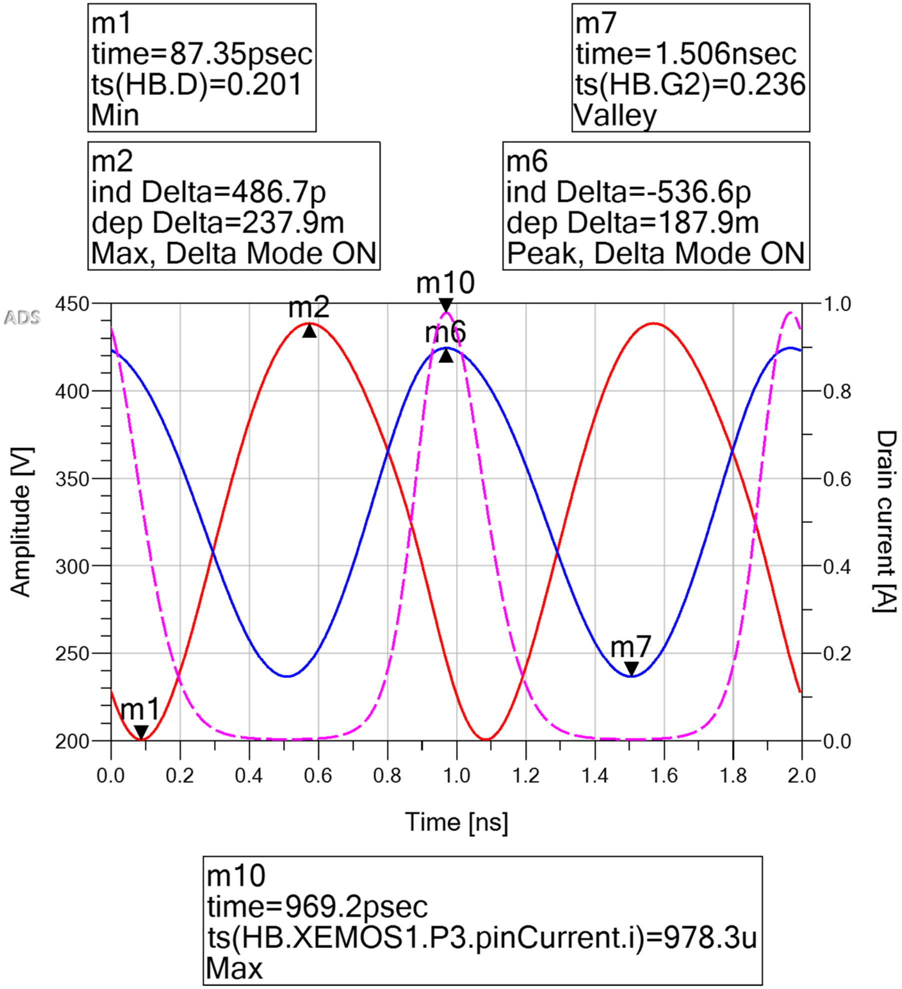
Pierce Oscillator Example in WI – Transient Simulations

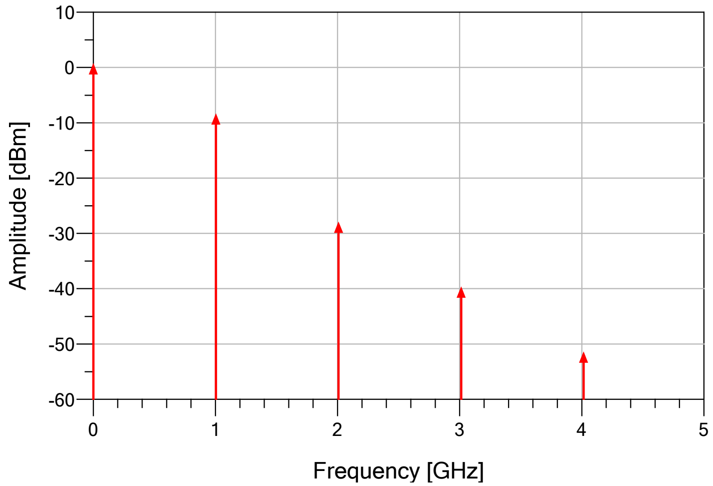




- Transient simulations performed with an ideal exponential transconductor
- The amplitude is slightly larger than 100mV (119 mV). This comes from the fact that Q_L is not that large generating harmonics which is in contradiction with the assumption of a sinusoidal gate voltage
- The above theory is based on the fundamental component only assuming a large Q and hence that the harmonics are negligible

Pierce Oscillator Example in WI – HB Simulations

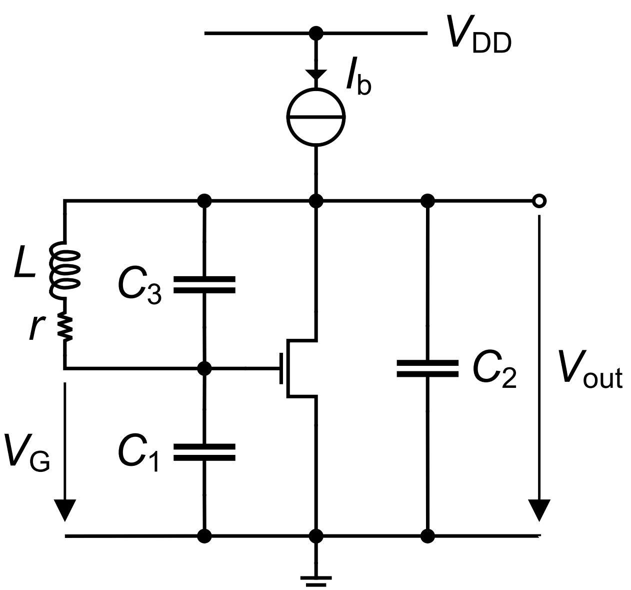




- Harmonic balance SS simulations performed with an ideal exponential transconductor
- Consistent with transient simulations
- The amplitude is slightly larger than 100mV (119 mV)

Pierce Oscillator in Strong Inversion

- The same analysis can be carried out assuming the transistor is operating in **strong inversion**
- It can be handled analytically as long as the oscillation amplitude A is assumed to be smaller than the overdrive voltage $V_G - V_{T0}$ in order for the current to remain positive **avoiding any current clipping**
- In this case the gate voltage and the drain current are given by



$$V_G(t) = V_{G0} + A \cdot \cos(\omega_0 t)$$

$$I_D(t) = I_{spec} \cdot \left(\frac{V_G(t) - V_{T0}}{2nU_T}\right)^2 = I_{spec} \cdot \left(\frac{V_{G0} - V_{T0} + A \cdot \cos(\omega_0 t)}{2nU_T}\right)^2$$

$$= \frac{I_{spec}}{4} \cdot \left(v_{gt0} + x \cdot \cos(\omega_0 t)\right)^2 \quad \text{for} \quad x \leq v_{gt0}$$

with $I_{spec} \triangleq 2n\beta U_T^2$, $\quad v_{gt0} \triangleq \dfrac{V_{G0} - V_{T0}}{nU_T}$, $\quad x \triangleq \dfrac{A}{nU_T}$

Pierce Oscillator in Strong Inversion

- Normalizing and developing the quadratic function leads to

$$i_d(t) = \frac{I_D(t)}{I_{spec}} = \frac{1}{4} \cdot \left(v_{gt0} + x \cdot \cos(\omega_0 t)\right)^2 = \frac{1}{4} \cdot \left(v_{gt0}^2 + 2 v_{gt0} \cdot x \cdot \cos(\omega_0 t) + x^2 \cdot \cos^2(\omega_0 t)\right)$$

$$= \underbrace{\frac{1}{4} \cdot \left(v_{gt0}^2 + \frac{x^2}{2}\right)}_{i_{dc}} + \underbrace{\frac{v_{gt0} \cdot x}{2}}_{i_{d(1)}} \cdot \cos(\omega_0 t) + \frac{x^2}{8} \cdot \cos(2\omega_0 t)$$

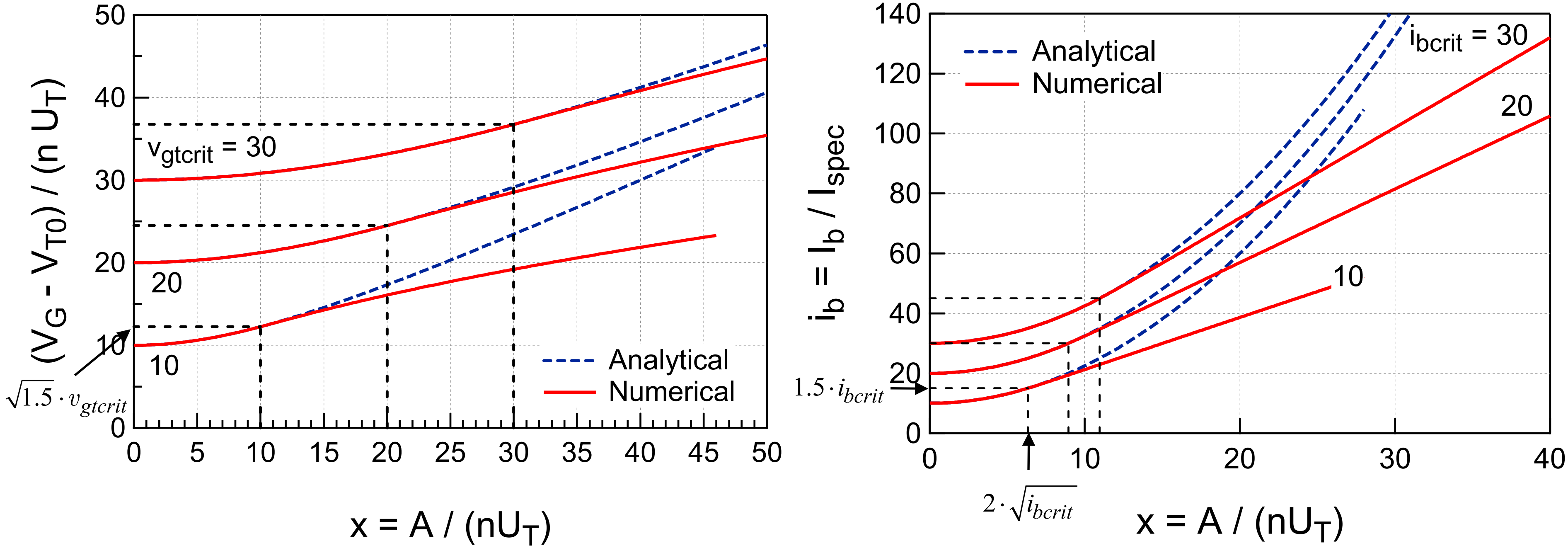
- The normalized **dc current** for $x < v_{gt0}$ is given by

$$i_{dc} \triangleq \frac{I_{dc}}{I_{spec}} = \frac{I_b}{I_{spec}} = \frac{1}{4} \cdot \left(v_{gt0}^2 + \frac{x^2}{2}\right) = i_0 + \frac{x^2}{8} \quad \text{with} \quad i_0 \triangleq \frac{I_0}{I_{spec}} = \left(\frac{v_{gt0}}{2}\right)^2$$

- Where i_0 is the dc current for zero amplitude

$$i_0 \triangleq \frac{I_0}{I_{spec}} = \left(\frac{v_{gt0}}{2}\right)^2 \quad \text{with} \quad I_0 \triangleq I_D\big|_{x=0} = I_{spec} \cdot \left(\frac{V_{G0} - V_{T0}}{2nU_T}\right)^2 = I_{spec} \cdot \left(\frac{v_{gt0}}{2}\right)^2$$

Bias Voltage and Current



- The required normalized bias overdrive voltage v_{gt} (normalized bias current i_b or inversion factor) for a given critical overdrive voltage v_{gtcrit} (critical current i_{bcrit}) assuming $x < v_{gtcrit}$ is given by

$$v_{gt} \triangleq \frac{V_G - V_{T0}}{nU_T} = \sqrt{v_{gtcrit}^2 + \frac{x^2}{2}} \quad \text{or} \quad i_b \triangleq \frac{I_b}{I_{spec}} = \left(\frac{v_{gt}}{2}\right)^2 = i_{bcrit} + \frac{x^2}{8}$$

Transconductance for the Fundamental

- The fundamental component for $x < v_{gt0}$ is given by

$$I_{D(1)} = I_{spec} \cdot \frac{v_{gt0}}{2} \cdot x \quad \text{for} \quad x \le v_{gt0}$$

- The transconductance for the fundamental component is then given by

$$G_{m(1)} = \frac{I_{D(1)}}{A} = \frac{I_{spec}}{A} \cdot \frac{v_{gt0}}{2} \cdot x = \frac{I_{spec}}{nU_T} \cdot \frac{v_{gt0}}{2} = \frac{I_{spec}}{nU_T} \cdot \sqrt{i_0} \quad \text{for} \quad x \le v_{gt0}$$

- From the dc constraint, we get the relation between i_0 and v_{gt0} which should both decrease with the amplitude x

$$i_b = \left(\frac{v_{gt}}{2}\right)^2 = \left(\frac{v_{gt0}}{2}\right)^2 + \frac{x^2}{8} = i_0 + \frac{x^2}{8} \quad \rightarrow \quad i_0 = i_b - \frac{x^2}{8} \quad \text{or} \quad v_{gt0} = \sqrt{v_{gt}^2 - \frac{x^2}{2}}$$

- The transconductance for the fundamental then becomes

$$G_{m(1)} = \frac{I_{spec}}{nU_T} \cdot \sqrt{i_b - \frac{x^2}{8}} = \frac{I_{spec}}{nU_T} \cdot \sqrt{\left(\frac{v_{gt}}{2}\right)^2 - \frac{x^2}{8}} = \frac{I_{spec}}{nU_T} \cdot \frac{1}{2} \cdot \sqrt{v_{gt}^2 - \frac{x^2}{2}}$$

Transconductance for the Fundamental

- The small-signal transconductance is given by

$$G_m = \frac{I_{spec}}{nU_T} \cdot \frac{v_{gt}}{2} = \frac{I_{spec}}{nU_T} \cdot \sqrt{i_b} \quad \text{with} \quad v_{gt} \triangleq \frac{V_G - V_{T0}}{nU_T} \quad \text{and} \quad i_b = \left(\frac{v_{gt}}{2}\right)^2$$

- The transconductance for the fundamental normalized to the small-signal transconductance is then given by

$$\frac{G_{m(1)}}{G_m} = \sqrt{1 - \frac{x^2}{8 i_b}} = \sqrt{1 - \frac{1}{2}\left(\frac{x}{v_{gt}}\right)^2} = \sqrt{1 - \frac{1}{2}\left(\frac{A}{V_G - V_{T0}}\right)^2} \quad \text{for} \quad x \le v_{gt}$$

- The critical condition is then given by

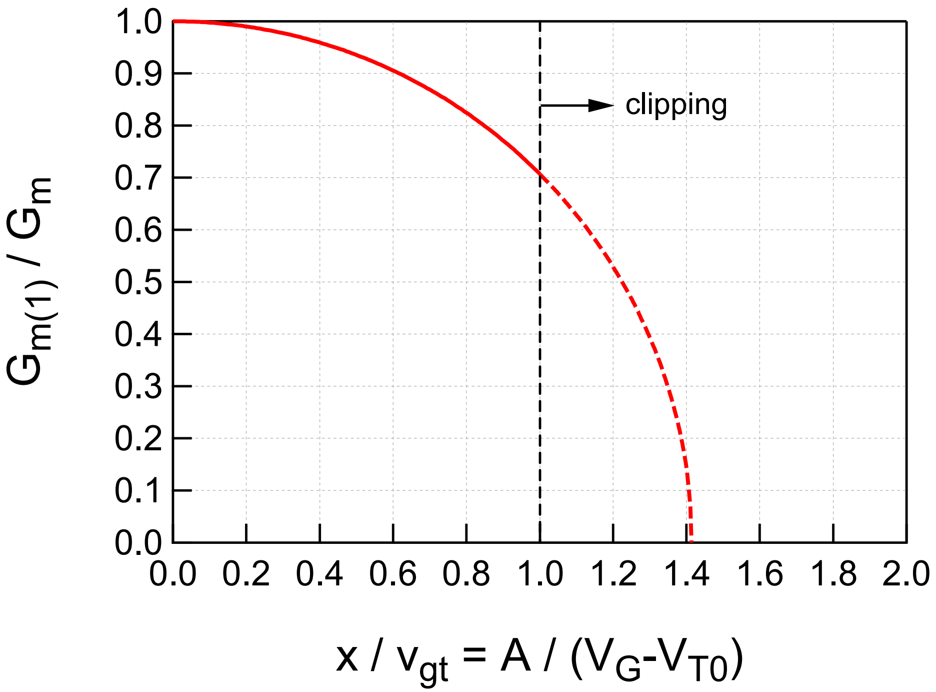
$$G_{m(1)} = G_{mcrit} \quad \rightarrow \quad \frac{I_{spec}}{nU_T} \cdot \frac{1}{2} \cdot \sqrt{v_{gt}^2 - \frac{x^2}{2}} = \frac{I_{spec}}{nU_T} \cdot \frac{v_{gtcrit}}{2} \quad \rightarrow \quad \sqrt{v_{gt}^2 - \frac{x^2}{2}} = v_{gtcrit}$$

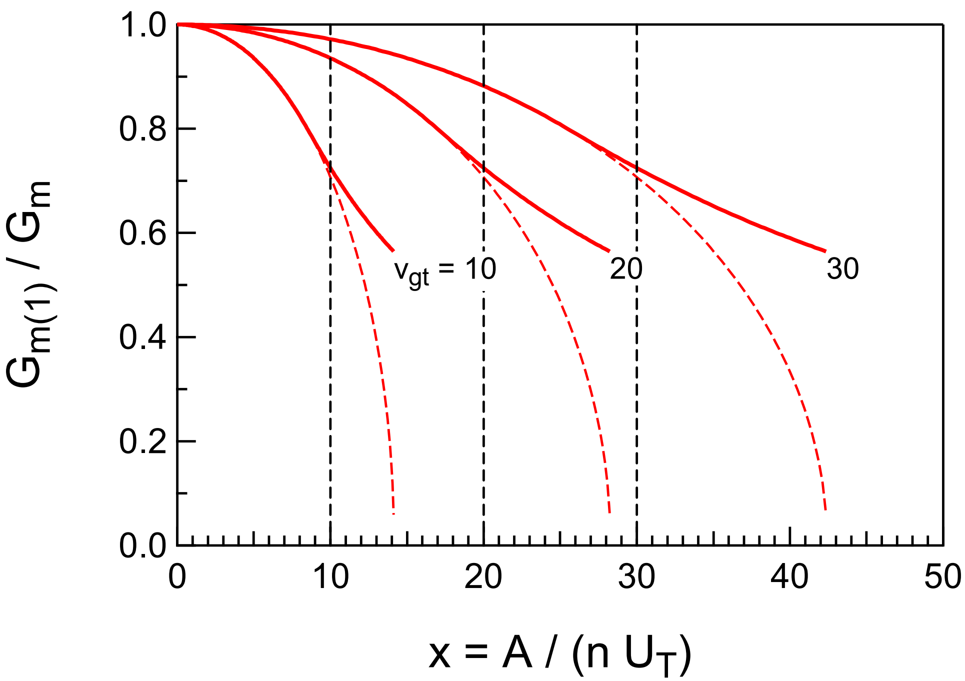
or in terms of currents $\frac{I_{spec}}{nU_T} \cdot \sqrt{i_b - \frac{x^2}{8}} = \frac{I_{spec}}{nU_T} \cdot \sqrt{i_{bcrit}} \quad \rightarrow \quad i_b - \frac{x^2}{8} = i_{bcrit}$

- Since there is no current clipping and no bias shift, it is not surprising to find that

$$v_{gtcrit} = v_{gt0} = \sqrt{v_{gt}^2 - \frac{x^2}{2}} \quad \text{and} \quad i_{bcrit} = i_0 = i_b - \frac{x^2}{8}$$

Transconductance for the Fundamental





$$\frac{G_{m(1)}}{G_m} = \sqrt{1-\frac{x^2}{8i_b}} = \sqrt{1-\frac{1}{2}\left(\frac{x}{v_{gt}}\right)^2} = \sqrt{1-\frac{1}{2}\left(\frac{A}{V_G - V_{T0}}\right)^2} \quad \text{for} \quad x \le v_{gt}$$

with $v_{gt} \triangleq \dfrac{V_G - V_{T0}}{nU_T}$ and $x \triangleq \dfrac{A}{nU_T}$

Pierce Oscillator Example – Bias in SI

- If we assume the transistor operates in strong inversion (with $n = 1.3$) and choose the operating overdrive voltage as $V_G - V_{T0} = 300\ mV$ and the oscillation amplitude as $A = 200\ mV$, we can then calculate the normalized amplitude as

$$v_{gt} \triangleq \frac{V_G - V_{T0}}{nU_T} = 9.12 \quad i_b \triangleq \frac{I_b}{I_{spec}} = \left(\frac{v_{gt}}{2}\right)^2 = 20.8 \quad x \triangleq \frac{A}{nU_T} = 6.08$$

- The critical overdrive and critical current can then be calculated as

$$v_{gtcrit} = 2\sqrt{i_{bcrit}} = \sqrt{v_{gt}^2 - \frac{x^2}{2}} = 8.045 \quad \text{or} \quad i_{bcrit} = \left(\frac{v_{gtcrit}}{2}\right)^2 = i_b - \frac{x^2}{8} = 16.18$$

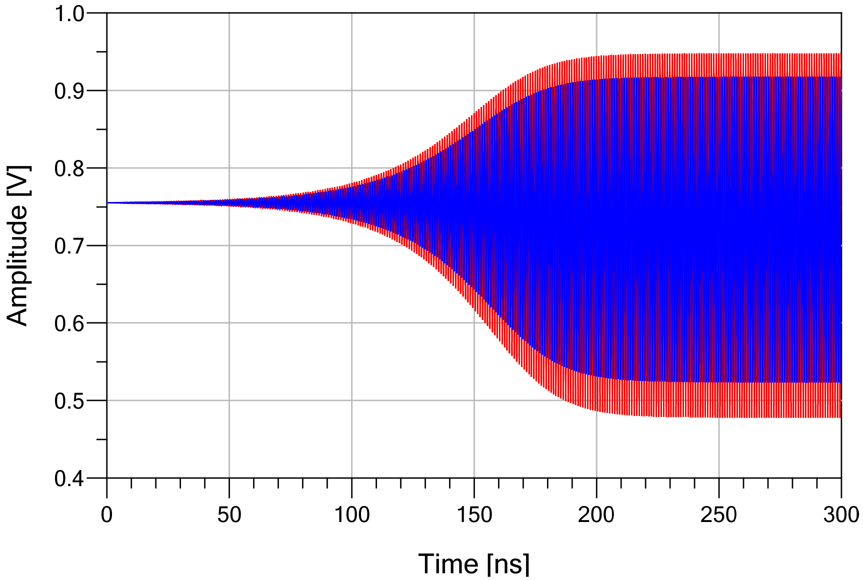
- The specific current is then obtained as

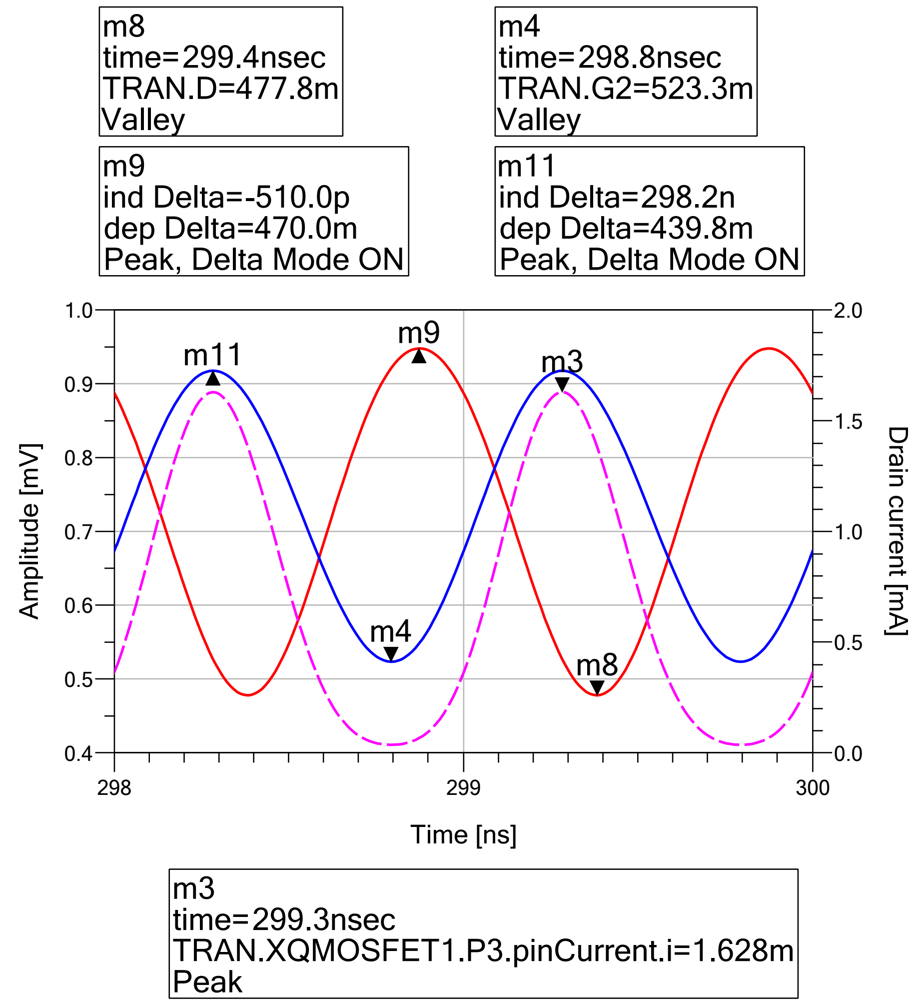
$$I_{spec} = \frac{2nU_T \cdot G_{mcrit}}{v_{gtcrit}} = \frac{nU_T \cdot G_{mcrit}}{\sqrt{i_{bcrit}}} = 32.9\ \mu A \quad \rightarrow \quad \frac{\beta}{2n} = \frac{I_b}{(V_G - V_{T0})^2}$$

- The critical current and the required bias current are given by

$$I_{crit} = I_{spec} \cdot i_{bcrit} = 532.875\ \mu A \quad I_b = I_{spec} \cdot i_b = I_{spec} \cdot \left(\frac{v_{gt}}{2}\right)^2 = 685.125\ \mu A$$

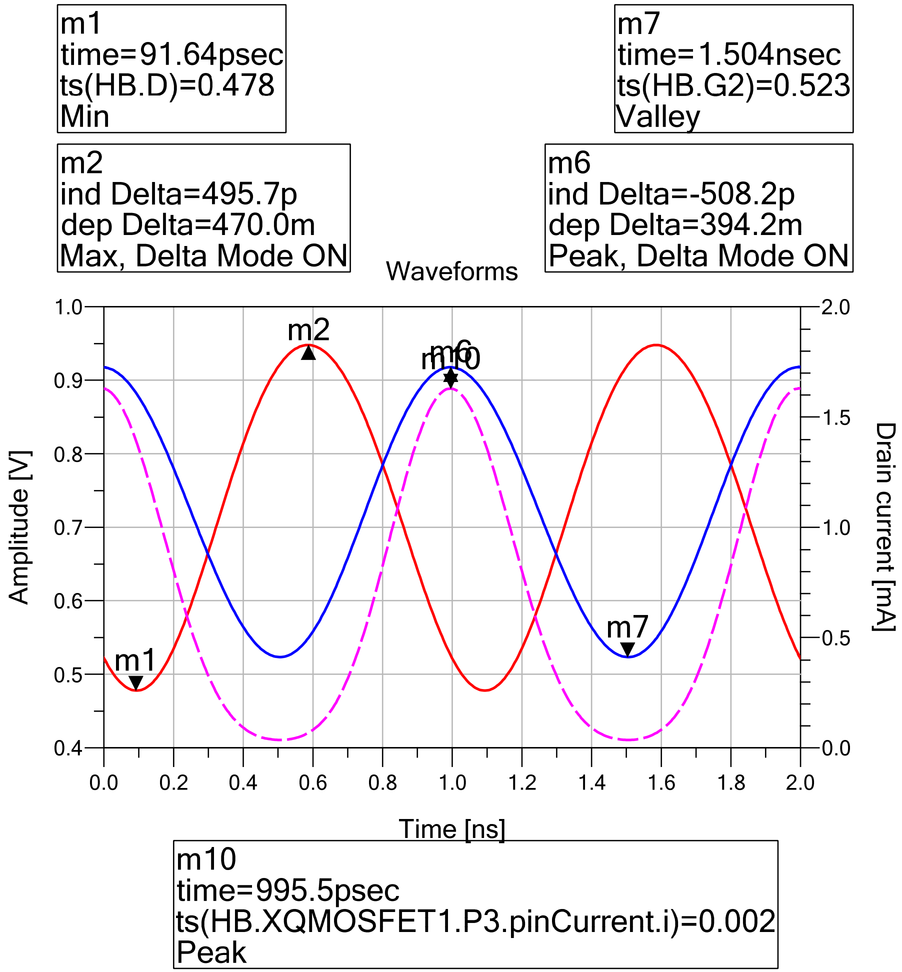
Pierce Oscillator Example in SI – Transient Simulations

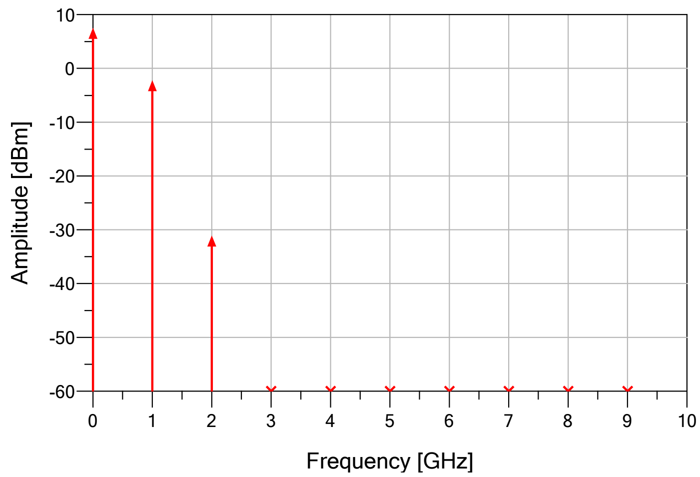




- Transient simulations performed with an ideal quadratic transconductor
- The amplitude at the drain (235 mV) and at the gate (220 mV) are slightly higher than 200mV
- The above theory is based on the fundamental component only assuming a large Q and hence that the harmonics are negligible

Pierce Oscillator Example in SI – HB Simulations





- Harmonic balance SS simulations performed with an ideal quadratic transconductor
- The amplitude at the gate is exactly 200mV, whereas the amplitude at the drain is slightly larger (235mV)

Design Procedure (Strong Inversion)

- From the Q of the tank and the capacitances, deduce the required G_{mcrit}
- Choose an appropriate critical overdrive voltage $V_{Gcrit} - V_{T0}$ (normalized form v_{gtcrit}) or critical inversion factor $i_{bcrit} = \left(v_{gtcrit}/2\right)^2$ which correspond to the average overdrive and bias current at the operating point
- We can now find the specific current according to

$$I_{spec} = \frac{2nU_T \cdot G_{mcrit}}{v_{gtcrit}} = \frac{nU_T \cdot G_{mcrit}}{\sqrt{i_{bcrit}}}$$

- Calculate the desired normalized amplitude $x = A/(nU_T)$
- For the chosen normalized critical overdrive voltage (or inversion factor) and normalized amplitude, deduce the normalized overdrive voltage or bias current from

$$v_{gt} \triangleq \frac{V_G - V_{T0}}{nU_T} = \sqrt{v_{gtcrit}^2 + \frac{x^2}{2}} \quad \text{or} \quad i_b \triangleq \frac{I_b}{I_{spec}} = \left(\frac{v_{gt}}{2}\right)^2 = i_{bcrit} + \frac{x^2}{8}$$

- The bias current I_b required for the desired amplitude is then given by

$$I_b = I_{spec} \cdot i_b = I_{spec} \cdot \left(\frac{v_{gt}}{2}\right)^2$$

Design Procedure (from Weak to Strong Inversion)

- From the Q of the tank and the capacitances, deduce the required G_{mcrit}
- Choose an appropriate inversion factor i_{bcrit}
- Calculate the minimum critical bias current $I_{bcritmin}$

$$I_{bcritmin} = n \cdot U_T \cdot G_{mcrit}$$

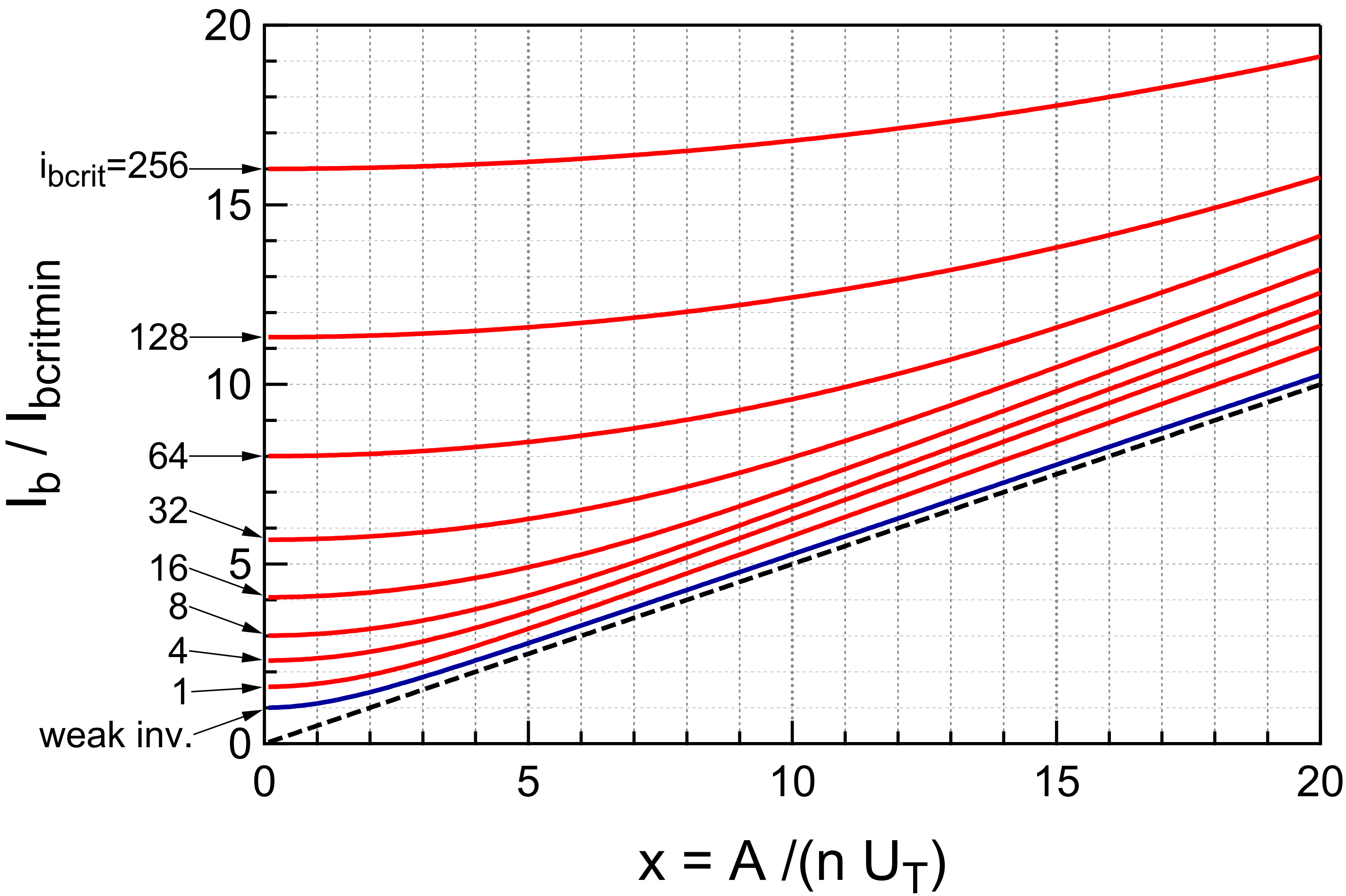
- Calculate the corresponding specific current

$$I_{spec} = \frac{I_{bcritmin}}{\sqrt{i_{bcrit}} \cdot \left(1 - \exp\left[-\sqrt{i_{bcrit}}\right]\right)}$$

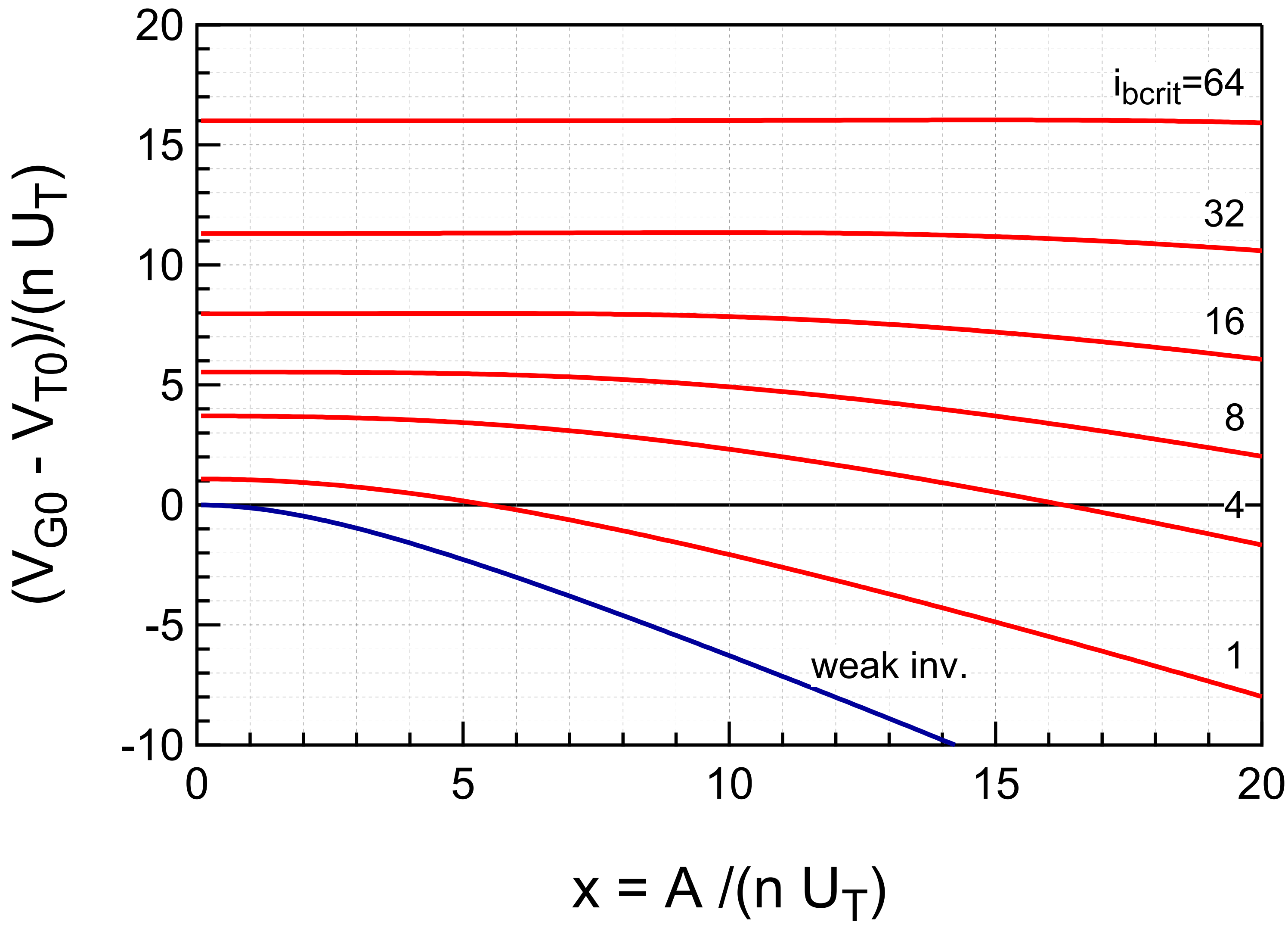
- Calculate the desired normalized amplitude $x = A/(nU_T)$
- For the chosen inversion factor i_{bcrit} and normalized amplitude x, deduce the normalized bias current $I_b/I_{bcritmin}$ from the abacus (next slide)
- Deduce the actual bias current

$$I_b = I_{bcritmin} \cdot \frac{I_b}{I_{bcritmin}}$$

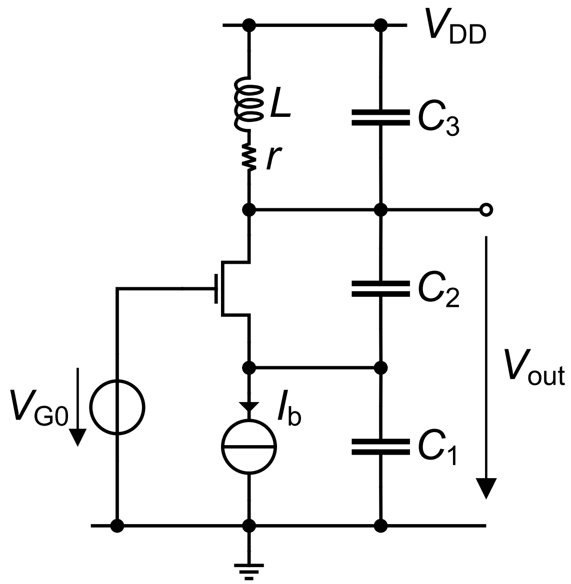
Bias Current from Weak to Strong Inversion

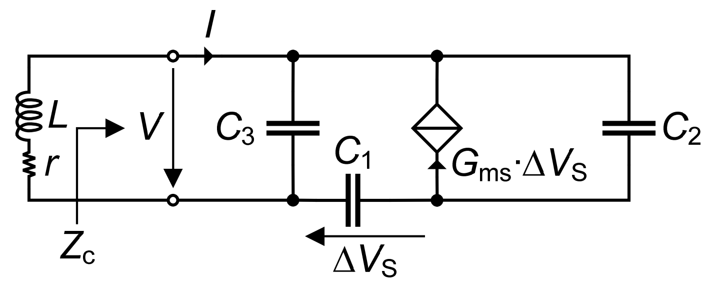


Bias Shift from Weak to Strong Inversion



The Colpitts Oscillator – Circuit Impedance





$$Z_c = -\frac{G_{ms} + j\omega\left(C_1 + C_2\right)}{\omega^2\left(C_1C_2 + C_1C_3 + C_2C_3\right) - j\omega G_{ms}C_3}$$

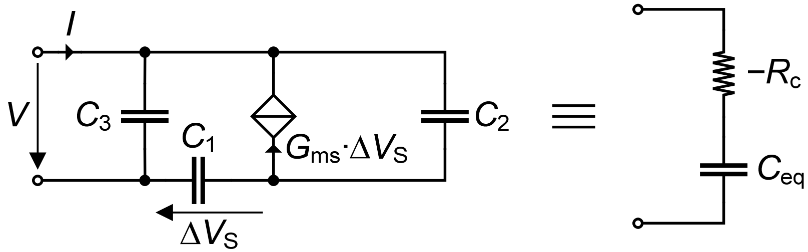
- Analysis almost identical to the Pierce except that G_m is replaced with G_{ms}

$$R_c = \frac{G_{ms}C_1C_2}{\left(G_{ms}C_3\right)^2 + \omega^2\left(C_1C_2 + C_1C_3 + C_2C_3\right)^2}$$

$$X_c = \frac{G_{ms}^2C_3 + \omega^2\left(C_1 + C_2\right)\left(C_1C_2 + C_1C_3 + C_2C_3\right)}{\omega\left[\left(G_{ms}C_3\right)^2 + \omega^2\left(C_1C_2 + C_1C_3 + C_2C_3\right)^2\right]}$$

The Colpitts Oscillator – Critical Transconductance

- For $G_{ms} \ll (\omega_0/C_3)(C_1C_2 + C_1C_3 + C_2C_3)$, R_c and X_c simplify to



$$R_c \cong \frac{G_{ms}C_1C_2}{\omega^2\left(C_1C_2+C_1C_3+C_2C_3\right)^2} \qquad X_c \cong \frac{1}{\omega C_{eq}}$$

with $C_{eq} = C_3 + C_{12} = \dfrac{C_1C_2+C_1C_3+C_2C_3}{C_1+C_2}$

- The oscillation frequency is approximated by $\omega_0 \cong \dfrac{1}{\sqrt{L\cdot C_{eq}}}$

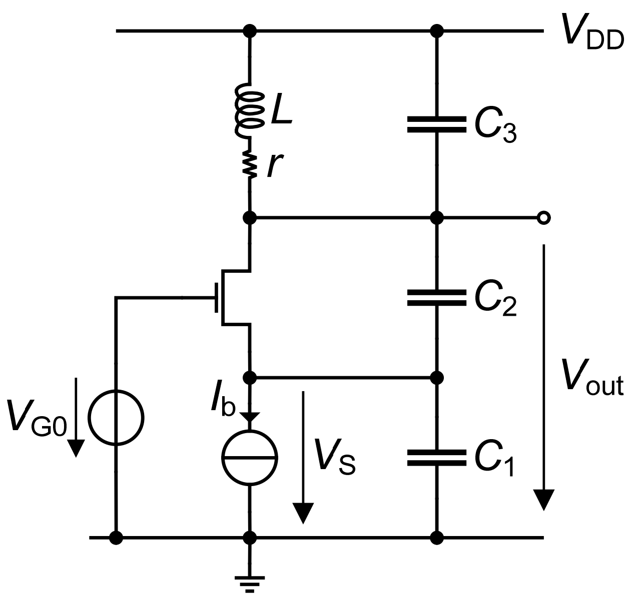
- And the critical (source) transconductance is given by

$$G_{mscrit} = \omega_0 C_2 \frac{\alpha_1 Q_L}{2\alpha_3}\left[1-\sqrt{1-\left(\frac{2\alpha_3}{\alpha_1 Q_L}\right)^2(\alpha_1+1)\left(1+\alpha_1+\frac{\alpha_1}{\alpha_3}\right)}\right]$$

$$\cong \frac{1}{r}\frac{\alpha_1}{2\alpha_3^2}\left[1-\sqrt{1-\left(\frac{2\alpha_3(\alpha_1+1)}{\alpha_1 Q_L}\right)^2}\right] \cong \frac{1}{r}\frac{(\alpha_1+1)^2}{\alpha_1 Q_L} = \frac{\omega_0}{Q_L}(C_1+C_2)\left(1+\frac{C_3}{C_{12}}\right)$$

where $\alpha_1 \triangleq \dfrac{C_1}{C_2}$ $\quad \alpha_3 \triangleq \dfrac{C_3}{C_2}$

Nonlinear Analysis of the Collpits Oscillator (weak inv.)



$$A = \Delta V_S = \frac{C_2}{C_1 + C_2} \cdot \Delta V_{out}$$

- In the case of the Collpits oscillator the source voltage can be assumed to be sinusoidal

$$V_S(t) = V_{S0} - A \cdot \cos(\omega_0 t)$$

- If the transistor is biased in weak inversion, the drain current is then given by

$$I_D(t) = I_{D0} \cdot e^{\frac{V_{G0} - nV_S(t)}{nU_T}} = I_{D0} \cdot e^{\frac{V_{G0} - nV_{S0} + nA \cdot \cos(\omega_0 t)}{nU_T}}$$

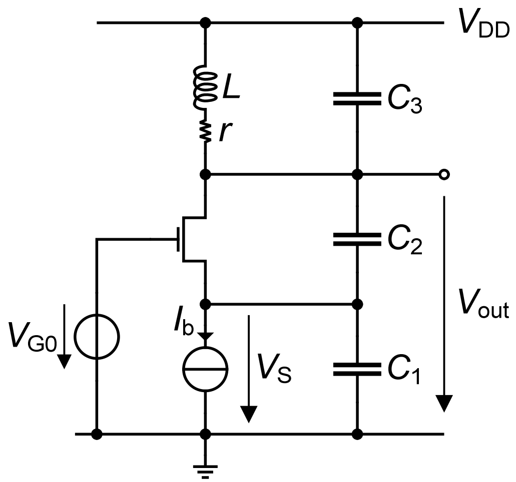
$$= I_0 \cdot e^{\frac{A \cdot \cos(\omega_0 t)}{U_T}} = I_0 \cdot e^{x \cdot \cos(\omega_0 t)}$$

with $I_0 \triangleq I_{D0} \cdot e^{\frac{V_{G0} - nV_{S0}}{nU_T}} = I_{spec} \cdot e^{\frac{V_{G0} - V_{T0} - nV_{S0}}{nU_T}}$

and $x \triangleq \frac{A}{U_T}$

- **Same analysis than the Pierce oscillator** and hence the results and normalized plots of the Pierce oscillator also apply to the Colpitts oscillator

Example – The Colpitts Oscillator



$$f_0 = 1\,GHz, Q_L = 10, C_1 = C_2 = 1\,pF, C_3 = 1\,pF$$

$$\omega_0 \cong \frac{1}{\sqrt{L(C_3 + C_{12})}} \Rightarrow L \cong \frac{1}{\omega_0^2 (C_3 + C_{12})} = 16.9\,nH$$

$$r = \frac{\omega_0 L}{Q_L} = 10.6\,\Omega$$

$$G_{mscrit} \cong \frac{\omega_0}{Q_L}(C_1 + C_2)\left(1 + \frac{C_3}{C_{12}}\right) = 3.8\frac{mA}{V}$$

- Since the inductance Q_L is not very high, the above approximation is not very accurate. The exact solution is then given by

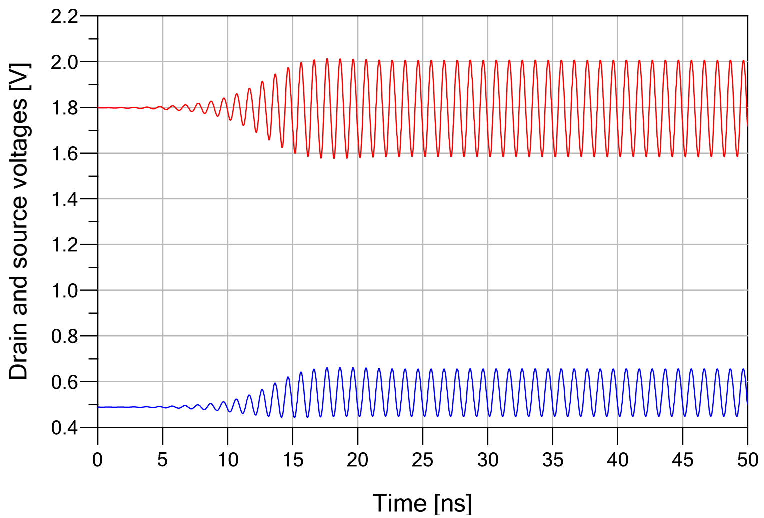
$$G_{mscrit} = \omega_0 C_2 \frac{\alpha_1 Q_L}{2\alpha_3}\left[1 - \sqrt{1 - \left(\frac{2\alpha_3}{\alpha_1 Q_L}\right)^2 (\alpha_1 + 1)\left(1 + \alpha_1 + \frac{\alpha_1}{\alpha_3}\right)}\right] = 4\frac{mA}{V}$$

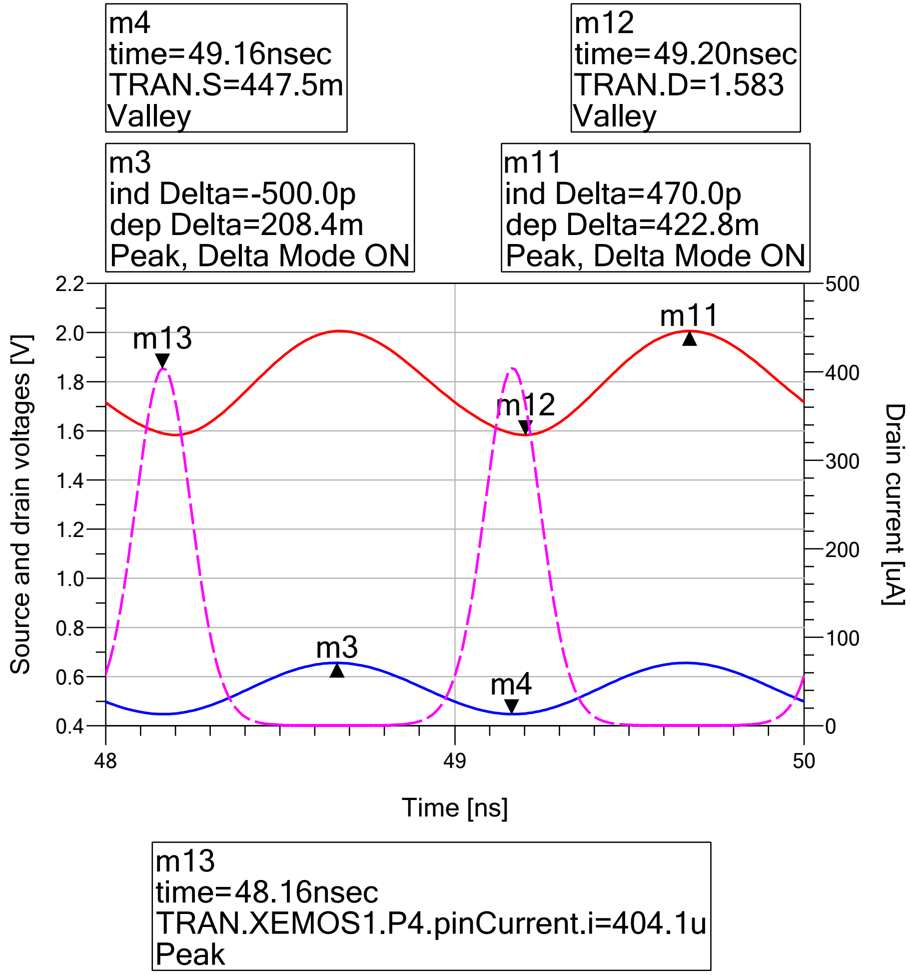
- The inductance value is then found from

$$L = \frac{X_c(\omega_0, G_{mscrit})}{\omega_0}$$

- This leads to $L = 17.256\,nH$ and $r = 10.8\,\Omega$

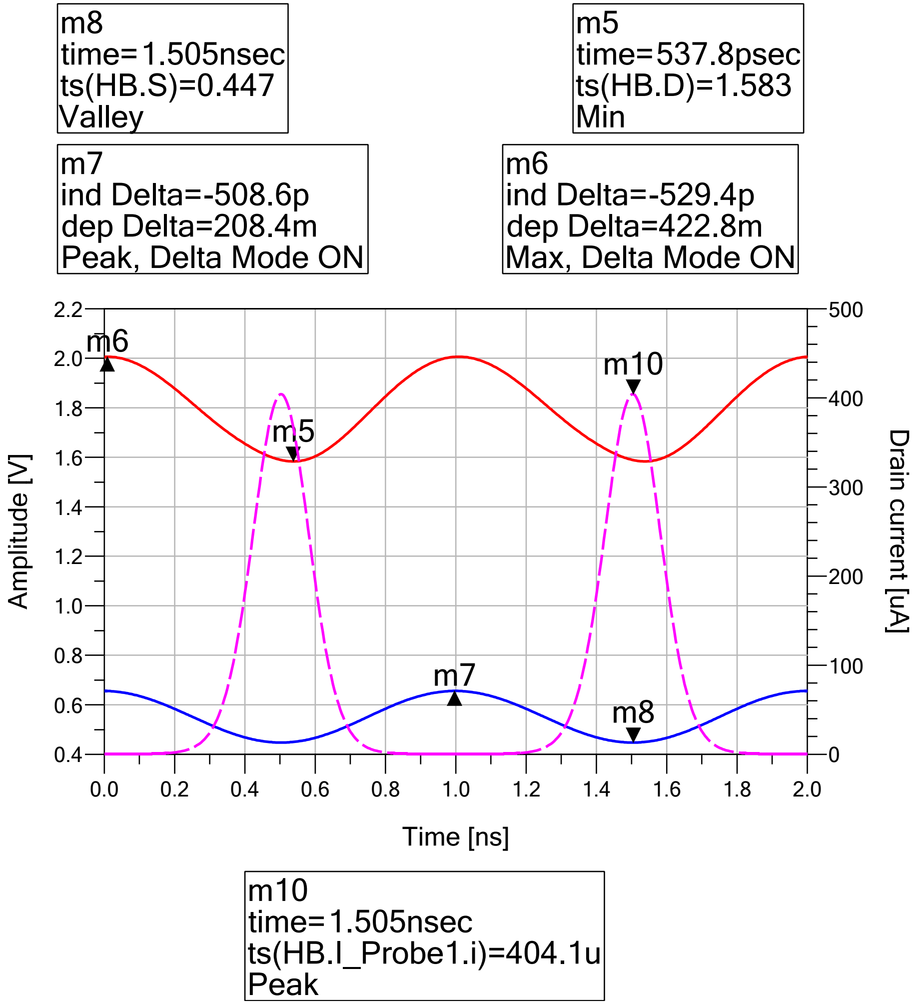
Colpitts Oscillator Example in WI – Transient Simulations

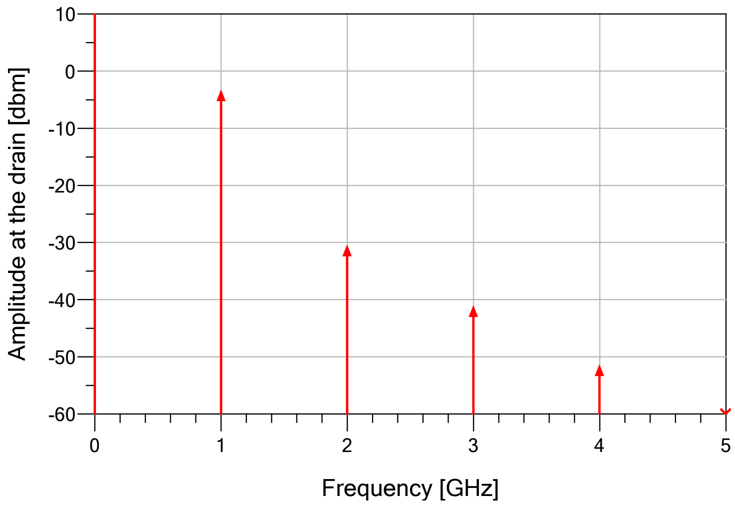




- Transient simulations performed with an ideal exponential transconductor
- The amplitude is almost exactly 100mV
- The above theory is based on the fundamental component only assuming a large Q and hence that the harmonics are negligible

Colpitts Oscillator Example in WI – HB Simulations

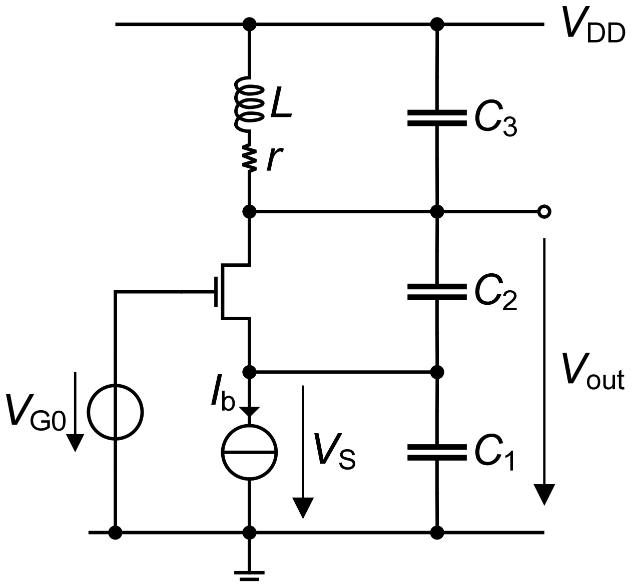




- Harmonic balance SS simulations performed with an ideal quadratic transconductor
- The amplitude at the gate is exactly 200mV, whereas the amplitude at the drain is slightly larger (235mV)

Colpitts Oscillator in Strong Inversion

- Similar analysis than for the Pierce oscillator. In this case we have



$$A = \Delta V_S = \frac{C_2}{C_1 + C_2} \cdot \Delta V_{out}$$

$$V_S(t) = V_{S0} - A \cdot \cos(\omega_0 t)$$

$$I_D(t) = I_{spec} \cdot \left(\frac{V_{P0} - V_S(t)}{2U_T} \right)^2$$

$$= I_{spec} \cdot \left(\frac{V_{P0} - V_{S0} - A \cdot \cos(\omega_0 t)}{2U_T} \right)^2$$

$$= \frac{I_{spec}}{4} \cdot \left(v_{ps0} - x \cdot \cos(\omega_0 t) \right)^2 \quad \text{for} \quad x \le v_{ps0}$$

- The results and plots obtained for the Pierce oscillator in strong inversion can be used for the Colpitts oscillator accounting for the different normalization given by

$$v_{ps0} \triangleq \frac{V_{P0} - V_{S0}}{U_T} = \frac{V_{G0} - V_{T0} - nV_{S0}}{nU_T}, \quad x \triangleq \frac{A}{U_T}$$

Colpitts Oscillator in Strong Inversion

- Normalizing and developing the quadratic function leads to

$$i_d(t) = \frac{I_D(t)}{I_{spec}} = \frac{1}{4} \cdot \left(v_{ps0} - x \cdot \cos(\omega_0 t)\right)^2 = \frac{1}{4} \cdot \left(v_{ps0}^2 - 2 v_{ps0} \cdot x \cdot \cos(\omega_0 t) + x^2 \cdot \cos^2(\omega_0 t)\right)$$

$$= \underbrace{\frac{1}{4} \cdot \left(v_{ps0}^2 + \frac{x^2}{2}\right)}_{i_{dc}} - \underbrace{\frac{v_{ps0} \cdot x}{2}}_{-i_{d(1)}} \cdot \cos(\omega_0 t) + \frac{x^2}{8} \cdot \cos(2\omega_0 t)$$

- The normalized dc current for $x < v_{ps0}$ is given by

$$i_{dc} \triangleq \frac{I_{dc}}{I_{spec}} = \frac{I_b}{I_{spec}} = \frac{1}{4} \cdot \left(v_{ps0}^2 + \frac{x^2}{2}\right) = i_0 + \frac{x^2}{8} \quad \text{with} \quad i_0 \triangleq \frac{I_0}{I_{spec}} = \left(\frac{v_{ps0}}{2}\right)^2$$

- The normalized fundamental component for $x < v_{ps0}$ is given by

$$i_{d(1)} \triangleq \frac{I_{D(1)}}{I_{spec}} = \frac{v_{ps0} \cdot x}{2} \quad \text{for} \quad x \le v_{ps0}$$

Transconductance for the Fundamental (Colpitts)

- The fundamental component for $x < v_{ps0}$ is given by

$$I_{D(1)} = I_{spec} \cdot \frac{v_{ps0}}{2} \cdot x \quad \text{for} \quad x \le v_{ps0}$$

- The transconductance for the fundamental component is given by

$$G_{m(1)} = \frac{I_{D(1)}}{A} = \frac{I_{spec}}{A} \cdot \frac{v_{ps0}}{2} \cdot x = \frac{I_{spec}}{U_T} \cdot \frac{v_{ps0}}{2} = \frac{I_{spec}}{U_T} \cdot \sqrt{i_0} \quad \text{for} \quad x \le v_{ps0}$$

- From the dc constraint we get

$$i_b = \left(\frac{v_{ps}}{2}\right)^2 = \left(\frac{v_{ps0}}{2}\right)^2 + \frac{x^2}{8} = i_0 + \frac{x^2}{8} \quad \rightarrow \quad i_0 = i_b - \frac{x^2}{8} \quad \text{or} \quad v_{ps0} = \sqrt{v_{ps}^2 - \frac{x^2}{2}}$$

- The transconductance for the fundamental then becomes

$$G_{ms(1)} = \frac{I_{spec}}{U_T} \cdot \sqrt{i_b - \frac{x^2}{8}} = \frac{I_{spec}}{U_T} \cdot \frac{1}{2} \cdot \sqrt{v_{ps}^2 - \frac{x^2}{2}}$$

Transconductance for the Fundamental (Colpitts)

- The small-signal source transconductance is given by

$$G_{ms} = \frac{I_{spec}}{U_T} \cdot \frac{v_{ps}}{2} = \frac{I_{spec}}{U_T} \cdot \sqrt{i_b} \quad \text{with} \quad v_{ps} \triangleq \frac{V_P - V_S}{U_T} \quad \text{and} \quad i_b = \left(\frac{v_{ps}}{2}\right)^2$$

- The transconductance for the fundamental normalized to the small-signal transconductance is then given by

$$\frac{G_{ms(1)}}{G_{ms}} = \sqrt{1 - \frac{x^2}{8 i_b}} = \sqrt{1 - \frac{1}{2}\left(\frac{x}{v_{ps}}\right)^2} = \sqrt{1 - \frac{1}{2}\left(\frac{A}{V_P - V_S}\right)^2} \quad \text{for} \quad x \le v_{ps}$$

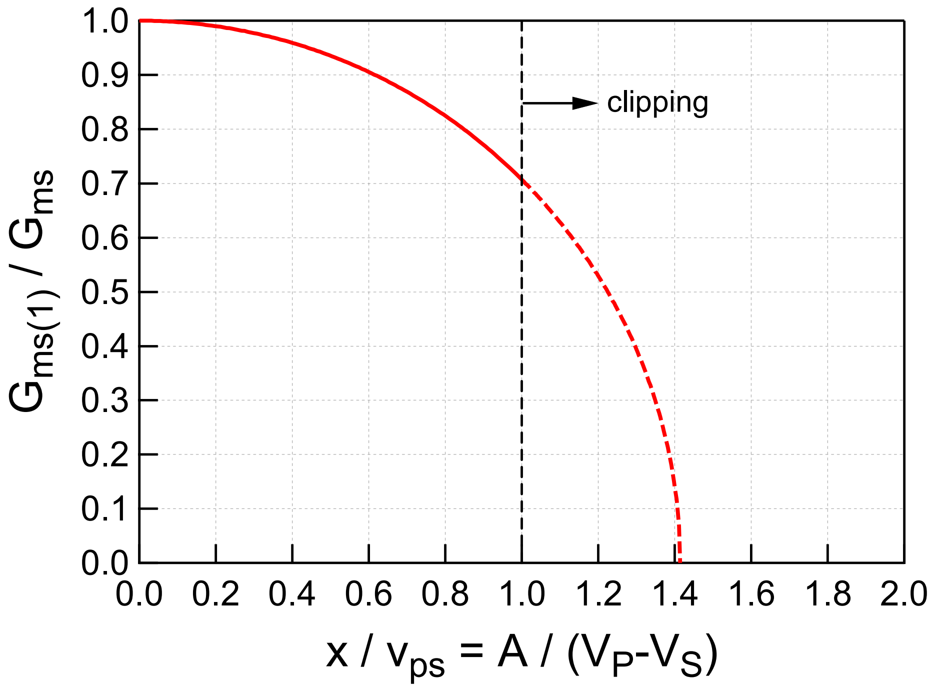
- The critical condition is then given by

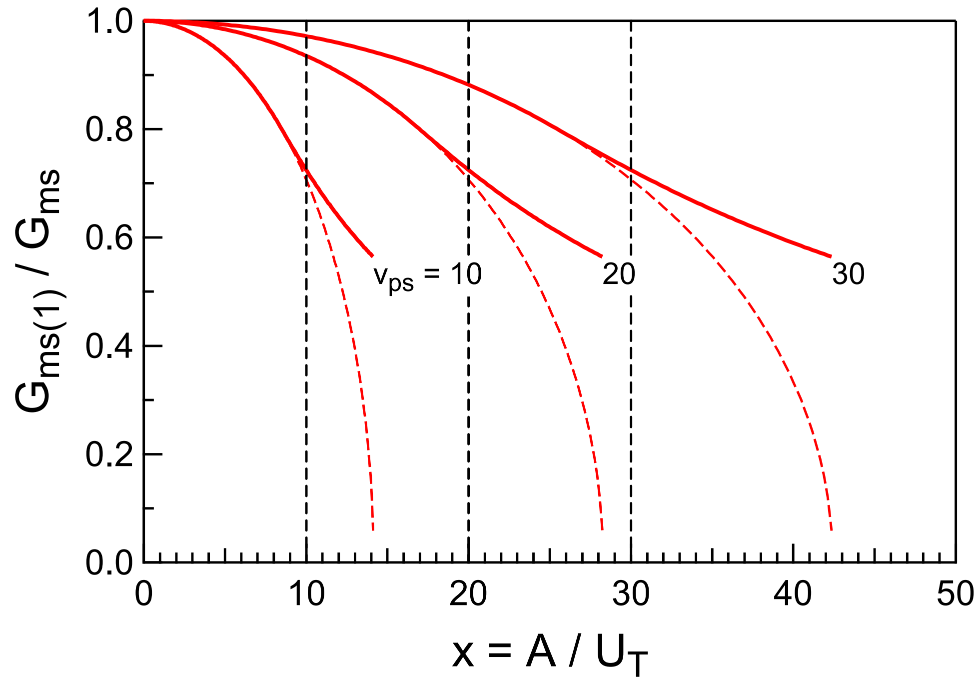
$$G_{ms(1)} = G_{mscrit} \quad \rightarrow \quad \frac{I_{spec}}{U_T} \cdot \frac{1}{2} \cdot \sqrt{v_{ps}^2 - \frac{x^2}{2}} = \frac{I_{spec}}{U_T} \cdot \frac{v_{pscrit}}{2} \quad \rightarrow \quad \sqrt{v_{ps}^2 - \frac{x^2}{2}} = v_{pscrit}$$

or in terms of currents

$$\frac{I_{spec}}{U_T} \cdot \sqrt{i_b - \frac{x^2}{8}} = \frac{I_{spec}}{U_T} \cdot \sqrt{i_{bcrit}} \quad \rightarrow \quad i_b - \frac{x^2}{8} = i_{bcrit}$$

Transconductance for the Fundamental (Colpitts)

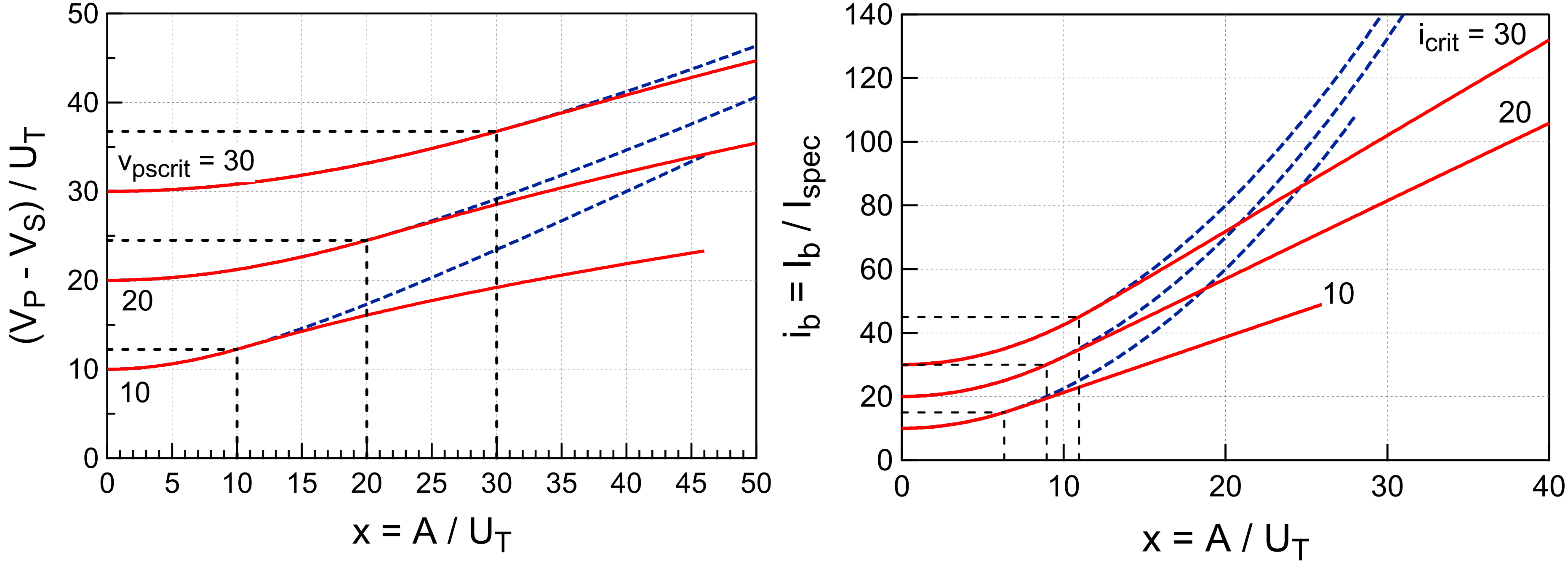




$$\frac{G_{ms(1)}}{G_{ms}} = \sqrt{1-\frac{x^2}{8i_b}} = \sqrt{1-\frac{1}{2}\left(\frac{x}{v_{ps}}\right)^2} = \sqrt{1-\frac{1}{2}\left(\frac{n \cdot A}{V_G - V_{T0} - n \cdot V_S}\right)^2} \quad \text{for} \quad x \le v_{ps}$$

with $v_{ps} \triangleq \dfrac{V_P - V_S}{U_T} \cong \dfrac{V_G - V_{T0} - n \cdot V_S}{n \cdot U_T}$ and $x \triangleq \dfrac{A}{U_T}$

Bias Voltage and Current (Colpitts)



- The required bias overdrive voltage v_{ps} (bias current i_b) for a given critical saturation voltage v_{pscrit} (critical current i_{bcrit}) assuming $x < v_{ps0}$ is given by

$$v_{ps} \triangleq \frac{V_P - V_S}{U_T} \cong \frac{V_G - V_{T0} - n \cdot V_S}{n U_T} = \sqrt{v_{pscrit}^2 + \frac{x^2}{2}} \quad \text{or} \quad i_b \triangleq \frac{I_b}{I_{spec}} = \left(\frac{v_{ps}}{2}\right)^2 = i_{bcrit} + \frac{x^2}{8}$$

Colpitts Oscillator Example – Bias in SI

- If we assume the transistor operates in strong inversion (with $n = 1.3$) and choose the operating saturation voltage as $V_P - V_S = 300\ mV$ and the output oscillation amplitude as $\Delta V_{out} = 200\ mV$ corresponding to an amplitude of the source voltage of $A = \Delta V_S = 100\ mV$, we can then calculate the normalized amplitude as

$$v_{ps} \triangleq \frac{V_P - V_S}{U_T} = 11.6 \quad i_b \triangleq \frac{I_b}{I_{spec}} = 33.6 \quad x \triangleq \frac{A}{U_T} = 3.86$$

- The critical overdrive and critical current can then be calculated as

$$v_{pscrit} = \sqrt{v_{ps}^2 - \frac{x^2}{2}} = 11.27 \quad \text{or} \quad i_{bcrit} = i_b - \frac{x^2}{8} = 31.7$$

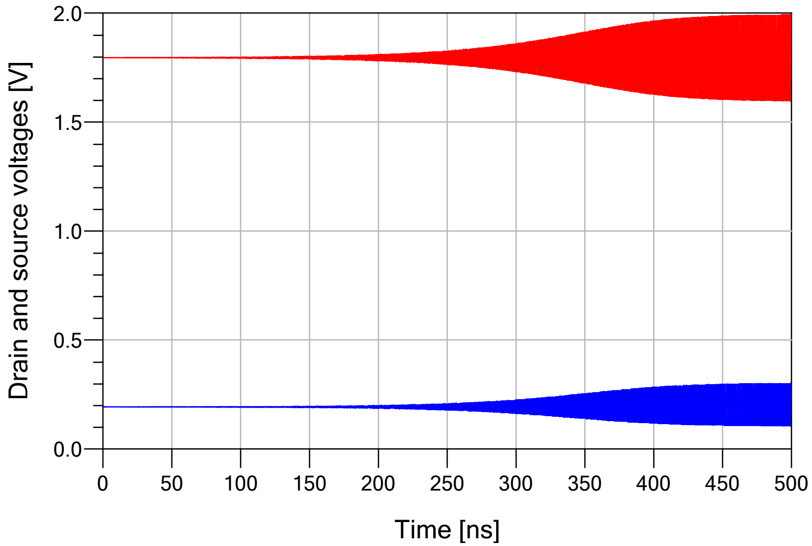
- The specific current is then obtained as

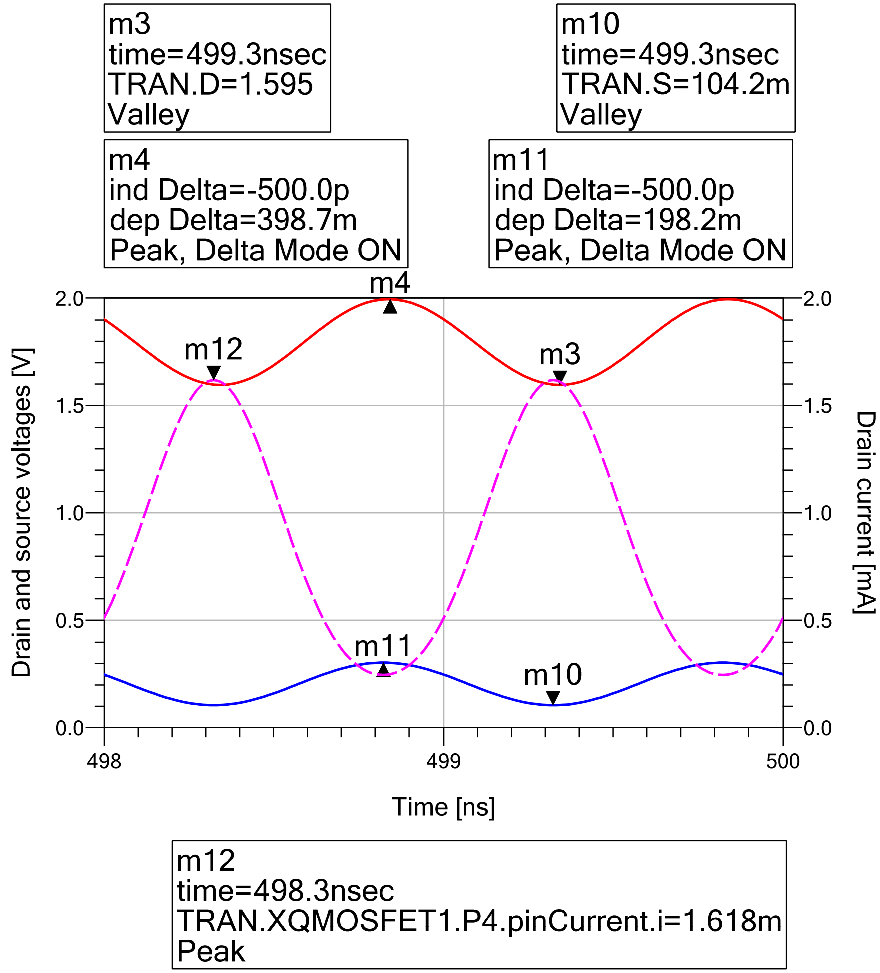
$$I_{spec} = \frac{2U_T \cdot G_{mscrit}}{v_{pscrit}} = \frac{U_T \cdot G_{mscrit}}{\sqrt{i_{bcrit}}} = 18.5\ \mu A$$

- The critical current and the required bias current are given by

$$I_{crit} = I_{spec} \cdot i_{bcrit} = 587.2\ \mu A \quad I_b = I_{spec} \cdot i_b = I_{spec} \cdot \left(\frac{v_{ps}}{2}\right)^2 = 621.7\ \mu A$$

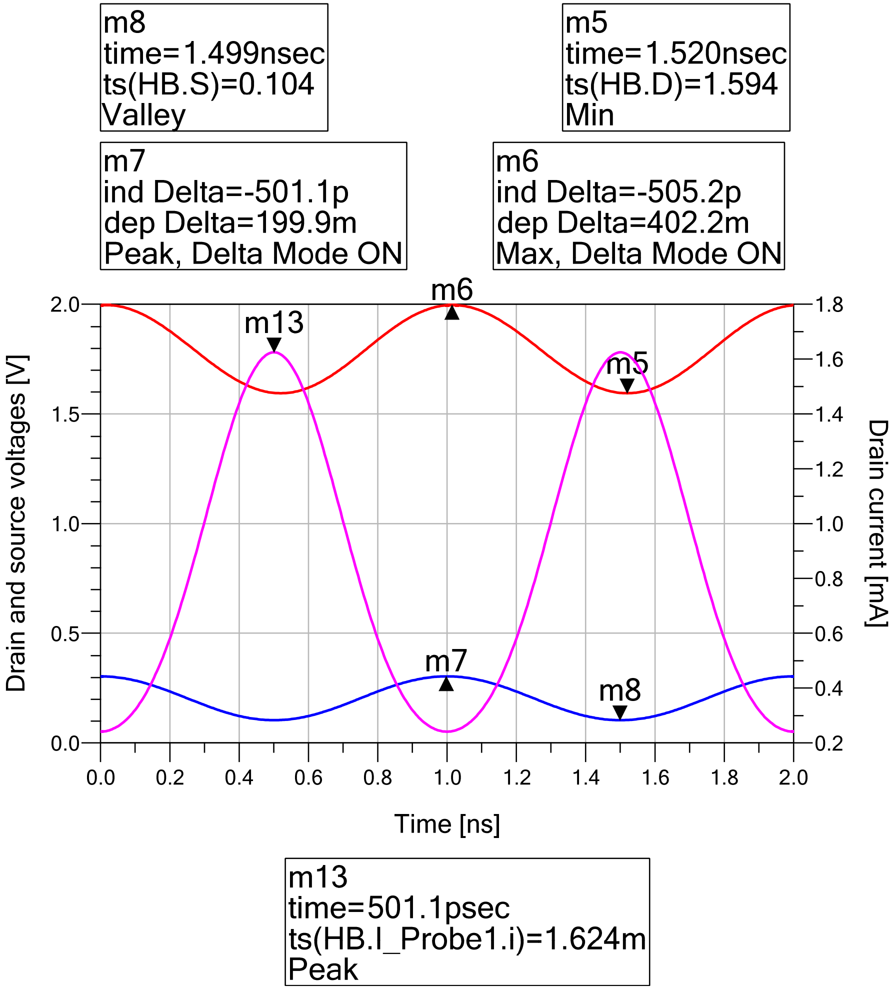
Colpitts Oscillator Example in SI – Transient Simulations

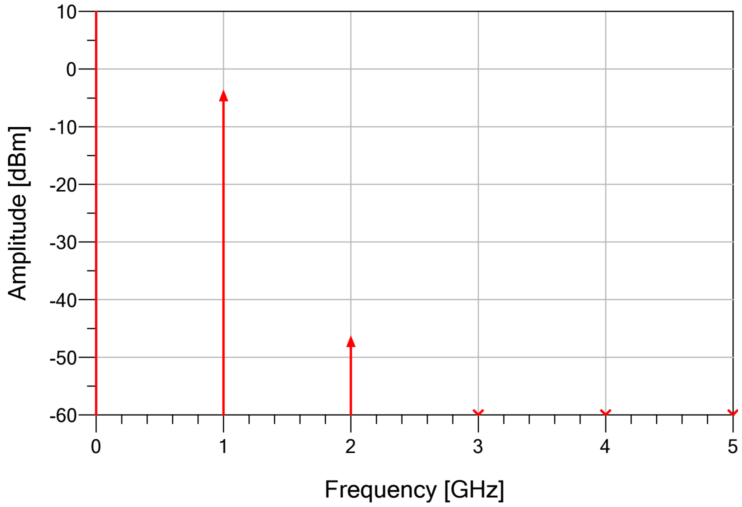




- Transient simulations performed with an ideal quadratic transconductor
- The amplitude at the source (99mV) and at the drain (199mV) are almost exactly 100mV and 200mV
- The above theory is based on the fundamental component only assuming a large Q and hence that the harmonics are negligible

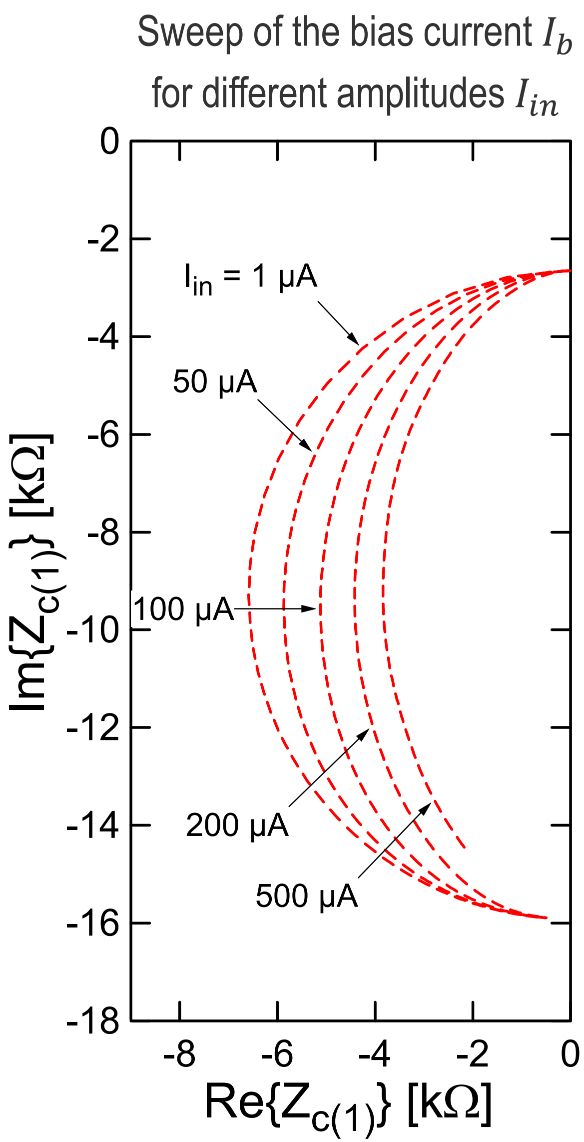
Colpitts Oscillator Example in SI – HB Simulations

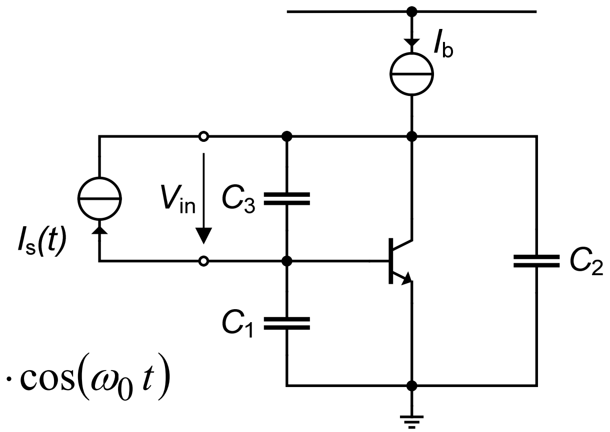




- Harmonic balance SS simulations performed with an ideal quadratic transconductor
- The amplitude at the gate is exactly 200mV, whereas the amplitude at the drain is slightly larger (235mV)

Nonlinear Effects



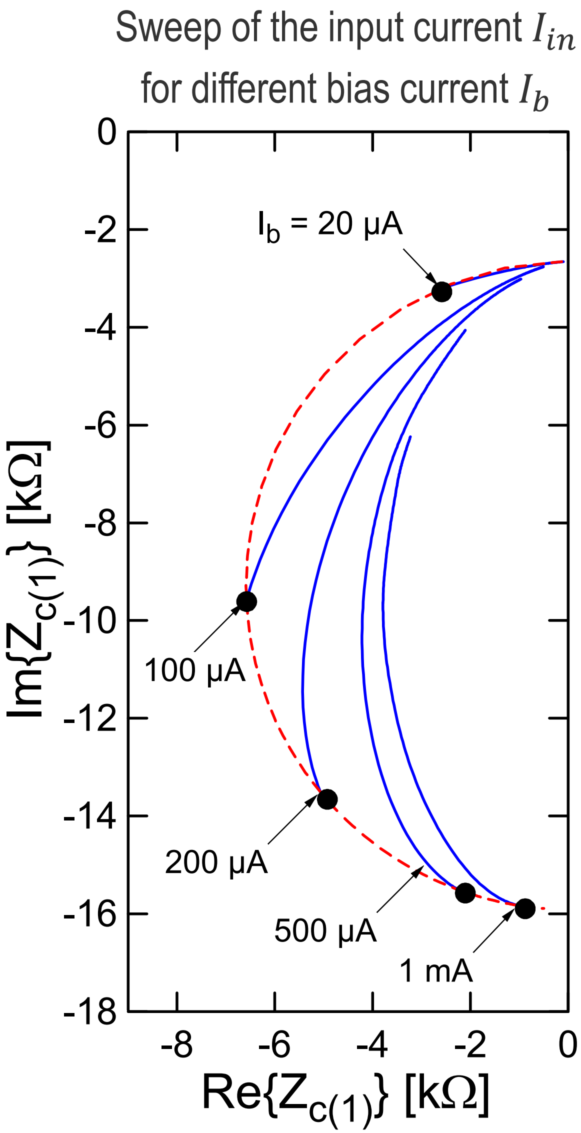


$$I_s(t) = I_{in} \cdot \cos(\omega_0 t)$$

$$Z_c = \frac{V_{in}}{I_{in}} \quad \text{and} \quad Z_{c(1)} = \frac{V_{in(1)}}{I_{in}}$$

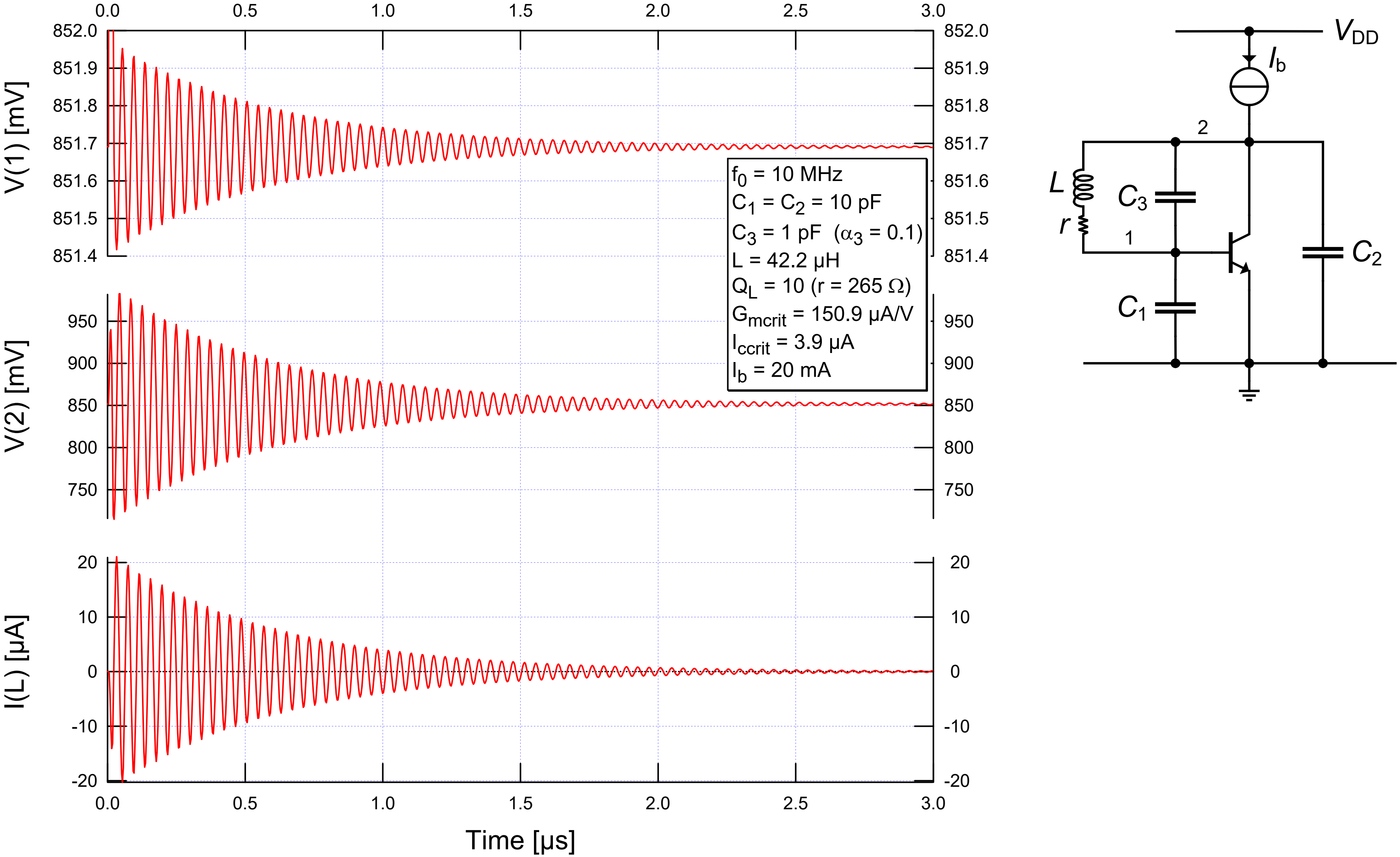
- Plot of the input impedance $Z_{in(1)}$ evaluated for the fundamental component versus the bias current I_b swept from 1 µA to 1.7 mA for different current amplitudes I_{in} (1µA, 50µA, 100µA, 200µA and 500µA)
- For small amplitudes ($I_{in} = 1\ \mu A$), we get the circle obtained from the linear analysis, but for large amplitudes, the locus starts to deviate from the circle obtained for small amplitude due to nonlinear effects

Nonlinear Effects



- Plot of the input impedance $Z_{in(1)}$ evaluated for the fundamental component versus the current amplitude I_{in} swept from 1µA to 1mA for different bias currents I_b (20µA, 100µA, 200µA, 500µA and 1mA)
- The locus always starts on the circle obtained for small amplitude with a direction tangent to the circle and then deviates from it due to nonlinear effects
- The actual operating point can be quite far from the one obtained with the linear analysis
- There may eventually be **no operating point** when the bias current becomes too large, even though the small-signal analysis would show an intersection

Unstable Point at Very Large Bias Current

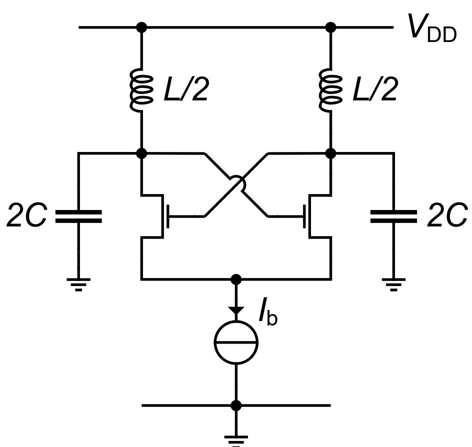


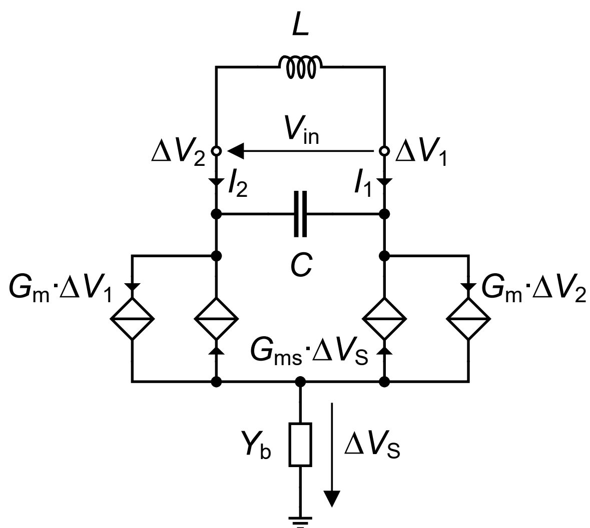
Outline

- General considerations
- The 3-points oscillator
- The cross-coupled pair oscillator

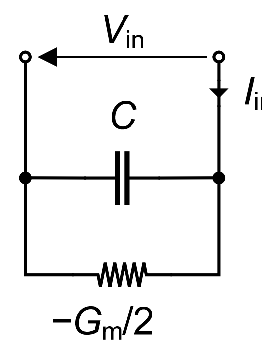
The Cross-coupled Pair Oscillator – Principle

- A balanced LO signal is most often required
- Can be generated by the cross-coupled pair oscillator shown below





$$\Delta V_S = \frac{G_m}{G_{ms} + Y_b/2} \cdot \frac{\Delta V_1 + \Delta V_2}{2}$$



- If fully balanced operation is assumed $\quad \Delta V_1 = -\Delta V_2 = \frac{V_{in}}{2} \rightarrow \Delta V_S = 0$
- And hence

$$Y_c = \frac{I_{in}}{V_{in}} = -\frac{G_m}{2} + j\omega C$$

Critical G_m

- The circuit impedance Z_c is then given by

$$Z_c = \frac{1}{Y_c} = \frac{1}{-G_m/2 + j\omega C} = -\frac{G_m/2}{(G_m/2)^2 + (\omega C)^2} - \frac{j\omega C}{(G_m/2)^2 + (\omega C)^2}$$

- And hence

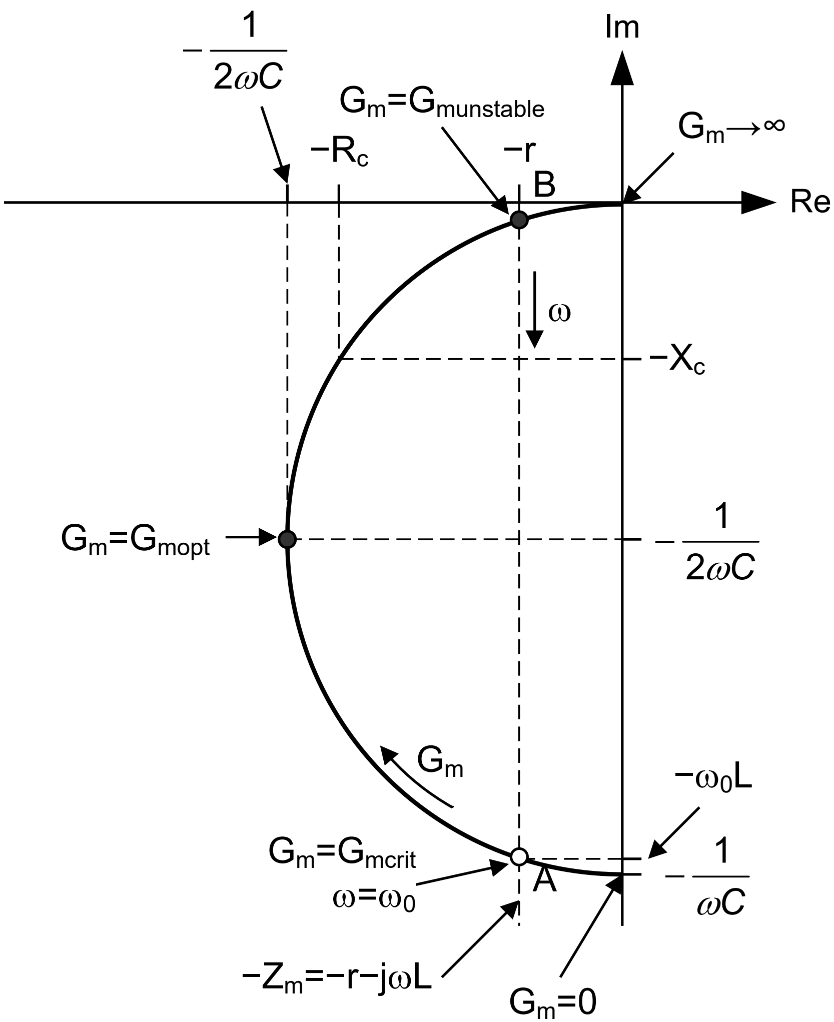
$$R_c = -\mathrm{Re}\{Z_c\} = \frac{G_m/2}{(G_m/2)^2 + (\omega C)^2} \quad \text{and} \quad X_c = -\mathrm{Im}\{Z_c\} = \frac{\omega C}{(G_m/2)^2 + (\omega C)^2}$$

- Solving for G_{mcrit} and ω_0 results in

$$\frac{X_c(\omega_0, G_{mcrit})}{R_c(\omega_0, G_{mcrit})} = Q_L \quad \Rightarrow \quad \frac{\omega_0 C}{G_{mcrit}/2} = Q_L \quad \Rightarrow \quad G_{mcrit} = \frac{2\omega_0 C}{Q_L}$$

$$X_c(\omega_0, G_{mcrit}) = \omega_0 L \quad \Rightarrow \quad \omega_0 = \frac{\omega_{LC}}{\sqrt{1 + \frac{1}{Q_L^2}}} \quad \text{with} \quad \omega_{LC} = \frac{1}{\sqrt{LC}}$$

Circuit Impedance

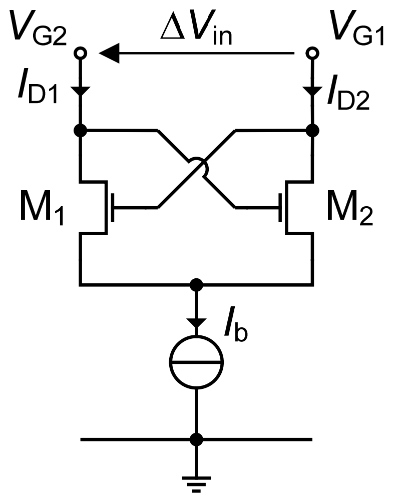


$$Z_c = \frac{-G_m/2 - j\omega C}{(G_m/2)^2 + (\omega C)^2}$$

$$G_{mcrit} = \frac{2C\omega_0}{Q_L}$$

$$\omega_0 = \frac{\omega_{LC}}{\sqrt{1+\frac{1}{Q_L^2}}} \quad \text{with} \quad \omega_{LC} = \frac{1}{\sqrt{LC}}$$

Large-signal Analysis (weak inversion)



- The differential current in WI is given by

$$\Delta I_{out} = I_{D2} - I_{D1} = -I_b \cdot \tanh\left(\frac{\Delta V_{in}}{2nU_T}\right)$$

- The output waveform for a sinusoidal differential voltage is then given by

$$\Delta V_{in}(t) = A \cdot \cos(\omega_0 t)$$

$$\Delta i_{out}(t) = \frac{\Delta I_{out}(t)}{I_b} = -\tanh\left(x \cdot \cos(\omega_0 t)\right) \quad \text{with} \quad x = \frac{A}{2nU_T}$$

- The output waveform is periodic and can be developed in a Fourier series

$$\Delta i_{out}(t) = \frac{\Delta I_{out}(t)}{I_b} = \sum_{\substack{n=1 \\ n\ odd}}^{+\infty} a_n(x) \cdot \cos(n\omega_0 t) = \sum_{\substack{n=1 \\ n\ odd}}^{+\infty} a_n(x) \cdot \cos(n\varphi)$$

with

$$a_n(x) = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \tanh\left(x \cdot \cos(\varphi)\right) \cdot \cos(n\varphi) \cdot d\varphi$$

Large-signal Analysis (weak inversion)

- We are mostly interested in the fundamental component given by

$$a_1(x) = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \tanh\left(x \cdot \cos(\varphi)\right) \cdot \cos(\varphi) \cdot d\varphi$$

- Unfortunately there is no analytical solution for this integral
- For $x \ll 1$, it can nevertheless be approximated by

$$a_1(x) \cong \frac{2\,\mathrm{I}_{B1}(x)}{\mathrm{I}_{B0}(x) + \mathrm{I}_{B2}(x)}$$

- For large signal amplitudes, the output waveform becomes a square wave and hence for $x \gg 1$ we have

$$a_1(x) \cong \frac{4}{\pi} \quad \text{for } x \gg 1$$

Transconductance for the Fundamental

- The transconductance for the fundamental component is then given by

$$G_{m(1)} = \frac{I_{D(1)}}{A} = \frac{a_1(x) \cdot I_b}{A} = \frac{I_b}{2nU_T} \cdot \frac{a_1(x)}{x} = G_m \cdot \frac{a_1(x)}{x}$$

with $G_m = \dfrac{I_b}{2nU_T}$

- Or in normalized form

$$\frac{G_{m(1)}}{G_m} = \frac{a_1(x)}{x} \cong \frac{2\,\mathrm{I}_{B1}(x)}{x \cdot \left(\mathrm{I}_{B0}(x) + \mathrm{I}_{B2}(x) \right)}$$

Transconductance for the Fundamental