Chapter 4

VOTING PROCEDURES

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Contents

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Abstract

Voting procedures focus on the aggregation of individuals' preferences to produce collective decisions. In practice, a voting procedure is characterized by ballot responses and the way ballots are tallied to determine winners. Voters are assumed to have clear preferences over candidates and attempt to maximize satisfaction with the election outcome by their ballot responses. Such responses can include strategic misrepresentation of preferences.

Voting procedures are formalized by social choice functions, which map ballot response profiles into election outcomes. We discuss broad classes of social choice functions as well as special cases such as plurality rule, approval voting, and Borda's point-count method. The simplest class is voting procedures for two-candidate elections. Conditions for social choice functions are presented for simple majority rule, the class of weighted majority rules, and for what are referred to as hierarchical representative systems.

The second main class, which predominates in the literature, embraces all procedures for electing one candidate from three or more contenders. The multicandidate elect-one social choice functions in this broad class are divided into nonranked one-stage procedures, nonranked multistage procedures, ranked voting methods, and positional scoring rules. Nonranked methods include plurality check-one voting and approval voting, where each voter casts either no vote or a full vote for each candidate. On ballots for positional scoring methods, voters rank candidates from most preferred to least preferred. Topics for multicandidate methods include axiomatic characterizations, susceptibility to strategic manipulation, and voting paradoxes that expose questionable aspects of particular procedures.

Other social choice functions are designed to elect two or more candidates for committee memberships from a slate of contenders. Proportional representation methods, including systems that elect members sequentially from a single ranked ballot with vote transfers in successive counting stages, are primary examples of this class.

Keywords

voting systems, voting paradoxes, social choice, Condorcet candidate, proportional representation

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1. Introduction

Voting procedures describe the manner in which the preferences of individuals are aggregated to produce a collective decision. The individuals, whom we refer to as *voters*, might be registered voters, legislators, trustees, committee members, jurors, or members of some other body whose decisions not only are binding on their members but often a larger community that the body represents. The alternatives among which the voters choose will be referred to as *candidates*. Depending on the context, candidates might be people running for office, passage or defeat of a bill, alternative budgets, applicants for a faculty position, or jury verdicts that a judge permits. Although we refer throughout the chapter to voters and candidates, it should be clear that voting procedures cover a multitude of voting situations that are often described in other ways.

As the term is used in this chapter, a *voting procedure* is defined by two characteristics. The first is the type of vote, or *ballot*, that is recognized as admissible by the procedure. This could range from an open show of hands in an assembly to an anonymous best-to-worst ranking of all the candidates by the voters (a secret ballot). We denote by $\mathcal{B}(A)$ the *set of admissible ballots* for a given procedure in which A is the set of feasible candidates. Given A, it is assumed that every voter selects a member of $\mathcal{B}(A)$ as his or her vote, or ballot. When there are $n \geqslant 2$ voters indexed by $i = 1, 2, \ldots, n$, and voter i selects $d_i \in \mathcal{B}(A)$, the n-tuple $d = (d_1, d_2, \ldots, d_n)$ is the *ballot response profile*. If there are no restrictions on voting patterns, any n-tuple in $\mathcal{B}(A)^n$ might occur as the ballot response profile. If each voter can vote for only one candidate, for example, a ballot response profile would indicate the candidate for whom each of the n voters voted.

The second defining characteristic of a voting procedure is how votes are counted to determine a winner or winners. For this purpose, we need a concrete rule that aggregates the individual responses in a ballot response profile into a collective choice or measure (possibly numerical). The criterion by which an outcome is chosen depends on the collective measure. The rule that whichever of two candidates obtains a simple majority is a familiar voting procedure. Less familiar is a procedure that involves successive elimination of lowest-vote candidates and transfers of their votes to candidates that remain in contention, but it is one that is used in both public and private elections.

Several important topics are not discussed in this chapter. One is the determination of eligible voters, which may involve registration, committee membership, or random choice, as in some jury-selection procedures. Another is the determination of feasible candidates or official nominees. We will also not discuss agenda formation, use of polls, campaign finance, or ballot-stuffing and other forms of election fraud. In addition, strategizing by candidates in campaigns, which can be influenced by the voting procedure, will not be treated here; it is discussed in Brams and Davis (1973, 1974, 1982), Lake (1979), Enelow and Hinich (1984, 1990), Cox (1984, 1987a,b, 1997), Snyder (1989), Riker (1986, 1996), Coughlin (1992) and Myerson (1995a), among other places.

On the other hand, we will pay attention to strategies that *voters* employ in making ballot choices. Their strategic voting can substantially affect election outcomes and, on occasion, subvert the intention of a voting procedure to treat voters and candidates fairly. The topic of voter strategy, which is also discussed in Chapters 6, 10 and 11 in this Volume, and Chapters 15, 23 and 25 in Volume 2 of this Handbook, is often tied to voter preferences, and we will make that tie-in here.

In the next section, we present our basic assumptions about voter preferences over the candidates. We define a voter preference profile as an n-tuple of voter preference orders indexed in the same way as the ballots in a ballot response profile. Because the structures of ballots and preference orders can be quite dissimilar, we presume no obvious connection between them. This may even be true when the ballot set $\mathcal{B}(A)$ is the same as the set of preference orders on A, because some voters might cast ballots different from their true preference orders in order to secure an outcome they prefer to that produced by sincere or honest voting.

The theme of strategic voting is a theme of individual or subgroup choice within a process of group choice. Roughly speaking, if a voting procedure sometimes allows a voter to secure a preferred outcome by voting in a way that, in isolation, clearly contradicts his or her true preferences, the procedure is said to be *susceptible to strategic manipulation*. A voting procedure that is not susceptible to strategic manipulation is said to be *strategyproof*. Common voting procedures used in selecting between two candidates are essentially strategyproof, but most procedures involving three or more candidates are not. The following example, which is motivated by Black (1958) and Farquharson (1969), illustrates the latter point. The approach to strategic voting discussed here was pioneered by Farquharson (1969).

Example 1.1. We consider a common legislative voting process in which m candidates, ordered as $a_1a_2 \cdots a_m$, are voted on in a succession of $m-1$ pairwise simple majority votes. The first vote is between a_1 and a_2. For $j \geqslant 1$, the $j+1$st vote is between a_{j+1} and the winner of the jth vote. The winner of the final vote is the overall winner. It is often true that a candidate's chances of being the overall winner increase the later it enters the process, for it then has to defeat fewer other candidates to emerge victorious. There are exceptions, however. Consider three candidates, a, b, and c, and three voters, 1, 2, and 3, who have the voter preference profile

$$(abc, cab, bca).$$

This signifies that 1 prefers a to b to c, 2 prefers c to a to b, and 3 prefers b to c to a. Assume that all voters know one another's preferences and that successive majority voting applies with voting order abc (or, equivalently, bac). The first vote is between a and b:

(i) if a wins, then c wins the second vote because 2 and 3 will vote for c over a;

(ii) if b wins, then b wins the second vote because 1 and 3 will vote for b over c. Because b or c is the overall winner, and voters 1 and 3 prefer b to c, it is in their interests to prevent (i) by ensuring that b defeats a on the first vote. Consequently, both 1 and 3 will vote for b on the first vote, even though 1 prefers a to b. In effect, voter 1 manipulates the procedure by voting strategically – in apparent contradiction to his or her true preferences – to secure a preferred outcome. The vote by 1 for b on the first vote, and for his or her preferred candidate on the second vote, is an example of what Farquharson (1969) calls a sophisticated strategy. Similar reasoning shows that a wins if the voting order is *cab* (or *acb*), and c wins if the voting order is *bca* (or *cba*). Whichever order is used, the last candidate is a sure loser under strategic voting. ■

In Example 1.1 with voting order *abc*, a ballot in $\mathcal{B}(\{a,b,c\})$ can be defined as a triple (α, β, γ) in which α denotes a first vote (a, b, or ϕ, where ϕ denotes abstention), $\beta \in \{a, c, \phi\}$ for the second vote when a wins the first vote, and $\gamma \in \{b, c, \phi\}$ when b wins the first vote. The aggregation rule tallies the votes in each position and specifies the winner by simple majority comparisons. We have ignored tied-vote outcomes, which can be factored into the decision criterion if a tie-breaking rule is adopted.

After we discuss individual preferences in the next section, we formalize the notion of an aggregation rule and its decision criterion by defining a social choice function as a mapping from ballot response profiles to subsets of feasible candidates. Particular types of social choice functions are examined in ensuing sections. Section 3 focuses on binary or two-candidate voting procedures, Sections 5 through 9 discuss social choice functions for elections of one candidate from among three or more contenders, and Sections 10 and 11 consider procedures for electing two or more candidates.

Section 4 provides further introduction to elections among three or more candidates. We will see that there are deep mathematical results behind the enduring fascination with multiple-candidate voting procedures. Briefly put, while there are innumerable voting procedures for such elections, all are flawed. The difficulties arise from two observations. The first, due to Arrow (1950, 1951), is that no social choice function for three or more candidates simultaneously satisfies a few conditions that can be viewed as desirable properties of such functions. The second, due to Gibbard (1973) and Satterthwaite (1975), says that all reasonable voting procedures for three or more candidates are susceptible to strategic manipulation. More recently, Saari (2001b), Chapter 25 in Volume 2 of this Handbook, argues that elections can be "chaotic."

The effects of these results on the theory of voting have parallels to the effects on physics of Heisenberg's uncertainty principle and the effects on logic and the foundations of mathematics of Gödel's incompleteness and undecidability results. The challenge is not to design a perfect voting system, which is impossible, but rather to identify those procedures that reflect the desires of voters in as faithful a manner as possible. Among other things, we would like a voting procedure to encourage sincere balloting (based on true preferences), be relatively immune to strategic manipulation, and avoid egregious anomalies or paradoxes, such as the negative responsiveness paradox that occurs when increased support for a candidate turns it from a winner into a loser. Our analyses of voting procedures in later sections will pay close attention to these matters.

In doing this, we follow a tradition that goes back to the development of procedures for conducting democratic elections in ancient Greece and Rome [Stavely (1972)]. Among others, Aristotle, in his *Politics*, gave considerable attention to better and worse forms of government, including representative democracies.

Many centuries later, two Frenchmen, Jean-Charles de Borda and the Marquis de Condorcet, argued on a more modest level for rather different election rules that still bear their names. Their differences are recounted by Young (1988, 1995), Moulin (1988a) and Saari (1995a). Borda (1781) [translated by de Grazia (1953)], for example, advocated the ranked voting procedure in which ballots are complete rankings of the m candidates, and $m-1, m-2, \ldots, 1, 0$ points are awarded to the best-to-worst candidates on each ballot. The candidate with the greatest point total wins. When it was pointed out to Borda that his procedure is quite susceptible to strategic manipulation, he is said to have replied that his scheme was intended only for honest men [Black (1958, p. 238)].

Condorcet (1785) took the position that if, based on ranked ballots, one candidate would defeat every other candidate in pairwise simple majority comparisons, this candidate, called the *majority* or *Condorcet candidate*, should be elected. He showed not only that Borda's scheme can violate this rule but also that there are ballot response profiles with a majority candidate who would not be elected by *any* point-assignment method that awards more points to a top-ranked candidate than a second-ranked candidate, more points to a second-ranked than a third-ranked candidate, and so forth (Section 9.3). Indeed, Condorcet pointed out that there may be no majority candidate, as in (abc, cab, bca) of Example 1.1, but he was unclear about how to proceed when this occurs. The nonexistence of a majority candidate has come to be known as Condorcet's paradox, the paradox of cyclical majorities, and *the* paradox of voting.

The writings of Borda and Condorcet initiated a huge literature on voting procedures [McLean and Urken (1995)]. Prominent nineteenth century examples include Nanson's (1883) extensive review of voting procedures and Hare's (1861) book on the election of representative legislatures. Hare's system, which is more widely known as the method of *single transferable vote* (STV) or *instant runoff* (in Australia, it is known as the *alternative vote*), was proposed as a way to ensure the representation of significant minorities. With various modifications, STV has been adopted throughout the world to elect public officials and representative assemblies, but in some jurisdictions it has been abandoned, including several cities in the USA. We discuss it further in Section 11.

Two of the most significant developments of the twentieth century are Arrow's (1950, 1951) celebrated "impossibility theorem" and the analysis of elections and voting procedures by methods of game theory, as represented in Farquharson (1969), Peleg (1984), Brams (1985), Moulin (1988a, 1994), Coughlin (1992), and chapters in this Handbook that deal with strategic voting.

The discussion of voting procedures that follows is indebted to a host of predecessors, including many cited above. We assume throughout that voters act to maximize the satisfaction of their preferences, subject to the rules of voting and what voters believe about other voters' preferences and likely behaviors. In analyzing voting procedures, we will pay special attention to both their successes and failures in producing social choices that are responsive to the preferences of voters.

2. Voter preferences and social choice functions

We presume that there are $n \geqslant 2$ voters, indexed by $i = 1, 2, \ldots, n$, and a set X of two or more candidates. Unless we note otherwise, X is taken to be finite with cardinality $|X|$, and voter i is assumed to have a preference *weak order* $\succsim_i$ on X, $i = 1, 2, \ldots, n$, so that $\succsim_i$ is *transitive* ($x \succsim_i y$ and $y \succsim_i z$ imply $x \succsim_i z$) and *complete* ($x \succsim_i y$ or $y \succsim_i x$ for all $x, y \in X$). The interpretation of $x \succsim_i y$ is that voter i prefers x to y or is indifferent between x and y. We denote the asymmetric (strict preference) part of $\succsim_i$ by $\succ_i$, and the symmetric (indifferent) part of $\succsim_i$ by $\sim_i$:

$$x \succ_i y \text{ if } x \succsim_i y \text{ and not } (y \succsim_i x),$$
$$x \sim_i y \text{ if } x \succsim_i y \text{ and } \quad y \succsim_i x.$$

It follows from weak order that $x \sim_i x$, $(x \succsim_i y, y \succ_i z) \Rightarrow x \succ_i z$, $(x \succ_i y, y \succsim_i z) \Rightarrow x \succ_i z$, and that exactly one of $x \succ_i y$, $y \succ_i x$, and $x \sim_i y$ holds when $x \neq y$. Moreover, the indifference relation $\sim_i$ on X is an *equivalence relation* (reflexive: $x \sim_i x$; symmetric: $x \sim_i y \Rightarrow y \sim_i x$; transitive: $x \sim_i y$ and $y \sim_i z$ imply $x \sim_i z$) that partitions X into r indifference classes $X_1, X_2, \ldots, X_r$ such that $\sim_i$ holds within each X_j and $x \succ_i y$ or $y \succ_i x$ whenever x and y are in different classes. The classes can be ordered by preference as $X_1 \succ_i X_2 \succ_i \cdots \succ_i X_r$, where $A \succ_i B$ means that $a \succ_i b$ for all $(a, b) \in A \times B$.

When $x \sim_i y \Leftrightarrow x = y$, for all $x, y \in X$, we refer to $\succsim_i$ or its asymmetric part $\succ_i$ as a *linear order* or *strict ranking* and abbreviate $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_m$ as $x_1 x_2 \cdots x_m$ (with i implicit or explicit). The three-candidate set $X = \{a, b, c\}$ with $|X| = 3$ admits 13 weak orders, including $a \sim b \sim c \sim a$, of which six are linear orders, namely *abc*, *acb*, *bac*, *bca*, *cab* and *cba*.

An n-tuple $v = (\succsim_1, \succsim_2, \ldots, \succsim_n)$ of weak orders on X, one for each voter, is a *voter preference profile*. We let V denote the nonempty set of voter preference profiles that are considered as possible voter preference profiles in a particular situation. If $|X| = 3$ and there are no restrictions on V apart from weak order, then $|V| = 13^n$; if all voter preference relations are assumed to be strict rankings, then $|V| = 6^n$. If $X = \{a, b, c\}$, if voter preference relations are strict rankings, and if preferences are *single-peaked* in the order *abc* (so that b is never least preferred), then $|V| = 4^n$.

Three inputs determine the domain of a social choice function. The first is the number $k \geqslant 1$ of candidates to be chosen by a voting procedure. The second is a nonempty set $\mathcal{X}$ of subsets of X, each of which might arise as the feasible set of candidates or the official set of nominees. We require $|A| \geqslant k$ for every $A \in \mathcal{X}$. The third input is the set D_A of ballot response profiles that can occur for each $A \in \mathcal{X}$. Each member of D_A is a ballot response profile $d = (d_1, d_2, \ldots, d_n)$, where $d_i \in \mathcal{B}(A)$ for each voter and $\mathcal{B}(A)$ is the set of admissible ballots. The *domain* of a social choice function is the set

$$\mathcal{D} = \bigcup_{A \in \mathcal{X}} \{(A, d) \colon d \in D_A\}$$

of all ordered pairs (A, d) of a set A of feasible candidates and a ballot response profile for that set.

A *social choice function* is a mapping F from a domain $\mathcal{D}$ into the family 2^X of subsets of X such that, for all $(A, d) \in \mathcal{D}$,

$$F(A, d) \subseteq A \quad \text{and} \quad |F(A, d)| \geqslant k.$$

We interpret $F(A, d)$ as the subset of feasible candidates chosen by the voting procedure for situation (A, d). It is assumed to contain at least k candidates. When $|F(A, d)| = k$ for all $(A, d) \in \mathcal{D}$, we say that F is *decisive*. By not imposing decisiveness, we allow for the possibility that a choice set contains more candidates than the precise number to be elected, i.e., we admit the possibility of unresolved ties. This may be inimical to practical necessity, but it has the technical advantage of sidestepping issues of tie-breaking procedures.

We now define several conditions for social choice functions that will be involved in discussions of specific voting procedures. We begin with domain aspects and then consider anonymity, neutrality, and Pareto-dominance properties.

Given k as defined above, we refer to F as a *choose-k* social choice function. We focus on choose-1 procedures until the final two sections.

The cardinality of feasible candidate sets might also be fixed, and when $|A| = m$ for all $A \in X$, we say that F is an *m-ary* social choice function. When F is m-ary, we assume that $m > k$. A voting procedure designed to choose two of five nominees is a 5-ary choose-2 procedure.

Most procedures have the same ballot-set structure for all $A \in X$ of the same cardinality. When this is true, we say that the ballot sets are *similar*. More precisely, given $A, B \in \mathcal{X}$ for which $|A| = |B|$, a ballot $d_0 \in \mathcal{B}(A)$, and a one-to-one mapping σ from A onto B, we denote by $\sigma(d_0)$ the action of σ on d_0 caused, in effect, by replacing every instance of a in d_0 by $\sigma(a)$, for all $a \in A$. For example, if $A = \{a, b, c\}$, $B = \{a, b, x\}$, $\sigma(a) = b$, $\sigma(b) = x$ and $\sigma(c) = a$, and if d_0 is the strict ranking bac of A, we have $\sigma(d_0) = xba$. Then the ballot sets are *similar* if

$$\mathcal{B}(B) = \{\sigma(d_0) \colon d_0 \in \mathcal{B}(A)\}$$

for all $A, B \in \mathcal{X}$ for which $|A| = |B|$ and all bijections σ from A onto B. Note that this definition is nonvacuous even when $\mathcal{X}$ contains only one set, say $\mathcal{X} = \{A\}$. It says, for example, that if $A = \{a, b, c\}$ and if abc is a ballot, then all permutations of abc are ballots. A situation in which similarity fails occurs when all ballots are weak orders or linear orders that are single-peaked with respect to a fixed left-to-right ordering of the candidates.

Along with similar ballot sets, most voting procedures of practical interest allow each voter independently to select any member of $\mathcal{B}(A)$ as his or her ballot, in which case $D_A = \mathcal{B}(A)^n$. We will assume this in all voting procedures discussed later, and in the rest of the present section.

One implication of unrestricted ballot response profiles is that if $d = (d_1, \ldots, d_n)$ is in $\mathcal{B}(A)^n = D_A$, and if ρ is any one-to-one mapping from $\{1, \ldots, n\}$ onto $\{1, \ldots, n\}$, then ρd, defined as $(d_{\rho(1)}, \ldots, d_{\rho(n)})$, is also in D_A. We say that F is *anonymous* if

$$F(A, \rho d) = F(A, d)$$

for all $(A, d) \in \mathcal{D}$ and all ρ from $\{1, \ldots, n\}$ onto $\{1, \ldots, n\}$. Anonymity says that the social choice set is invariant to permutations of ballots among voters, and it therefore embodies the notion that all voters are treated equally. It fails if there is a tie-breaking chairperson or if some voters' ballots are weighted more heavily than others.

The term *neutrality* is used to convey the notion that all feasible *candidates* are treated equally. Assume that ballot sets are similar. Given $A, B \in X$ with $|A| = |B|$ and a mapping σ from A onto B, let $\sigma(d) = (\sigma(d_1), \ldots, \sigma(d_n))$ for $d \in D_A$, and let $\sigma(A') = \{\sigma(a): a \in A'\}$ for every $A' \subseteq A$. Then $\sigma(d) \in D_B$ and $\sigma(A') \subseteq B$. We say that F is *locally neutral* if

$$F(A, \sigma(d)) = \sigma(F(A, d))$$

for all $(A, d) \in \mathcal{D}$ and all permutations σ on A, and that it is *globally neutral*, or simple *neutral*, if

$$F(B, \sigma(d)) = \sigma(F(A, d))$$

for all $A, B \in \mathcal{X}$ for which $|A| = |B|$, all $d \in D_A$, and all σ from A onto B. Neutrality says that if the ballot response profile for B is obtained from the ballot response profile for A under the action of σ, then a is in A's choice set if and only if $\sigma(a)$ is in B's choice set. It fails for the binary voting procedure in which a challenger needs a two-thirds majority to replace an incumbent. Because the challenger has a bigger hurdle in replacing the incumbent than the incumbent has in staying in office, they are not treated equally.

Pareto dominance is usually defined with respect to voters' true preferences. We consider an alternative definition based on ballot response profiles. To do this, it is necessary to have a sense in which a ballot reveals that a voter prefers a to b even when, because of strategic voting, the voter may actually prefer b to a. For example, if a voter votes for a but not b on a nonranked ballot, or ranks a ahead of b on a ranked ballot, we may say that the voter *expresses* a preference for a over b. We denote this by ad_ib for voter i, and for $a, b \in A$ and $d \in D_A$ write adb if ad_ib for $i = 1, \ldots, n$. Because adb indicates unanimous preference for a over b, we consider the following dominance condition for choose-1 voting procedures:

$$\{a, b \in A, (A, d) \in \mathcal{D}, adb\} \Rightarrow b \notin F(A, d). \tag{2.1}$$

A similar condition for $k \geqslant 2$ may be inconsistent with $|F(A, d)| \geqslant k$. We can, however, require that $a \in F(A, d)$ when adb and $b \in F(A, d)$, so that if a dominated candidate is in the choice set, then every feasible candidate that dominates it is also in the choice set. This reduces to the former choose-1 condition for $k = 1$ when F is decisive. We will see in Section 7.3 that it is violated by the type of multistage voting procedure defined in Example 1.1.

3. Voting procedures for two candidates

We assume throughout this section that F is a binary choose-1 voting procedure with $X = \{a, b\}$ and $\mathcal{X} = \{X\}$. For algebraic convenience, we encode the nonempty subsets of X as follows:

1 signifies $\{a\}$
0 signifies $\{a, b\}$
-1 signifies $\{b\}$.

We take $\mathcal{B} = \{1, 0, -1\}$ and $D = \{1, 0, -1\}^n$, where 1, 0 and -1 denote a vote for a, an abstention (or a vote for both a and b), and a vote for b, respectively. The social choice function F, with $F(d) = F(\{a, b\}, d)$, maps D into $\{1, 0, -1\}$, where $F(d)$ equals 1, 0, and -1 according to whether a wins, a and b tie, and b wins, respectively.

The convenience of our encoding is seen in part by expressions of potential properties for F. Using terms defined in the preceding section, F is *decisive* if $F(d)$ never equals 0, F is *anonymous* if $F(d) = F(\rho d)$ for all $d \in D$ and mappings ρ from $\{1, \ldots, n\}$ onto $\{1, \ldots, n\}$, F is *neutral* if $F(-d) = -F(d)$ for all $d \in D$, and F satisfies the dominance condition (2.1) if $F(j, \ldots, j) = j$ for $j \in \{1, -1\}$.

Let $d \geqslant d'$ mean that $d_i \geqslant d'_i$ for all i, and let $d > d'$ mean that $d \geqslant d'$ and not $(d' \geqslant d)$. We then say that F is *monotonic* if

$$d \geqslant d' \Rightarrow F(d) \geqslant F(d') \quad \text{for all} \quad d, d' \in D,$$

and that F is *strongly monotonic* if it is monotonic and, for all $d > d'$ in D,

$$F(d') = 0 \Rightarrow F(d) = 1; \qquad F(d) = 0 \Rightarrow F(d') = -1.$$

Monotonicity conditions are often referred to as *nonnegative* or *positive responsiveness* conditions. They formalize the crucial idea for voting procedures that increased support for a candidate never hurts, and may help it to win.

The overwhelmingly most common binary choose-1 voting procedure is simple majority, with or without a tie-breaking rule. Let s: $\mathbb{R} \to \{1, 0, -1\}$ be the *sign function* defined by

$$s(r) = 1 \quad \text{if} \quad r > 0, \qquad s(0) = 0, \qquad s(r) = -1 \quad \text{if} \quad r < 0.$$

The *simple majority function* is defined by

$$F(d) = s\left(\sum_{i=1}^{n} d_i\right) \quad \text{for all} \quad d \in D.$$

If F is to be decisive and all tied votes are broken in a's favor, we can take

$$F(d) = s\left(\sum_{i=1}^{n} d_i + \tfrac{1}{2}\right) \quad \text{for all} \quad d \in D.$$

This violates neutrality because neutrality implies $F(0, \ldots, 0) = 0$ so long as abstentions are allowed.

May (1952) axiomatized the simple majority function in one of the earliest characterizations of a voting procedure by properties it possesses:

Theorem 3.1. *F is the simple majority function if and only if it is anonymous, neutral, and strongly monotonic.*

We next consider other binary procedures that relax one or more of May's conditions. Monotonicity, but not necessarily strong monotonicity, will apply throughout.

The most common voting procedures that violate anonymity are weighted majority functions. Let $w_i \geqslant 0$ denote a *weight for voter i*, let $W = \{w = (w_1, \ldots, w_n)\colon w_i \geqslant 0$ for all i and $\sum w_i > 0\}$, and denote by $\langle w, d \rangle$ the inner product $\sum w_i d_i$ of w and d. We say that F is a *weighted majority function* if there is a $w \in W$ such that

$$F(d) = s(\langle w, d \rangle) \quad \text{for all} \quad d \in D.$$

Simple majority, with $w_1 = w_2 = \cdots = w_n$, is the only anonymous weighted majority function. Simple majority with a tie-breaking chairperson, voter 1, can be characterized by $w = (3, 2, 2, \ldots, 2)$.

The usual setting for weighted majority is a representative body in which voters have different sized constituencies. A common concern for such bodies, which is intimately connected to notions of voting power and fair representation [Banzhaf (1965), Shapley and Shubik (1954), Dubey and Shapley (1979), Balinski and Young (1982, 2001), Felsenthal and Machover (1998), Chapter 8 in this Volume, Chapter 26 in Volume 2 of this Handbook)] and has precipitated many court cases, is what weights to assign voters.

Neutrality says that if d^1, $d^2 \in D$ and $d^1 + d^2 = (0, \ldots, 0)$, then $F(d^1) = 1 \Leftrightarrow F(d^2) = -1$. We extend this for weighted majority by saying that F is *strongly neutral* if for all $m \geqslant 2$ and $d^1, \ldots, d^m \in D$ for which $d^1 + \cdots + d^m = (0, \ldots, 0)$,

$$F(d^j) = 1 \quad \text{for some} \quad j \leqslant m \Leftrightarrow F(d^k) = -1 \quad \text{for some} \quad k \leqslant m.$$

The following characterization is from Fishburn (1973):

Theorem 3.2. *F is a weighted majority function if and only if it is monotonic, strongly neutral, and satisfies condition (2.1).*

The most common violators of neutrality are special majorities. An example is the decisive two-thirds majority rule defined by

$$F(d) = s\left(|\{i\colon d_i = 1\}| - \tfrac{2}{3}n + \tfrac{1}{4}\right),$$

where the challenger a wins if and only if more than two-thirds of the voters vote for a.

A generalization of weighted majority functions was introduced by Murakami (1966, 1968) and further studied by Fishburn (1971, 1973, 1979b), Fine (1972) and Keiding (1984). Fishburn refers to these procedures as *representative systems*. Beginning from the projections $p_i(d) = d_i$, the family of representative systems is defined recursively by nested hierarchies of weighted majorities under the s operator. With $s(r_1, r_2, \ldots, r_m) = s(\sum r_j)$, an example is

$$F(d) = s(s(d_1, \ldots, d_q), \quad s(d_{q+1}, \ldots, d_t), \quad s(d_{t+1}, \ldots, d_n)).$$

This characterizes a tricameral legislature in which each house votes by simple majority. The overall winner is determined by simple majority applied to the vote outcomes of the houses.

To be more precise, let $\mathcal{R}_0 = \{d_1, \ldots, d_n\}$ and for each positive integer t let

$$\mathcal{R}_t = \{s(f_1, \ldots, f_K)\colon K \in \{1, 2, \ldots\} \quad \text{and} \quad f_1, \ldots, f_K \in \mathcal{R}_{t-1}\}.$$

It is easily checked that $\mathcal{R}_1$ is the set of weighted majority functions and, because $s(f) = f$, we have $\mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \cdots$ We refer to $\mathcal{R} = \cup \mathcal{R}_i$ as the set of *representative systems* for n voters.

Complete characterizations of representative systems first appeared in Fishburn (1971) and Fine (1972). A key aspect of their characterizations is a relaxation of strong neutrality that we refer to as the dual partition condition. We say that voter i is *essential* if there is some $(d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n) \in \{1, 0, -1\}^{n-1}$ for which it is false that $F(d_1, \ldots, d_{i-1}, x, d_{i+1}, \ldots, d_n)$ is the same for each x in $\{1, 0, -1\}$. Then F satisfies the *dual partition condition* if $F(d^k) = 1$ for some $k \in \{1, \ldots, m\}$ whenever m is an odd positive integer,

$$d^1 > -d^2, \quad d^2 > -d^3, \ldots, d^{m-1} > -d^m, \quad d^m > -d^1,$$

and $\sum_{k=1}^{m} d_i^k > 0$ for some essential i.

Theorem 3.3. *F is a representative system if and only if it is monotonic, neutral and satisfies condition (2.1) and the dual partition condition.*

A proof of Theorem 3.3 is included in Fishburn (1973, Chapter 4).

Further studies of representative systems have focused on the number of hierarchical levels needed to express all $F \in \mathcal{R}$ for n voters. Let $\mu(n)$ denote the smallest t for n voters such that $\mathcal{R}_t = \mathcal{R}$. Fishburn (1979b) notes that $\mu(n) = n - 1$ for $1 \leqslant n \leqslant 4$, $\mu(5) = \mu(6) = 4$, $\mu(n) \leqslant n - 2$ for all $n \geqslant 6$, and μ is unbounded. He also conjectured that $\mu(n)/n \to 0$ as n gets large. Keiding (1984) confirmed the conjecture by proving that $\mu(n) \leqslant \log_2(n(n-1)) + 5$.

4. Introduction to voting procedures for three or more candidates

The monotonic binary voting procedures of the preceding section are strategyproof, because a voter can never help elect a preferred candidate by voting contrary to his or her true preferences. A very different picture of strategic voting emerges when there are three or more feasible candidates. While virtually all voting procedures for multicandidate elections are susceptible to strategic manipulation, we will see in ensuing sections that some are more manipulable than others.

In this section we indicate why elections with three or more candidates can be problematical. We begin with issues familiar to Borda and Condorcet and then consider Arrow's impossibility theorem from the perspective of voting procedures.

Example 4.1. Suppose ranked ballots are used in a four-candidate election among a, b, c, and x, and there are 13 voters with a ballot response profile in which

4 voters have *axbc*
3 voters have *caxb*
6 voters have *bcax*.

Majority comparisons of expressed preferences show that

a has a 7 to 6 majority over b
b has a 10 to 3 majority over c
c has a 9 to 4 majority over a,

so majorities cycle among a, b, and c: there is no candidate that beats the other two in pairwise contests. (As for x, everyone prefers a to x, c has a 9 to 4 majority over x, but x has a 7 to 6 majority over b.) Although there is no majority candidate, making Condorcet's choice criterion inapplicable, if we count the number of majority wins for each candidate, a and c come out on top with two apiece. Because c is preferred to a by nearly 70% of the voters, c would appear to be the best social choice from a Condorcet perspective.

What about Borda? When 3, 2, 1 and 0 points are awarded to the candidates in best-to-worst order for each voter, the Borda point totals for a, b, c, and x are 24, 22, 21, and 11, respectively, so a is the top Borda candidate. While everyone prefers a to x, however, one might reasonably suspect that a's supporters nominated x to inflate a's Borda score.

Why? Suppose x is disqualified for any of the following reasons: x is certified as a ringer or Doppelgänger; the rules edit out all Pareto-dominated candidates before the Borda count is taken; x drops dead before the votes are counted. With x deleted, the Borda point totals for a, b, and c are 11, 16, and 12, respectively. Not only is b the top Borda candidate, but a also comes in last. ■

Arrow's seminal contribution was to show that there is no way around the inherent difficulties of situations like Example 4.1. His theorem has several formulations and interpretations. The one that follows seems most congruent with our emphasis on voting procedures.

Let $\mathcal{X}_2$ denote the family of two-candidate subsets of X with $|X| \geqslant 3$, and assume that every member of $\mathcal{X}_2$ is a potential feasible set, so $\mathcal{X}_2 \subseteq \mathcal{X}$. For every $\{x,y\} \in \mathcal{X}_2$, assume $D_{\{x,y\}} = \{\{x\},\{x,y\},\{y\}\}^n$, as in the preceding section. We do *not* assume that the restriction of F to any $\{x,y\} \times D_{\{x,y\}}$ is anonymous, neutral, or monotonic, and we do not presume that the same voting procedure is used for every $\{x,y\} \in \mathcal{X}_2$. We do, however, require F on $\{A \times D_A\colon A \in \mathcal{X}_2\}$ to satisfy two conditions, (2.1) for Pareto dominance and a minimal nondictatorship condition:

A1: *For every* $\{x,y\} \in \mathcal{X}_2$, $F(\{x,y\},(j,\ldots,j)) = j$ *for* $j \in \{\{x\},\{y\}\}$;

A2: *For every* $i \in \{1,\ldots,n\}$, there is an $\{x,y\} \in \mathcal{X}_2$ *and a* $d \in D_{\{x,y\}}$ *such that* $d_i = \{x\}$ and $y \in F(\{x,y\},d)$.

A2 says that for every voter there is at least one binary situation in which the voter votes for x but x is not the sole member of the social choice set, as might be expected if all the other voters voted for y.

Conditions A1 and A2 are undemanding and accommodate a wide variety of behaviors for F on $\{A \times D_A\colon A \in \mathcal{X}_2\}$. The thrust of Arrow's theorem is that all such behaviors are incompatible with a certain transitivity constraint on relationships among binary choices.

To formulate his constraint, let $\{a,b,c\}$ be a three-candidate subset of X, and let d^1, d^2, and d^3 be binary ballot response profiles in $D_{\{a,b\}}$, $D_{\{a,c\}}$, and $D_{\{b,c\}}$, respectively. We say that the triple (d^1,d^2,d^3) is *consistent* if, for every $i \in \{1,\ldots,n\}$,

$$d_i^1 = d_i^2 = \{a\} \quad \text{or} \quad d_i^1 = d_i^3 = \{b\} \quad \text{or} \quad d_i^2 = d_i^3 = \{c\}.$$

The sense of this definition can be seen by the fact that if, for example, $d_i^1 = d_i^2 = \{a\}$, then voter i's ballots in the three binary cases are consistent with the hypothesis that his or her true preferences on $\{a,b,c\}$ are $a \succ_i b \succ_i c$ or $a \succ_i c \succ_i b$ or $a \succ_i b \sim_i c$, and that he or she votes according to these preferences in each case. Arrow's transitivity constraint can be expressed as follows:

A3: For all three-candidate subsets $\{a,b,c\}$ of X, and all consistent $(d^1,d^2,d^3) \in D_{\{a,b\}} \times D_{\{a,c\}} \times D_{\{b,c\}}$, if

$$a \in F(\{a,b\},d^1) \quad \text{and} \quad b \in F(\{b,c\},d^3),$$

then $a \in F(\{a,c\},d^2)$.

Given consistency, $a \in F(\{a,b\},d^1)$ and $b \in F(\{b,c\},d^3)$ say that *a is socially as good as b*, and *b is socially as good as c* according to F: the conclusion, $a \in F(\{a,c\},d^2)$, completes the transitivity triad by asserting that *a is socially as good as c*.

Theorem 4.2. *If* $|X| \geqslant 3$ *and* $\mathcal{X}_2 \subseteq \mathcal{X}$, *then F cannot satisfy all three of* A1, A2 *and* A3.

Arrow's theorem has been interpreted in various ways, but the interpretation we prefer in the voting context is that it is unrealistic to suppose that binary voting outcomes should be transitive in the sense of A3. A corollary is that even when voters have weak preference orders and vote nonstrategically, there is no obvious or compelling way to ground social choices (from feasible sets of three or more candidates) on binary comparisons, whether determined by simple majority or in other ways.

Theorem 4.2, or rather the original versions of Arrow's theorem in Arrow (1950, 1951), have motivated a vast amount of research on multicandidate elections. There are now several dozen Arrow-type impossibility theorems that address a wide array of social choice situations, but all have the same theme of the collective incompatibility of conditions which, taken separately, seem reasonable and appealing. The books by Kelly (1978) and Fishburn (1987), and Chapters 1, 2 and 3 in this Volume and Chapter 17 in Volume 2 of this Handbook, focus on these theorems.

We conclude our discussion of impossibility theorems by recalling the theorem of Gibbard (1973) and Satterthwaite (1975) for strategyproof social choice functions. The question they addressed is whether it is possible to design a *decisive* choose-1 social choice function F on $\mathcal{D} = \{X\} \times V$ that is nondictatorial and strategyproof when $|X| \geqslant 3$ and V is the set of all n-tuples of weak orders on X. Unlike Theorem 4.2, $\mathcal{X}$ has only one member, namely X. We let

$$X^* = \{x \in X\colon F(X,v) = \{x\} \quad \text{for some} \quad v \in V\}$$

and say that F is *nondictatorial* if for every $i \in \{1, \ldots, n\}$ there is a $v = (\succsim_1, \ldots, \succsim_n)$ in V and $x,y \in X^*$ such that $x \succ_i y$ and $F(X,v) = \{y\}$. In addition, F is *strategyproof* if for all $v = (\succsim_1, \ldots, \succsim_n)$ and $v' = (\succsim'_1, \ldots, \succsim'_n)$ in V and all $i \in \{1, \ldots, n\}$,

$$(\succsim_j = \succsim'_j \quad \text{for all} \quad j \neq i) \Rightarrow F(X,v) \succsim_i F(X,v').$$

This implies that a voter can never unilaterally obtain a preferred outcome by voting contrary to his or her true preferences:

Theorem 4.3. *Suppose F is a decisive social choice function on $\{X\} \times V$, where V is the set of all n-tuples of weak orders on X, and $|X^*| \geqslant 3$. Then F cannot be both nondictatorial and strategyproof.*

When Example 4.1 and Theorems 4.2 and 4.3 are compounded many times over by related examples and theorems, it takes little imagination to conclude that there is no such thing as a completely satisfactory voting procedure for elections among three or more candidates.

In our ensuing discussion of specific multicandidate voting procedures, we make several simplifying assumptions with no significant loss of generality. We assume that F is an m-ary social choice function for $m \geqslant 3$, that all ballot sets $\mathcal{B}(A)$ for $|A| = m$ and $A \in \mathcal{X}$ are similar, that $D_A = \mathcal{B}(A)^n$ for each $A \in \mathcal{X}$, and that the same criterion for membership in $F(A, d)$ is used for all $A \in \mathcal{X}$.

We make a further concession to notation by taking X itself as the exemplary m-candidate set for the purpose of defining each procedure, and use A, B, C, S, and T to denote subsets of X. The ballot set will be denoted by $\mathcal{B}$ so that F is a mapping from $\{X\} \times \mathcal{B}^n$ into subsets of X with $|F(X,d)| \geqslant k$ for every $d = (d_1, \ldots, d_n)$ in $\mathcal{B}^n$.

5. Nonranked voting and dominated strategies

We assume henceforth that $|X| \geqslant 3$ and, until Section 10, that $k = 1$. This section and Section 6 focus on the strategic analysis of nonranked voting procedures with a single balloting stage. Section 7 will consider aspects of multistage nonranked voting, and Sections 8 and 9 then treat procedures that use ranked ballots. The following definition applies to this section and Section 6.

Definition 5.1. F on $\{X\} \times \mathcal{B}^n$ with $|X| = m$ is a *nonranked voting procedure* if there is a nonempty subset M of $\{0, 1, \ldots, m-1\}$ that includes at least one $j > 0$ such that $\mathcal{B} = \{A \subset X\colon |A| \in M\}$ and, for all $x \in X$ and $d \in \mathcal{B}^n$,

$$x \in F(X,d) \Leftrightarrow |\{i\colon x \in d_i\}| \geqslant |\{i\colon y \in d_i\}| \quad \text{for all} \quad y \in X\,. \tag{5.1}$$

Most M we consider include 0, which denotes abstention. We exclude m from M (a vote for all candidates) because it has the same effect on Equation (5.1) as an abstention. According to Equation (5.1), x is in the choice set if and only if as many voters vote for x as for any other candidate. An example of a single-stage nonranked procedure that does *not* adhere to Definition 5.1 is given at the end of Section 6.

5.1. Examples

Although nonranked voting procedures limit the ability of voters to express their preferences by their votes, they are the most widely used procedures for multicandidate elections. Some examples follow.

Plurality voting has $M = \{0, 1\}$, so each nonabstaining voter votes for exactly one candidate. It is commonly used in single-winner elections and sometimes in multiple-winner elections. The main criticisms of plurality voting concern its severe limitation on the expression of voter preferences, the dispersion of votes that it produces across ideologically similar candidates – rendering them vulnerable to other candidates, particularly on the ideological extremes, who have no opposition – and the extent to which it encourages voters to vote for candidates other than their favorites when their favorites have no real chance of winning. Axioms for plurality voting are given by Richelson (1978) and Roberts (1991).

Vote for no more than t has $M = \{0, 1, \ldots, t\}$. This is sometimes used for choose-1 elections with $1 \leqslant t \leqslant k$, but it is more common for choose-k (often $k = t$) elections. When $k < t$, it is called limited voting. $M = \{0, k\}$ is a more restrictive procedure.

Approval voting has $M = \{0, 1, \ldots, m - 1\}$, so a voter can vote for any proper subset of X. It was named by Weber (1977), axiomatized by Fishburn (1978a) and Sertel (1978) with a variable-n formulation (see Section 9.2), and is extensively analyzed with comparisons to other procedures, including plurality voting, in Brams and Fishburn (1978, 1983) and Merrill (1988). The analysis in this section and the next is adapted from Brams and Fishburn (1983) and Brams (1994). Approval voting has been adopted by several professional societies for elections of their officers.

Negative voting, which allows each voter either to vote for one candidate or to vote *against* one candidate, is tantamount to $M = \{1, m - 1\}$ or to $M = \{0, 1, m - 1\}$ when abstentions are allowed. It was proposed by Boehm (1976) and is analyzed in Brams (1977, 1978, 1983) and Felsenthal (1989).

5.2. *Voter preferences*

Additional definitions and assumptions for voter preferences are needed for our ensuing analysis of strategic voting. We denote a voter's weak preference order on X by $\succsim$, with $A \succsim B$ for nonempty $A, B \subseteq X$ if $a \succsim b$ for all $(a, b) \in A \times B$. Strict preference $\succ$ and indifference $\sim$ are defined in the usual ways, and $A \succ B$ if $a \succ b$ for all $(a, b) \in A \times B$. As in Section 2, the ordered indifference classes determined by $\succsim$ are $X_1 \succ X_2 \succ \cdots \succ X_r$.

We say that $\succsim$ is *unconcerned* if $r = 1$, *dichotomous* if $r = 2$, *trichotomous* if $r = 3$, and *multichotomous* if $r \geqslant 4$. A *voter* is *unconcerned* if $r = 1$, and is otherwise *concerned*. In the latter case, with $r \geqslant 2$, we say also that $\succsim$ is *concerned*.

A subset $A \subseteq X$ is *high for* $\succsim$ if

$$x \in A \cap X_j \Rightarrow X_i \subseteq A \quad \text{for all} \quad i < j,$$

and is *low for* $\succsim$ if

$$x \in A \cap X_j \Rightarrow X_i \subseteq A \quad \text{for all} \quad i > j.$$

Every subset of X is both high and low for an unconcerned voter, but only X and the empty set $\emptyset$ are both high and low for a concerned voter. It is easily seen that A is low for $\succsim$ if and only if its complement $X \setminus A$ is high for $\succsim$. The high sets for trichotomous $\succsim$ on $\{x_1, \ldots, x_5\}$ with

$$X_1 = \{x_1\}, \quad X_2 = \{x_2, x_3\}, \quad \text{and} \quad X_3 = \{x_4, x_5\},$$

are X, $\emptyset$, $\{x_1\}$, $\{x_1,x_2\}$, $\{x_1,x_3\}$, $\{x_1,x_2,x_3\}$, $\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_3,x_5\}$. These can be interpreted as the voting strategies this voter would consider viable – they are not dominated by any other strategies, as we will see in Section 5.3 and describe further in Section 6.1.

Assumptions that go beyond weak order are used in our analysis. They concern preferences between subsets viewed as vote outcomes or social choice sets. For a voter with weak order $\succsim$ on X, APB means that outcome A is strictly preferred to outcome B, and ARB means that the voter considers A at least as good as B. We assume without further notice that $\{a\}P\{b\} \Rightarrow a \succ b$, $\{a\}R\{b\} \Rightarrow a \succsim b$, and that APB and BRA cannot both hold. In addition, we assume the following for all $a, b \in X$ and all $A, B, C \subseteq X$:

Assumption P. If $a \succ b$, then $\{a\}P\{a,b\}P\{b\}$.

Assumption R. If $A \cup B$ and $B \cup C$ are not empty, and if $a \succsim b$, $b \succsim c$ and $a \succsim c$ for all $a \in A$, $b \in B$ and $c \in C$, then $(A \cup B)R(B \cup C)$.

Assumption P asserts that if candidate a is preferred to candidate b, then outcome $\{a\}$ is preferred to the tied outcome $\{a, b\}$, which is preferred in turn to $\{b\}$. This is uncontroversial if the voter believes that, however a tie might be broken, each of a and b has positive probability of winning when $F(X, d) = \{a, b\}$. Assumption R asserts that outcome $A \cup B$ is at least as good as $B \cup C$ when $A \succsim B$, $B \succsim C$ and $A \succsim C$ for the nonempty pairs from $\{A, B, C\}$.

5.3. *Dominance between strategies*

A *strategy* in the present context is any $A \subseteq X$, and a voter chooses strategy A if he or she votes for all $a \in A$ and no other candidate. We defer consideration of feasible strategies in $\mathcal{B}$ to Section 6 and focus here on a notion of dominance that applies to all strategies and is therefore applicable to all nonranked voting procedures.

Roughly speaking, strategy S dominates strategy T for a voter if he or she likes the outcome of S as much as the outcome of T in every possible circumstance, and strictly prefers S's outcome to T's outcome in at least one circumstance. To make this precise, we define a *contingency* as a function f that assigns a nonnegative integer to each candidate. A contingency is interpreted as specifying the number of votes each candidate receives from all voters *other than* the voter for whom dominance is being defined.

Call the latter voter the *focal voter*. Given a contingency f and a strategy S for the focal voter, let $\boldsymbol{F}(S,f)$ denote the outcome of the vote when the focal voter chooses

S, presuming that the votes in S count. Thus, with $S(a) = 1$ if $a \in S$, and $S(a) = 0$ otherwise,

$$a \in \boldsymbol{F}(S,f) \Leftrightarrow f(a) + S(a) \geqslant f(b) + S(b) \quad \text{for all} \quad b \in X.$$

Although different strategies may be preferred under different contingencies, some strategies are uniformly as good as, or better than, other strategies, regardless of the contingency. That is, one strategy may dominate another.

Definition 5.2. Given P and R for a voter with weak order $\succsim$ on X, strategy S *dominates* strategy T for this voter, or $S \operatorname{dom} T$ for $\succsim$, if $\boldsymbol{F}(S,f)\,\boldsymbol{R}\boldsymbol{F}(T,f)$ for all possible contingencies f, and $\boldsymbol{F}(S,f)\,\boldsymbol{P}\boldsymbol{F}(T,f)$ for at least one contingency.

It may be noted that if n is very small, say $n = 2$, then a contingency which demonstrates $\boldsymbol{F}(S,f)\,\boldsymbol{P}\boldsymbol{F}(T,f)$ might be unavailable because it presumes more than one other voter. However, even modest values of n avoid this possibility, and we shall ignore it in what follows.

Assumption R implies that an unconcerned voter will be indifferent between all outcomes as well as all individual candidates. Because Definition 5.2 requires $\boldsymbol{F}(S,f)\,\boldsymbol{P}\boldsymbol{F}(T,f)$ for some f to obtain $S \operatorname{dom} T$, it follows that there is no dominance for an unconcerned voter. The following theorem characterizes dominance between strategies for all concerned voters.

Theorem 5.3: Dominance. *Suppose $\succsim$ is concerned and Assumptions P and R hold. Then $S \operatorname{dom} T$ for $\succsim$ if and only if $S \neq T$, $S \setminus T$ is high for $\succsim$, $T \setminus S$ is low for $\succsim$, and neither $S \setminus T$ nor $T \setminus S$ is the set of all candidates.*

Proofs of Theorem 5.3 and results in the next section are given in Brams and Fishburn (1978). The intuition behind Theorem 5.3 is that because dominance is based on all contingencies, and the focal voter votes for all candidates in $S \cap T$ when he or she uses S or T, S dominates T for $\succsim$ if and only if $S \setminus T$ dominates $T \setminus S$ for $\succsim$. That is, dominance shows up in the nonoverlapping candidates.

Although the conclusion of Theorem 5.3 is predicated on Assumptions P and R, the necessary and sufficient conditions for $S \operatorname{dom} T$ use only $\succsim$ and not P or R explicitly. This greatly simplifies the identification of dominated strategies for a voter.

For example, if $X = \{a, b, c\}$ and $\succsim$ is the trichotomous linear order *abc*, Theorem 5.3 says that strategy $\{a\}$, under which the voter votes only for his or her most preferred candidate, dominates strategies $\{c\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$, and the abstention strategy. Moreover, these are the only strategies that $\{a\}$ dominates, whereas $\{a, b\}$ dominates these strategies and $\{b\}$ also.

Continuing with preference order *abc*, we illustrate the applicability of Theorem 5.3 to plurality and approval voting. Under approval voting, the theorem says that if the voter considers voting for b (second choice), he or she should also vote for a (first choice) because $\{a, b\} \operatorname{dom} \{b\}$. That is, $\{a, b\}$ is as good as, and sometimes better than, $\{b\}$. However, under plurality voting, a vote for b alone could be the voter's best strategy since in this case $\{b\}$ is not dominated by any other feasible strategy. Exhaustive enumeration for approval voting shows that there is no contingency in which $\{b\}$ induces a better outcome than $\{a, b\}$ [Brams (1978, pp. 199–202), Brams (1983, pp. 38–41)]. Fortunately, Theorem 5.3 relieves one of the necessity of checking all contingencies for which the focal voter's vote might affect the outcome.

6. Strategic analysis of nonranked voting

Let $M \subseteq \{0, 1, \ldots, m-1\}$ denote the nonranked voting procedure characterized by M in Definition 5.1. We say that strategy S is *feasible for M* if $|S| \in M$, i.e., if $S \in \mathcal{B}$ for M. We assume that, when M is used, a ballot is counted if and only if it is a feasible strategy. All other nonabstaining ballots are thrown out or treated like abstentions.

6.1. Admissible strategies

An admissible strategy is a feasible strategy that is not dominated by another feasible strategy.

Definition 6.1. Strategy S is *admissible for M* and $\succsim$ if S is feasible for M and there is no strategy T that is also feasible for M and has $T \operatorname{dom} S$ for $\succsim$.

As seen above, a strategy such as $\{b\}$ with preference ranking *abc* that is feasible for two or more nonranked voting procedures can be admissible for some procedures and inadmissible for others. Because of this, and because our analysis of strategic voting will be based on the assumption that nonabstaining voters use only admissible strategies, it is useful to have a theorem that characterizes all admissible strategies for every M and every concerned $\succsim$. To facilitate the statement of the admissibility theorem to follow, let

$$H(\succsim) = X_1, \quad \text{the subset of most preferred candidates under } \succsim,$$

$$L(\succsim) = X_r, \quad \text{the subset of least preferred candidates under } \succsim.$$

The admissibility theorem may seem complex, but as later corollaries will make clear, it is not difficult to apply to particular voting procedures. Moreover, comparisons among procedures will show that they possess striking differences that bear on their susceptibility to strategic manipulation.

Theorem 6.2: Admissibility. *Suppose $\succsim$ is concerned and Assumptions P and R hold. Then strategy S is admissible for M and $\succsim$ if and only if S is feasible for M and either C1 or C2 (or both) holds:*

C1: Every candidate in $H(\succsim)$ is in S, and it is impossible to partition S into nonempty subsets S_1 and S_2 such that S_1 is feasible for M and S_2 is low for $\succsim$.

C2: No candidate in $L(\succsim)$ is in S, and there is no nonempty $A \subseteq X$ disjoint from S such that $A \cup S$ is feasible for M and A is high for $\succsim$.

Because the abstention strategy satisfies neither C1 nor C2, it is never admissible for a concerned voter. A vote for all m candidates, which was omitted from the formal Definition 5.1, would likewise be inadmissible if it were permitted.

We now consider some corollaries of Theorem 6.2 for particular voting procedures, beginning with approval voting:

Corollary 6.3. Strategy S is admissible for approval voting and concerned $\succsim$ if and only if S contains all candidates in $H(\succsim)$ and none in $L(\succsim)$.

Hence, concerned voters use an admissible strategy under approval voting if and only if they vote for every one of their most preferred candidates and never vote for a least preferred candidate. If $m = 4$ and a voter has linear preference order *abcx*, then his or her admissible strategies are $\{a\}$, $\{a, b\}$, $\{a, c\}$ and $\{a, b, c\}$.

Corollary 6.4. A voter has a unique admissible strategy under approval voting if and only if his or her $\succsim$ is dichotomous. This unique strategy is the voter's subset of most preferred candidates.

Thus, if a voter has dichotomous preferences with $X_1 = \{a, b, c\}$ and $X_2 = \{x, y\}$, then $\{a, b, c\}$ is his or her unique dominant and admissible strategy under approval voting.

It is instructive to compare approval voting with plurality voting and negative voting with respect to feasible and admissible strategies. We assume that abstentions are allowed in all cases, so negative voting is equivalent to approval voting when $m = 3$. When $m \geqslant 3$, approval voting has $2^m - 1$ feasible strategies, which is the number of subsets of X, minus X itself. By contrast, plurality voting allows $m + 1$ different choices (a vote for one of the m candidates or an abstention), and negative voting allows $2m + 1$ strategies (a vote for or against a candidate or an abstention). Other nonranked voting procedures allow between $m + 1$ and $2^m - 1$ different strategies.

The following corollaries of Theorem 6.2 identify the admissible strategies for plurality and negative voting. In Corollary 6.6, $\bar{a}$ denotes the strategy in which the voter votes for all candidates other than candidate a or, equivalently, casts a vote against a.

Corollary 6.5. Strategy $\{a\}$ is admissible for plurality voting and concerned $\succsim$ if and only if a is not in $L(\succsim)$.

Corollary 6.6. Suppose $m \geqslant 4$ and $\succsim$ is concerned. Then:
(i) strategy $\{a\}$ is admissible for negative voting if and only if the voter strictly prefers a to at least two other candidates;
(ii) strategy $\bar{a}$ is admissible for negative voting if and only if the voter strictly prefers at least two other candidates to a.

Corollaries 6.3, 6.5 and 6.6 can be used to identify and compare sets of admissible strategies for various preference orders under approval, plurality, and negative voting. Suppose, for example, that $X = \{a, b, c, d\}$ (where d is a candidate, not a ballot response profile) and $a \sim b \succ c \succ d$, which we write as *(ab)cd* with parentheses

Table 1
Numbers of admissible voting strategies for three procedures with four candidates

Concerned weak order		Number of admissible strategies for		
		Approval voting	Negative voting	Plurality voting
Dichotomous	$a(bcd)$	1	1	1
	$(abc)d$	1	1	3
	$(ab)(cd)$	1	4	2
Trichotomous	$(ab)cd$	2	4	3
	$ab(cd)$	2	4	2
	$a(bc)d$	4	2	3
Multichotomous	$abcd$	4	4	3

enclosing candidates between which the voter is indifferent. The admissible strategies for weak order $(ab)cd$ are:

(1) *Approval voting*: $\{a, b\}$, $\{a, b, c\}$. These are the only feasible strategies that contain all the voter's most preferred, and none of his or her least preferred, candidates.

(2) *Plurality voting*: $\{a\}$, $\{b\}$, $\{c\}$. These are the only feasible nonabstention strategies that do not contain the voter's least preferred candidate.

(3) *Negative voting*: $\{a\}$, $\{b\}$, $\bar{c}$, $\bar{d}$. These are the only feasible strategies in which the voter strictly prefers the candidate to at least two others, or strictly prefers at least two others to the barred candidate.

Table 1 shows the numbers of admissible strategies for all concerned $\succsim$ for four candidates for the aforementioned three voting procedures. It is clear that the relative numbers of admissible strategies for the three procedures are very sensitive to the specific form of $\succsim$. For example, approval voting offers voters more admissible strategies than the others when $\succsim$ is $a(bc)d$ but fewer when $\succsim$ is $(ab)cd$. Hence, although the number of feasible strategies increases exponentially in m for approval voting but only linearly in m for plurality and negative voting, the number of *admissible* strategies under approval voting is comparable to that of the other procedures and should not overwhelm voters with a wealth of viable options.

6.2. Sincere voting and strategyproofness

We use the following notions of sincere voting and strategyproofness for nonranked voting procedures to facilitate comparisons among procedures in terms of their ability to elicit true preferences of voters.

Definition 6.7. Let $\succsim$ be a concerned preference order on X. Then strategy S is *sincere* for $\succsim$ if S is high for $\succsim$; voting procedure M is *sincere* for $\succsim$ if all admissible strategies for M and $\succsim$ are sincere; strategy S is *strategyproof* for M and $\succsim$ if it is the *only* admissible strategy for M and $\succsim$ (in which case it must be sincere); and voting system M is *strategyproof* for $\succsim$ if S is strategyproof for M and $\succsim$.

Sincere strategies are essentially ballots that directly reflect the true preferences of a voter. If $\succsim$ is $abcd$, then $\{a, c\}$ is not sincere because a and c are not the voter's two most preferred candidates. Because it is desirable that democratic voting procedures be based on true preferences, and sincere strategies foster the expression of such preferences, voting procedures that encourage sincerity are important. They are also important to individual voters, for if a procedure is sincere, voters will vote for *all* candidates ranked above the lowest-ranked candidates they consider acceptable. Thus, in our example, they would not vote for c without also voting for a and b, and they would not vote for b without also voting for a.

For the seven preference orders on four candidates in Table 1, approval voting is sincere in six cases (only $abcd$ is excluded), negative voting is sincere in four cases, and plurality voting is sincere in only the first three cases. These results follow easily from Corollaries 6.3, 6.5 and 6.6.

It is no accident that approval voting is "more sincere" than the others in Table 1. The following theorem demonstrates that approval voting is the uniquely most sincere nonranked voting procedure among those characterized in Definition 5.1.

Theorem 6.8. *If $\succsim$ is dichotomous, then every voting procedure M is sincere for $\succsim$. If $\succsim$ is trichotomous, then approval voting is sincere for $\succsim$, and this is the only procedure that is sincere for every trichotomous $\succsim$. If $\succsim$ is multichotomous, then no M is sincere for $\succsim$.*

No procedure is sincere when $\succsim$ is multichotomous because, for every M and every $\succsim$ with indifference classes $X_1 \succ X_2 \succ \cdots \succ X_r$, $r \geqslant 4$, there is an admissible strategy that is not sincere. When there are relatively few candidates, however, it is reasonable to expect that many voters will have dichotomous or trichotomous preference orders. Indeed, Theorem 6.8 says that when voters do not (or cannot) make finer distinctions, approval voting is the most sincere of all nonranked voting procedures, and this result extends to voters with multichotomous preferences [Fishburn (1978b)].

Even if a voting procedure is sincere for $\succsim$, it is not strategyproof for $\succsim$ if it allows more than one admissible strategy. Like sincerity, strategyproofness seems desirable for voting procedures. If voters have a strategyproof strategy, they will never have an incentive to deviate from it, even when they know the result of all other votes. Such a strategy dominates all other feasible strategies, so whatever contingency arises, a voter cannot be hurt, and may be helped, by choosing it.

Sincerity, on the other hand, does not imply such stability but asserts instead that whatever admissible strategy is chosen, whenever it includes voting for some candidate, it also includes voting for all candidates preferred to that one. In effect, a voting procedure is sincere if it never induces voters, for strategic reasons, to "abandon" a more preferred for a less preferred candidate.

Because the demands of strategyproofness are more stringent than those for sincere voting, strategyproofness is less likely to obtain than sincerity. Nevertheless, as with sincerity, approval voting is the uniquely most strategyproof of the procedures covered by Definition 5.1.

Theorem 6.9. *If $\succsim$ is dichotomous, then approval voting is strategyproof for $\succsim$, and this is the only procedure that is strategyproof for every dichotomous $\succsim$. If $\succsim$ is trichotomous or multichotomous, then no M is strategyproof for $\succsim$.*

Theorems 6.8 and 6.9 provide strong support for approval voting based on sincerity and strategyproofness, which can be extended to $k \geqslant 2$ for the election of committees [Fishburn (1981a)]. However, the limitations of these results also are important: strategyproofness depends entirely on dichotomous preferences; sincerity extends to trichotomous preferences, but it is a weaker criterion of nonmanipulability than strategyproofness. We note also that the manipulability and *stability* of approval voting (as measured by how sensitive outcomes, for a given voter preference profile, are to different ballot response profiles of admissible strategies) have provoked exchanges between Niemi (1984) and Brams and Fishburn (1985), between Saari and Van Newenhizen (1988a,b) and Brams, Fishburn and Merrill (1988a,b), and between Brams and Fishburn (2001) and Saari (2001a). The first paper in each pair is critical of approval voting, saying that approval voting is too sensitive to where voters draw the line between acceptable and unacceptable candidates, whereas the second paper responds to this criticism, saying that this sensitivity is desirable because it makes approval voting more responsive to voter preferences than voting procedures that allow the voter less leeway, either by choosing or by ranking candidates.

6.3. *Efficacy*

Another criterion that has been used to compare voting procedures concerns the ability of a voting strategy to change the outcome from what it would be if the voter in question abstained. We refer to this as the *efficacy* of a voting strategy and define it as the probability that a focal voter's ballot will affect the outcome, given that all possible ways that other voters can vote are equiprobable and ties are broken randomly [Fishburn and Brams (1981b,c), Brams and Fishburn (1983)].

In large electorates, the most efficacious approval voting strategies are for a focal voter to vote for either the top one or two candidates in three-candidate contests, and to vote for approximately the top half of all candidates when $m \geqslant 4$. When utilities are associated with the voter's preferences according to the expected-utility model [Fishburn (1970)], the voter's utility-maximizing strategy in large electorates is to vote for all candidates whose utilities exceed the average utility over all candidates. Hoffman (1982, 1983) and Merrill (1979, 1981, 1982, 1988) have independently derived similar results; in doing so, they consider criteria other than expected-utility maximization. A voter's utility-maximizing strategy can lead to substantially different expected-utility gains, depending on his or her utilities for the candidates. However, it can be shown that plurality voting gains are even more disparate [Fishburn and Brams (1981b,c, 1983), Rapoport and Felsenthal (1990)], so approval voting is more equitable in the sense of minimizing differences among voters.

As a case in point, plurality voting affords a dichotomous voter who equally likes four candidates but despises a fifth in a five-candidate race little opportunity to express his or her preferences, compared with a voter who greatly prefers one candidate to all others. Approval voting, on the other hand, is equitable to both – allowing the first voter to vote for his or her top four choices, the second to vote for his or her top choice – despite the extreme differences in their utilities. In general, not only is a voter able to be more efficacious under approval than plurality voting, but he or she cannot suffer as severe utility losses under the former procedure.

6.4. Plurality ballots and the median outcome

We conclude our discussion of single-stage nonranked voting with an example that does not fit Definition 5.1 because it has a continuum of candidates and uses a different selection criterion. The example features single-peaked preferences, which are discussed at greater length in Chapter 13 in Volume 2 of this Handbook..

Example 6.10. A nine-member committee is to decide how much of next year's budget to devote to some activity. We assume that each member has an ideal amount, with preference decreasing as one moves away from the ideal in either direction. Suppose the committee decides as follows: each member writes down one amount on a slip of paper, and the median of the ballot amounts becomes the collective choice. It is easily seen that each member has a unique dominant strategy, namely to vote for his or her ideal. The unique dominant strategies are sincere and the system itself is strategyproof in this restricted context. A complete characterization of unique dominant strategies for voting procedures in a general context is given in Dasgupta, Hammond and Maskin (1979). ■

7. Nonranked multistage voting: successive elimination

A *multistage nonranked voting procedure* is a procedure which, in a succession of nonranked ballots, eliminates candidates at each stage, or after each ballot, until a winner is determined. The number of ballots can be fixed or variable, depending on the procedure's rules. While there are many such procedures in use today, most are similar to one of the following three types.

7.1. Examples

Plurality with a runoff starts with a plurality-voting ballot; then it determines a winner by a simple majority vote between the top two candidates from the first ballot. The second ballot is often avoided if the top plurality candidate gets a sufficiently large percentage of the vote on the first ballot, say 40% or 50%, in which case that candidate wins outright. Other nonranked procedures could be used on the first ballot, but plurality voting is by far the most common. Requiring a maximum of two ballots, plurality with a runoff is effective in reducing a large field of candidates quickly and hence is popular in public elections, in which several ballots are impractical.

Plurality with successive elimination uses plurality voting on a succession of ballots until one candidate, the winner, gets more than 50% of the vote. After each ballot that requires a successor, some candidates may withdraw voluntarily from the race or be eliminated by a low-vote rule such as "lowest-person out". But successive votes are sometimes allowed without any reduction of the still-in-contention set, and some procedures even allow new candidates to enter during the process. Plurality with successive elimination often leads to fierce politicking during the balloting, with arm twisting, backroom deals, and the like. Examples of the procedure are analyzed by Brams and Fishburn (1981, 1983) and Fishburn, Fishburn and Hagy (1992). The latter paper describes instances in which dozens of time-consuming votes were taken to elect a candidate, leading some to refer to the procedure as "election by exhaustion"; such contests were common in U.S. national party conventions in the 19th and early 20th centuries [Brams (1978, Chapter 2)].

Successive majority voting uses a series of simple majority votes between subsets of X. The first vote is between subsets A and B for which $A \cup B = X$ and both $A \setminus B$ and $B \setminus A$ are nonempty. If A wins the first vote, the candidates in $B \setminus A$ are eliminated and, if $|A| \geqslant 2$, a second vote is taken between A_1 and A_2 for which $A_1 \cup A_2 = A$ and both $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are nonempty. If B wins the first vote, the candidates in $A \setminus B$ are eliminated; if $|B| \geqslant 2$, a second vote is taken between B_1 and B_2 for which $B_1 \cup B_2 = B$ and both $B_1 \setminus B_2$ and $B_2 \setminus B_1$ are nonempty. At each vote, the remaining candidates not in the winning subset are eliminated. The process continues until a single candidate, the final winner, remains. Example 1.1 gives an example. The first vote, between a_1 and a_2, is viewed as a vote between $A = \{a_1, a_3, \ldots, a_m\}$ and $B = \{a_2, a_3, \ldots, a_m\}$ since the winning subset remains in contention after this vote. Succeeding votes can be interpreted similarly as votes between two subsets of the candidates not yet eliminated.

7.2. Binary multistage voting

Farquharson (1969) discusses a generalization of successive majority voting that we refer to as *binary multistage voting*. The votes proceed between subsets as described in the preceding paragraph with the following modifications: abstentions are not permitted, each vote is decisive, and, when $m \geqslant 4$, the decision rule in each stage need not be simple majority. However, the decision rules must be monotonic, nondictatorial, and responsive to every voter's vote: see Farquharson (1969, p. 14) for precise definitions. To be satisfied, his conditions require at least three voters.

The two subsets and the decision rule used for each potential binary vote are specified in advance. A strategy, or ballot in our previous terms, says which of the two subsets the voter votes for in each case that might arise. Our previous definitions of sincere, admissible, and strategyproof strategies are patterned after Farquharson's definitions. He assumes that voters have linear preference orders on X. A strategy is *sincere* if, in any vote between subsets A and B, the voter votes for A if the top candidate in $A \setminus B$ is preferred to the top candidate in $B \setminus A$, and votes for B when the reverse obtains. A strategy is *admissible* if it is not dominated by another strategy, where dominance is based on contingencies, as in Section 5. And a strategy is *straightforward* (strategyproof by our earlier definition) if it is the unique admissible strategy.

One further definition leads to Farquharson's main theorem for straightforwardness. When A and B are nonempty subsets of X, $\{A, B\}$ *separates* a voter's preference order $\succsim$ if either the least preferred candidate in A bears $\succsim$ to the most preferred candidate in B, or the least preferred candidate in B bears $\succsim$ to the most preferred candidate in A.

Theorem 7.1. *A binary multistage voting procedure is straightforward for a voter if and only if $\{A, B\}$ separates his or her linear preference order for every potential binary vote between subsets A and B that might arise during the voting.*

For example, if $m = 3$ and the first vote is between $\{a, b\}$ and $\{c\}$, then *abc*, *bac*, *cab* and *cba* have straightforward strategies (the separation in each case is shown by the slash: *ab/c*, *ba/c*, *c/ba*, and *c/ab*), but *acb* and *bca* do not because *c* divides *a* from *b* (no slash can separate these orders into $\{a, b\}$ and $\{c\}$). If the first vote is between $\{a, b\}$ and $\{b, c\}$, then only *abc* and *cba* have straightforward strategies (in these cases, there is a common element on both sides of the slashes: *ab/bc* and *cb/ba*). Farquharson notes that no binary multistage voting procedure can be straightforward for all linear orders.

He also introduced the term *sophisticated voting* to characterize voting strategies arrived at by recursive analysis when every voter knows the others' preferences and it is assumed that every voter uses an admissible strategy. A strategy is *primarily admissible strategy* if there is no other strategy which produces at least as good an outcome in every contingency where other voters use admissible strategies, and produces a better outcome in some such contingency. A strategy is *secondarily admissible strategy* if it is primarily admissible when all other voters use primarily admissible strategies. Continuation leads to ultimately admissible strategies, which are called *sophisticated strategies*.

Theorem 7.2. *When all voters have linear preference orders and know each other's preferences, every voter has a unique sophisticated strategy for every binary multistage voting procedure.*

The voting strategies described in Example 1.1 are sophisticated. As seen there, sophisticated strategies need not be sincere. Indeed, insincere sophisticated strategies are prime examples of a procedure's susceptibility to strategic manipulation.

Farquharson (1969, p. 43) notes that the conclusion of Theorem 7.2 does not extend to multistage nonranked voting when votes are taken for three or more competing subsets in some stages. This does not, of course, mean that such systems are free from strategic manipulation but only that voters need not have uniquely best strategies under the type of calculation that produces sophisticated strategies. For example, plurality with a runoff is rife with strategic possibilities: if one's favorite candidate cannot win, there may be several ways to defeat one's worst candidate, either by helping to prevent him or her from making the runoff, or by helping someone else win against him or her in the runoff.

7.3. Paradoxes

Although multistage nonranked voting procedures are very popular and can serve a group's practical needs, they are subject not only to strategic manipulation but also to a variety of anomalies, or paradoxes, that are often not recognized by their proponents. The paradoxes can arise under naive or sincere voting in the absence of strategic calculations; indeed, they can subvert a procedure's purpose of electing a candidate, in a democratic manner, that best serves the interests of a group.

An array of paradoxes for multicandidate voting procedures is described and analyzed by Riker (1958), Fishburn (1974a, 1981b, 1982), Niemi and Riker (1976), Doron and Kronick (1977), Doron (1979), Gehrlein (1983), Fishburn and Brams (1983), Saari (1984, 1987, 1989, 1994), Moulin (1988a), Brams, Kilgour and Zwicker (1998), Scarsini (1998) and Nurmi (1998a,b, 1999), among others. We illustrate five of these for multistage nonranked procedures under sincere voting.

The *dominated candidate paradox* [Fishburn (1974a)] occurs when *all* voters prefer another specific candidate to the winner. Suppose the 13-voter voter preference profile of Example 4.1 holds and successive majority voting is used with voting order *acbx*. The winners under sincere voting after the three votes are c, b, and x, respectively, so x wins the election. However, *all* voters prefer a to x. It follows that, when $m \geqslant 4$, successive majority voting does not satisfy the Pareto dominance condition (2.1). Theorem 7.2 applies to this case and, if all voters use their sophisticated strategies, c would be elected. Including sincere voting on the third vote, the sophisticated strategy of the 4 voters with ranking *axbc* is to vote for a whenever possible, and to vote for b if b faces c. The sophisticated strategies of the 9 voters with rankings *caxb* and *bcax* are to vote for c whenever possible, and to vote for a if a faces b. Only the latter voters, with ranking *bcax*, have insincere sophisticated strategies.

The *winner-turns-loser paradox* [Doron and Kronick (1977)] illustrates the failure of monotonicity that occurs under sincere voting when the winner would have been a loser if some voters had ranked this candidate higher in their preference orders, all else unchanged. An example for plurality with a runoff occurs with the 93-voter preference profile in which

27 voters have *abc*; 42 voters have *cab*; 24 voters have *bca*.

The first-vote plurality winners are a and c, with c beating a 66-to-27 in the runoff. If 4 of the first 27 voters had raised c from third place to first, the profile would have been

23 voters have *abc*
46 voters have *cab*
24 voters have *bca*.

Now the first-vote winners are b and c, with b beating c 47-to-46 in the runoff. Thus, c changes from a winner to a loser when it gains support.

Another anomaly, which is closely related to the winner-turns-loser paradox, is the *no-show paradox* [Fishburn and Brams (1983)]. It occurs when the addition of identical preference orders with candidate x ranked last changes the winner from another candidate to x. This occurs in the preceding example when we begin with the second profile, where b wins under plurality with a runoff. If we then add from two to 42 voters to the group with ranking *abc* who have c in last place, c wins. Additional analyses of the paradox are in Ray (1986), Moulin (1988b) and Holzman (1988–1989). The key to both this paradox and its predecessor is who gets scratched after the first vote. It does *not* depend on the Condorcet paradox, wherein majorities cycle, which happens to occur in these examples. The main three-candidate example in Fishburn and Brams (1983) has a majority or Condorcet candidate (see Section 8), but the no-show paradox still occurs when that candidate is scratched after the first vote for one of the voter preference profiles.

Our fourth paradox, the *multiple-districts paradox*, occurs when one candidate would win in each of a number of districts separately but loses the combined-districts election. A two-districts example in Fishburn and Brams (1983) shows for a three-candidate election conducted by plurality with a runoff, one candidate would win in each district but lose the overall combined-districts election. Moreover, each of the other candidates has a sizable majority over the candidate who would win in each separate district. We return to this phenomenon in Section 9, where its proscription is noted to be a central axiom of positional voting procedures like Borda's.

Our final paradox in this section (others will be discussed in Section 9.3) is the *multiple-election paradox* [Brams, Kilgour and Zwicker (1998), Scarsini (1998), Nurmi (1998b, 1999)]. Consider a referendum in which voters can vote either yes (Y) or no (N) on each proposition on the ballot. The paradox occurs when the set of propositions that wins, when votes are aggregated separately for each proposition (proposition aggregation), receives the fewest votes when votes are aggregated by combination (combination aggregation). As an example, suppose there are 3 propositions, so there are $2^3 = 8$ combinations because each voter can make one of two choices (i.e., Y or N) on each proposition. Suppose further that there are 13 voters who cast the following numbers of votes for each of the eight combinations:

YYY: 1 YYN: 1 YNY: 1 NYY: 1 YNN: 3 NYN: 3 NNY: 3 NNN: 0.

For example, YYN means a Y vote on the first and second propositions and an N vote on the third.

Notice that on each of the three propositions, N beats Y by 7 to 6 votes, so NNN wins according to proposition aggregation. However, NNN is the only combination that receives 0 votes, illustrating the dramatic difference that can arise between proposition aggregation and combination aggregation (YNN, NYN, and NNY tie for first place with 3 votes each). It turns out that the existence of this paradox implies that majorities cycle, but not vice versa, and actually occurred in the 1990 California general election, as did some variants of the paradox [Brams, Kilgour and Zwicker (1997)].

This paradox vividly illustrates the conflict that can occur between the two vote-aggregation procedures. Like the other paradoxes, it does not depend on either sincere or strategic voting: voters may be perfectly sincere in voting for their preferred position on every proposition, or they may be strategic (in some sense). The paradox says only that majority choices according to proposition aggregation may receive the fewest votes according to combination aggregation.

8. Condorcet choices and ranked voting

We continue to assume that $|X| \geqslant 3$ and $k = 1$ so that $F(d) = F(X, d)$ is a nonempty subset of X for every ballot response profile $d \in D$. In addition, V with typical member $v = (\succsim_1, \ldots, \succsim_n)$ is a set of voter preference profiles. The *Condorcet set*, or majority set, for $v \in V$ is

$$\begin{aligned}\mathrm{Con}(v) &= \{a \in X\colon\ |\{i\colon a \succsim_i b\}| \geqslant |\{i\colon b \succsim_i a\}| \quad \text{for all} \quad b \in X\}\\ &= \{a \in X\colon\ |\{i\colon a \succ_i b\}| \geqslant |\{i\colon b \succ_i a\}| \quad \text{for all} \quad b \in X\},\end{aligned}$$

and a candidate in $\mathrm{Con}(v)$ is a *Condorcet candidate* or majority candidate. $\mathrm{Con}(v)$ is empty if and only if every candidate can be beaten by another candidate in a sincere simple majority vote between the two, presuming a voter abstains if and only if he or she is indifferent between the two. When n is odd and every $\succ_i$ in v is a linear order, $|\mathrm{Con}(v)| \in \{0, 1\}$. If all voters are indifferent among all candidates, $\mathrm{Con}(v) = X$. We also denote by $>_M$ the *strict simple majority relation* on X induced by a profile v, so that $a >_M b$ if $|\{i\colon a \succ_i b\}| > |\{i\colon b \succ_i a\}|$, with $\mathrm{Con}(v) = \{a \in X\colon b >_M a$ for no $b \in X \setminus \{a\}\}$.

Our discussion of $\mathrm{Con}(v)$ is divided into three parts. The first considers combinatorial aspects of $>_M$ and $\mathrm{Con}(v)$. The second relates $\mathrm{Con}(v)$ to voting procedures defined in preceding sections, and the third examines Condorcet social choice functions, which are designed to elect a candidate which has a strict simple majority over every other candidate when such a candidate exists.

8.1. *Condorcet combinatorics*

Under this heading we describe studies devoted to the structure of $>_M$ and $\mathrm{Con}(v)$. As before, n is the number of voters and m is the number of candidates. In addition,

V_L will denote the set of all voter preference profiles for (m, n) in which every $\succsim_i$ is a linear order or strict ranking.

Some time ago, McGarvey (1953) showed that if n is large enough compared to m, then for *every* asymmetric binary relation on X there is a $v \in V_L$ that has this relation as its $>_M$. The question then arose as to the smallest n, denoted by $\sigma(m)$, for which this is true when $|X| = m$. Stearns (1959) proved that $\sigma(m) \leqslant m+1$ for odd m, $\sigma(m) \leqslant m+2$ for even m, and equality holds here when $m \in \{3,4,5\}$. He showed also that $\sigma(m) > [(\log 3)/2]m/(\log m)$, where log denotes the natural logarithm. Erdős and Moser (1964) then noted that $\sigma(m) \leqslant c_1 m/(\log m)$ for a fixed constant c_1. Precise values of $\sigma(m)$ are unknown beyond the first few m, and the question of whether $\sigma(m)(\log m)/m$ tends to a limit as $m \to \infty$ remains open.

Riker (1958, 1982) and Gehrlein (1983), among others, describe multicandidate cases in practice that probably had no Condorcet candidate. One technical approach to the likelihood of Condorcet's paradox focuses on the proportion $p(m, n)$ of the $(m!)^n$ profiles in V_L that *have* a Condorcet candidate that bears $>_M$ to every other candidate. If each voter independently chooses one of the $m!$ rankings at random (i.e., according to the uniform distribution), then $p(m, n)$ is the probability that one candidate has a strict majority over every other candidate. Early studies of $p(m, n)$ include Guilbaud (1952), Niemi and Weisberg (1968), and DeMeyer and Plott (1970), with later refinements by Gehrlein and Fishburn (1976, 1979). It is easily seen that $p(3,3) = 17/18$, but exact computations for $m > 3$ or $n > 3$ get complex very quickly. The most efficient method for three candidates [Gehrlein and Fishburn (1976)] uses

$$p(3, n) = 3^{-n+1} \sum \frac{n! 2^{-(n_2+n_3)}}{n_1! n_2! n_3! n_4!},$$

where $\sum$ is a triple sum with limits $\{0 \leqslant n_1 \leqslant (n-1)/2,\ 0 \leqslant n_2 \leqslant (n-1)/2 - n_1,\ 0 \leqslant n_3 \leqslant (n-1)/2 - n_1\}$ and $n_4 = n - n_1 - n_2 - n_3$. The most efficient method known for three voters [Gehrlein and Fishburn (1979)] uses

$$p(m, 3) = \sum_{m_1=0}^{m-1} \sum_{m_2=0}^{m-1-m_1} \frac{(m-1-m_1)!(m-1-m_2)!}{m!(m-1-m_1-m_2)!(m_1+m_2+1)}.$$

When $m \geqslant 4$ is even and n is odd, there is a nice recursion relation for $p(m, n)$. The simplest case [May (1971)] is

$$p(4, n) = 2p(3, n) - 1.$$

The recursion for $m = 6$ and n odd is

$$p(6, n) = 3p(5, n) - 5p(3, n) + 3,$$

and in general [Gehrlein and Fishburn (1976)]

$$p(m, n) = \sum^{m/2} c_{jm} p(2j-1, n),$$

where the c_{jm} coefficients are independent of n. No similar relationship holds for odd m. With limiting proportion

$$p(m) = \lim_{n \to \infty} p(m, n),$$

Guilbaud (1952) showed that

$$p(3) = \tfrac{3}{4} + \tfrac{3}{2\pi} \sin^{-1}\left(\tfrac{1}{3}\right) \doteq 0.91226,$$

and Niemi and Weisberg (1968) proved that $p(m)$ equals m times the $(m-1)$-dimensional normal positive orthant probability with all correlations equal to $\frac{1}{3}$. An approximation accurate within one-half of one percent for odd $m < 50$ is

$$p(m) \doteq \frac{9.33}{m + 9.53} + (0.63)^{(m-3)/2}.$$

A more complex but more accurate approximation appears in Gehrlein (1999). Many additional results along this line appear in Kelly (1974), Fishburn, Gehrlein and Maskin (1979) and the extensive reviews of Gehrlein (1983, 1997).

We now turn to restrictions on voter preferences which imply that $>_M$ is acyclic, or that $\mathrm{Con}(v)$ is nonempty. Based on the approach taken by Ward (1965) and Sen and Pattanaik (1969), let T denote a subset of the $m!$ linear orders on $X = \{1, 2, \ldots, m\}$, and define T to be *acyclic* if there do not exist $a, b, c \in X$ and three orders in T whose restrictions on $\{a, b, c\}$ are abc, cab, and bca. Interest in T stems from the following basic proposition, where $V(T)$ denotes the set of all nonempty finite lists (any number of voters) of linear orders in T and v_A for $A \subseteq X$ is the restriction to A of $v \in V(T)$.

Theorem 8.1. $\mathrm{Con}(v_A)$ *is nonempty for every* $v \in V(T)$ *and every* $A \subseteq X$ *with* $3 \leqslant |A| \leqslant m$ *if and only if* T *is acyclic.*

Several people, including Kim and Roush (1980), Abello and Johnson (1984), Abello (1991), Craven (1996) and Fishburn (1997), have considered how large T can be while providing the guarantee of Condorcet candidates given by Theorem 8.1. We let

$$f(m) = \max\{|T| : T \text{ is acyclic for } X = \{1, \ldots, m\}\},$$

and remark that T is acyclic if and only if, for all $a < b < c$ in X, the restrictions of T's orders to $\{a, b, c\}$ must exclude at least one order in each of the cyclic triples $\{abc, cab, bca\}$ and $\{acb, bac, cba\}$. Thus $f(3) = 4$. In addition, $f(4) = 9$ [Abello (1981), Raynaud (1982)] with acyclic

$$T = \{1234, 1324, 1342, 3124, 3142, 3412, 3421, 4312, 4321\},$$

$f(5) = 20$ [Fishburn (1997)] and, based on an example of Bernard Monjardet and a construction procedure in Craven (1996) and Fishburn (1997), $f(6) \geqslant 45$, $f(7) \geqslant 100$, and $f(8) \geqslant 222$. It is known also that the optimal pattern for $\max T$ undergoes a paradigm shift near $m = 10$, that $f(m) > (2.1708)^m$ for all large m, and, as proved by Raz (2000), that $f(m) < c_2^m$ for some constant c_2 and all m.

Nonmaximum but natural restrictions on voters' preferences that guarantee Condorcet candidates include single-peaked preferences [Galton (1907), Arrow (1951), Black (1958), Fishburn (1973)]. A typical case occurs when the candidates can be ordered along a line and each voter's preferences, left-to-right, increase up to a most preferred candidate and then decrease. With no loss of generality we use the natural order $12 \cdots m$ for $X = \{1, 2, \ldots, m\}$. The following definition generalizes weak order by allowing each $\succsim_i$ to be a *partial order*, which means that its asymmetric part $\succ_i$ is irreflexive and transitive.

Definition 8.2. A voter preference profile $v = (\succsim_1, \ldots, \succsim_n)$ of partial orders on $X = \{1, 2, \ldots, m\}$ is *single peaked* in the order $12 \cdots m$ if, for each $i \in \{1, \ldots, n\}$, there are unique $a_i, b_i \in X$ with $a_i \leqslant b_i$ such that, for all $x, y, z \in X$:
(i) $x < y \leqslant a_i \Rightarrow y \succ_i x$
(ii) $b_i \leqslant y < x \Rightarrow y \succ_i x$
(iii) $a_i \leqslant y \leqslant b_i \Rightarrow y \succsim_i x$
(iv) $(x < y < z, x \sim_i y, y \sim_i z) \Rightarrow x \sim_i z$.

The candidates in $[a_i, b_i]$ are voter i's *preference plateau*. The following theorem [Fishburn (1973, p. 108)] locates $\mathrm{Con}(v)$ as a nonempty interval of integers:

Theorem 8.3. *Suppose v is a single-peaked voter preference profile of partial orders as specified in Definition 8.2. Let $c_1, c_2, \ldots, c_{2n}$ be a rearrangement of the sequence $a_1, \ldots, a_n, b_1, \ldots, b_n$ with $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_{2n}$. Then*

$$\mathrm{Con}(v) = \{x \in X: c_n \leqslant x \leqslant c_{n+1}\}.$$

Strategyproofness for single-peaked preferences is discussed by Moulin (1980), Berga (1998), and Ching and Serizawa (1998). Other restrictions on voter preferences are discussed in Chapter 3 in this Volume, and Chapter 21 in Volume 2 of this Handbook.

8.2. Nonranked voting and Condorcet candidates

This section considers the propensities of nonranked voting procedures to elect Condorcet candidates when $\mathrm{Con}(v)$ is nonempty. As in Section 6, we let M denote the single-stage nonranked voting procedure characterized by M in Definition 5.1. We also let M^+ denote the two-stage runoff procedure in which the first vote is a procedure-M vote and the two candidates with the most votes on the first vote go against each other in the runoff, whose outcome is determined by simple majority voting. Thus, $\{0, 1\}^+$ is plurality with a runoff, and $\{0, 1, \ldots, m-1\}^+$ is approval voting with a runoff.

Inada (1964) proved that $\mathrm{Con}(\upsilon)$ is nonempty if all preference weak orders in profile υ are dichotomous. Using previous results, which presume Assumptions P and R in Section 5.2, we can prove more than this, namely that the use of admissible strategies under approval voting and dichotomous preferences always yields $\mathrm{Con}(\upsilon)$ as the outcome. Moreover, for every other M procedure, the use of admissible strategies may give an outcome that contains no Condorcet candidate. The following definition, in which preference orders are mapped into admissible strategies, will be used to express these results more precisely.

Definition 8.4. For any nonempty list $\upsilon = (\succsim_1, \ldots, \succsim_n)$ of preference orders, and every nonranked voting procedure M, let $\upsilon(M)$ be the set of all $d = (d_1, \ldots, d_n)$ in which d_i is an admissible strategy for M and $\succsim_i$. For every $d \in \upsilon(M)$, let $F(d) = F(X, d)$, the outcome for ballot response profile d under procedure M.

To illustrate, suppose $\upsilon = (abc, abc, c(ab))$: two voters prefer a to b to c; the other is indifferent between a and b and prefers c to both. Assume that M is the plurality procedure. By Corollary 6.5, each of the first two voters has two admissible strategies, $\{a\}$ and $\{b\}$, and the third voter has one. The outcomes of the $4 = 2 \times 2 \times 1$ members of $\upsilon(M)$ are $\{a\}$, $\{a, b, c\}$, $\{a, b, c\}$ and $\{b\}$.

Theorem 8.5. *If all preference orders in υ are dichotomous and M is the approval voting procedure, then $F(d) = \mathrm{Con}(\upsilon)$ for all $d \in \upsilon(M)$.*

In other words, if all voters have dichotomous preferences, their use of admissible strategies under approval voting invariably yields $\mathrm{Con}(\upsilon)$ as the outcome.

To see how plurality voting differs with dichotomous preferences, suppose $X = \{a, b, c\}$ with $2n + 1$ terms in υ: one is $a(bc)$, n are $b(ac)$, and n are $(ac)b$. Then $\mathrm{Con}(\upsilon) = \{a\}$. However, if as few as two of the n with $(ac)b$ vote for c (voting for either a or for c is an admissible strategy under plurality voting), then the outcome is $\{b\}$, which is disjoint from $\mathrm{Con}(\upsilon)$. The following theorem shows that a similar result holds for every procedure other than approval voting:

Theorem 8.6. *Suppose M is a nonranked voting procedure other than approval voting. Then there is a υ composed entirely of dichotomous preference orders and a $d \in \upsilon(M)$ such that $F(d)$ and $\mathrm{Con}(\upsilon)$ are disjoint.*

In contrast to the definitive picture for dichotomous preferences, comparisons among approval voting and other M procedures are less clear-cut when some voters partition X into three or more indifference classes. Reviews by Merrill (1988) and Nurmi (1987), based primarily on computer simulations [see also Bordley (1983), Chamberlin and Featherston (1986), Fishburn and Gehrlein (1976, 1977, 1982), Nurmi (1988), Regenwetter and Grofman (1998)], suggest that approval voting is generally as good as or better than other M procedures, particularly plurality voting [Nurmi and Uusi-Heikkilä (1985), Felsenthal and Maoz (1988)], in electing Condorcet candidates. Indeed, it compares favorably with most positional scoring procedures (Section 9) not only in terms of its Condorcet efficiency [Merrill (1985)] but also in terms of the "social utility" of elected candidates [Weber (1977, 1995), Merrill (1984, 1988)].

We now integrate runoff procedures of type M^+ into the picture. Generally speaking, these procedures are less sincere than M procedures, and plurality with a runoff is less sincere than approval voting with a runoff because the former is more restrictive when preferences are dichotomous. Neither runoff approval nor plurality is strategyproof, even when preferences are dichotomous. Runoff approval voting, especially, is susceptible to severe manipulation effects and is even more manipulable when preferences are not dichotomous. For example, if a sizable minority of voters has preference order *abc* and is fairly sure that a would beat c but lose to b in a runoff, these voters may well vote $\{a,c\}$ on the first vote in an attempt to engineer a runoff between a and c. Other examples, and more precise statements of results, are given in Fishburn and Brams (1981a) and Brams and Fishburn (1983).

Our next theorem considers Condorcet candidates under sincere voting. A *sincere* strategy for M^+ is a strategy whose M vote is sincere according to Definition 6.7 and whose vote in the runoff is sincere.

Theorem 8.7. *For both M and M^+ procedures, there exist sincere strategies that will elect a Condorcet candidate under approval voting but not necessarily under any other procedure.*

This is not to say, however, that *all* sincere strategies guarantee the election of a Condorcet candidate when one exists and approval voting is used. But it is possible to make this guarantee in some cases.

Example 8.8. Suppose $m = n = 3$ and $\upsilon = (xab, (ax)b, (bx)a)$. Then $\mathrm{Con}(\upsilon) = \{x\}$. Under approval voting, the only admissible strategies for voters 2 and 3 are $\{a,x\}$ and $\{b,x\}$, respectively. The first voter has two sincere admissible strategies, $\{x\}$ and $\{x,a\}$, and x wins for both. Hence, approval voting *must* elect x when voters use sincere admissible strategies.

Consider plurality, runoff plurality, and runoff approval voting. The following strategies are admissible and sincere for all three procedures on the first or only vote: 1 votes for x, 2 for a, and 3 for b. These do not guarantee the election of x under plurality, nor under a runoff procedure where the runoff pair could be $\{a,b\}$. ■

A second example demonstrates that when all voters use admissible but not necessarily sincere strategies, a Condorcet candidate's election may again be ensured only under approval voting.

Example 8.9. Suppose $m = 3$, $n = 4$, and

$$\upsilon = ((xa)b, (xa)b, xba, bxa),$$

so $\mathrm{Con}(\upsilon) = \{x\}$. All admissible strategies for approval voting are sincere (Theorem 6.8) and will elect x. But if the last two voters vote for b under plurality voting, x's election would not be ensured. Moreover, if in a runoff the first two voters vote for a on the first vote and the other two vote for b, neither runoff plurality nor runoff approval voting would elect x. Thus, not only may approval voting be the only procedure to guarantee the election of a Condorcet candidate when voters are restricted to sincere admissible strategies, but it also may be the only procedure to provide this guarantee when voters are less constrained and can use any admissible strategy. ■

We generalize these examples in the following theorem for plurality and approval voting, with or without a runoff. When x beats all other candidates by simple majority comparisons, x is a *strict* Condorcet candidate.

Theorem 8.10. *Suppose all voters use admissible strategies on the first vote (if there is a runoff) or only vote (if there is not) and, if there is a runoff, vote on the runoff if and only if they are not indifferent between its two candidates. Then, for all $m > 3$ and all voter preference profiles for m candidates,*

[x must be elected under runoff plurality voting]
$\Rightarrow$ *[x must be elected under runoff approval voting]*
$\Rightarrow$ *[x must be elected under plurality voting]*
$\Rightarrow$ *[x must be elected under approval voting]*
$\Rightarrow$ *[x must be a strict Condorcet candidate].*

Because examples can be constructed to show that the converse implications are false, the ability of a procedure to guarantee the election of a strict Condorcet candidate is highest for approval voting, next highest for plurality voting, third highest for runoff approval voting, and lowest (essentially nonexistent) for runoff plurality voting. Moreover, approval voting also encourages the use of sincere strategies. Because procedures with more complex choice criteria do not [Merrill and Nagel (1987), Merrill (1988)], voters need not resort to insincere strategies to elect Condorcet candidates, if they exist, under the approval voting procedure.

The primary mathematical analyses of the likelihood that a rule of type M or M^+ will choose a Condorcet candidate, given that one exists, appear in Gehrlein and Fishburn (1978a) and Gehrlein (1981, 1982, 1993, 1995), with a review in Gehrlein (1997, pp. 190–194). Under random choices of voter rankings for V_L and the presumption of sincere voting, the limiting conditional probabilities in n that plurality and runoff plurality will elect a Condorcet candidate for $m = 3$ are 0.7572 and 0.9629, respectively [Gehrlein and Fishburn (1978a), Gehrlein (1993)]. Given $m = 3$ with $n = 3j$, $j \in \{1, 3, 5, \dots\}$, Gehrlein (1982) obtained closed-form expressions for the conditional probability of electing the Condorcet candidate (given that one exists) under a different probabilistic preference order assignment called "impartial anonymous culture". The expression for plurality voting is

$$\frac{119n^4 + 1348n^3 + 5486n^2 + 10812n + 10395}{135(n+1)(n+3)^2(n+5)} \to \frac{119}{135} \doteq 0.8815,$$

and for runoff plurality is

$$\frac{523n^4 + 6191n^3 + 25117n^2 + 40749n + 22140}{540(n+1)(n+3)^2(n+5)} \to \frac{523}{540} \doteq 0.9685.$$

Gehrlein (1995, 1997) includes tables which show how such Condorcet likelihoods change as social homogeneity, measured by a parameter in a contagion model of choices of preferences, varies. As one would expect, for most voting procedures social homogeneity increases the likelihood that Condorcet candidates will be elected.

8.3. *Condorcet voting procedures*

A number of voting procedures have been proposed whose aim, in part, is to elect a strict Condorcet candidate x ($x >_M y$ for all $y \neq x$ in X) when one exists [Condorcet (1785), Fishburn (1977), Richelson (1979), Straffin (1980), Riker (1982), Dummett (1984), Schwartz (1986), Tideman (1987), Nurmi (1987, 1998a), Merrill (1988), Levin and Nalebuff (1995), Le Breton and Truchon (1997)]. Apart from the highly manipulable successive majority procedure of Section 7.1, none is widely used in practice. We therefore merely outline various procedures and leave most details of their axiomatic and strategic analysis to the references.

We simplify matters considerably by assuming that voters have linear preference orders and vote sincerely. In some cases, the ballot response profile d is taken to be equal to the voter preference profile υ, whereas other cases only require d to reveal certain aspects of υ, such as $>_M$ on X. We say that a procedure is a *Condorcet voting procedure* if $F(X,d) = \{x\}$ whenever x is a strict Condorcet candidate. We define an even dozen such procedures, which are partitioned into three groups according to the information needed to determine the social choice set $F(X,d)$, which we denote simply by F. The names of most procedures are explained more fully in Fishburn (1977).

A Condorcet voting procedure is a *C1 procedure* if $>_M$ is sufficient to determine F. We note six C1 procedures

(1) *Copeland's procedure* [Copeland (1951), Goodman (1954), Henriet (1985)] has $x \in F$ if x maximizes $|\{y\colon x >_M y\}| - |\{y\colon y >_M x\}|$. It always chooses a strict Condorcet candidate when one exists, but its F can be disjoint from a nonempty $\mathrm{Con}(\upsilon)$. The four-voter profile $\upsilon = (xyabc, xybac, cbaxy, yacbx)$ has $\mathrm{Con}(\upsilon) = \{x\}$ but $F = \{y\}$.

(2) *Miller's procedure* [Miller (1980, 1995), Shepsle and Weingast (1984), Epstein (1998)] takes $x \in F$ if x is in the *uncovered subset* of $(X, >_M)$, i.e., if whenever $y >_M x$ there is a $z \in X$ such that $z >_M y$ and $x \geqslant_M z$, where $x \geqslant_M z$ means that not ($z >_M x$) or that x beats or ties z.

(3) *Fishburn's procedure* [Fishburn (1977)] defines $>'_M$ by $a >'_M b$ if $z >_M a \Rightarrow z >_M b$ for all $z \in X$, and for some z, $a \geqslant_M z >_M b$. Then $x \in F$ if no y has $y >'_M x$. The relation $>'_M$ is a *strict partial order* (asymmetric and transitive), so $(X, >'_M)$ always has maximal candidates when X is finite. Fishburn's choice set is always included in Miller's choice set, and the inclusion can be proper. Another Condorcet voting procedure with this property is described by Dutta (1988).

(4) *Schwartz's procedure* [Good (1971), Schwartz (1972, 1974)] defines $>_M^*$ as the asymmetric part of the transitive closure of $>_M$, and takes $x \in F$ if no y has $y >_M^* x$. Like $>'_M$, $>_M^*$ is a strict partial order.

The next two definitions assume that $x >_M y$ or $y >_M x$ for all distinct $x, y \in X$, i.e., that $(X, >_M)$ is a *tournament*. This is always true under preceding assumptions when n is odd.

(5) *Banks's procedure* [Banks (1985), Miller, Grofman and Feld (1990)] takes $x \in F$ if x is the maximum candidate of a maximal transitive subtournament of $(X, >_M)$. In other words, if $(Y, >_M)$ is a $>_M$-linearly ordered subset of $(X, >_M)$ and no other linearly ordered subset $(Z, >_M)$ has $Y \subset Z$, then the candidate in Y that beats all others in Y is in F.

(6) *Slater's procedure* [Slater (1961), Laffond and Laslier (1991), Laslier (1997)] takes $x \in F$ if x is the maximum candidate in a linear ordering of X which requires the minimum number of *reversals* of pairs in $>_M$ to achieve linearity. Charon, Hudry and Woirgard (1997) define a 16-vertex tournament for which the choice sets of the Banks and Slater procedures are disjoint.

For $v \in V_L$ and distinct $x, y \in X$, let $v(x, y)$ denote the number of voters who prefer x to y, so $v(x, y) + v(y, x) = n$ when there are n voters. We refer to a Condorcet voting procedure as a *C2 procedure* if it is not a C1 procedure and the $v(x, y)$ counts suffice to determine F. We consider four C2 procedures:

(7) *Black's procedure* [Black (1958)] takes $F = \mathrm{Con}(v)$ if $\mathrm{Con}(v)$ is nonempty; otherwise, F is the set of Borda winners, so that $x \in F$ if $\sum_z v(x, z) \geqslant \sum_z v(y, z)$ for all $y \in X$.

(8) *Nanson's procedure* [Nanson (1907), Hoag and Hallett (1926), Black (1958), Nurmi (1989), McLean (1996)] is a Borda-elimination procedure that is a Condorcet voting procedure. Let $v(x, A) = \sum_{y \in A} v(x, y)$. Let $A_1 = X$ and for each $j \geqslant 1$ let $A_{j+1} = A_j \setminus \{x \in A_j\colon v(x, A_j) \leqslant v(y, A_j)$ for all $y \in A_j$, and $v(x, A_j) < v(y, A_j)$ for some $y \in A_j\}$. The A_j decrease to a nonempty limit set A^*, which is F for Nanson's procedure.

(9) *Condorcet's procedure* [Condorcet (1785), Black (1958)] is a maximin procedure. Let $v_*(x) = \min\{v(x, y)\colon y \in X \setminus \{x\}\}$. Then $x \in F$ if x maximizes $v_*(x)$ over X.

Young (1988) argues that the next procedure is more in keeping with Condorcet's intentions when there is no Condorcet candidate.

(10) *Kemeny's procedure* [Kemeny (1959)] takes $x \in F$ if x is the maximum candidate in a linear ordering L of X that maximizes $\sum\{v(a, b)L(a, b)\colon a, b \in X\}$, where $L(a, b) = 1$ if aLb and $L(a, b) = 0$ otherwise. This procedure is axiomatized by Young and Levenglick (1978); also see Young (1988, p. 1242).

Finally, we define a Condorcet voting procedure as a *C3 procedure* if it is neither a C1 nor C2 procedure. We note two C3 procedures:

(11) *Dodgson's procedure* [Dodgson (1876), Black (1958), Fishburn (1973)], named after the Rev. Charles Lutwidge Dodgson, a.k.a. Lewis Carroll, is based on the minimum number of reversals in the linear orders in v by which a candidate beats

or ties every other candidate under simple majority comparisons. Denote this minimum reversals number for x by $r(x, v)$, and let $r^*(x, v) = \lim_{N\to\infty} r(x, Nv)/N$, where Nv is v replicated N times (so Nv has Nn voters). Dodgson's procedure takes F as the set of candidates that minimize $r^*(x, v)$.

(12) *Young's procedure* [Young (1975b)] is based on the largest number of terms (voters) in a sublist of v for which a candidate beats or ties every other candidate under simple majority applied to the sublist. Denote this largest number for x by $l(x, v)$ and, if no sublist has the noted property for x, let $l(x, v) = 0$. Then let $l^*(x, v) = \lim_{N\to\infty} l(x, Nv)/N$. Young's procedure takes F as the set of candidates that maximize $l^*(x, v)$.

Along with the strict Condorcet property, all 12 procedures are anonymous, neutral, and *homogeneous* in the sense that F remains invariant to replications Nv of v. All were intended to yield a "good" choice set in the absence of a strict Condorcet candidate, but some seem better than others in this regard. Apart from the Miller, Banks, and Slater procedures, a comparative analysis of other properties these procedures do and do not satisfy is given in Fishburn (1977). For example, all except Schwartz's procedure satisfy the Pareto condition (2.1), and all except Nanson's and Dodgson's procedures are monotonic. Another appealing property is *Smith's Condorcet principle* [Smith (1973)], which says that if X can be partitioned into nonempty A and B such that $a >_M b$ for all $(a, b) \in A \times B$, then F contains no candidate in B. This is violated by procedures 7, 9, 11 and 12, but it holds for the others.

9. Positional scoring procedures and Borda choices

Along with $|X| = m \geqslant 3$ and $k = 1$, we assume throughout this section that the ballot set $\mathcal{B}$ is a set of linear orders or strict rankings of X, except when noted otherwise. We take $D = \mathcal{B}^n$ when n is fixed. It is convenient, however, in the present setting to extend our prior definition of a social choice function by letting n range over the positive integers with

$$D^+ = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3 \cup \cdots$$

and with F^+ defined on $\{X\} \times D^+$. The extended form is used in axiomatizations of positional scoring procedures, which we will consider shortly.

9.1. Positional scoring procedures

Positional scoring procedures include Borda's method and those in which the *differences* between the points awarded to candidates in successive positions on a voter's ballot are not equal. We denote by s_j the points awarded to a candidate in position j and refer to $s = (s_1, s_2, \ldots, s_m)$ as a *positional scoring vector*. It is assumed that $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_m$ and $s_1 > s_m$. Borda's method as described in Section 1 has $s = (m-1, m-2, \ldots, 1, 0)$.

For every $x \in X$, every $j \in \{1, \ldots, m\}$, and every ballot response profile $d = (d_1, \ldots, d_n)$ in $\mathcal{B}^n$, let $d(x,j)$ denote the number of voters who rank candidate x in jth position. Clearly, $d(x,1) + d(x,2) + \cdots + d(x,m) = n$. The *score* of candidate x for ballot response profile d with respect to positional scoring vector s is

$$s(x,d) = \sum_{j=1}^{m} s_j d(x,j).$$

The *positional scoring procedure for s* takes $F(X,d)$ or $F^+(X,d)$ as the subset of candidates that maximize $s(x,d)$ over X for each d in D or D^+.

The positional scoring procedures for s and s' are equivalent when there is a positive number $\alpha > 0$ and a real number β such that $s' = \alpha s + (\beta, \ldots, \beta)$; then $s'(x,d) = \alpha s(x,d) + \beta n$, and the maximizing subsets for s and s' are identical. Moreover, if s and s' are not so related, then in the extended formulation there will be a $d \in D^+$ at which $F^+(X,d)$ differs for s and s'. Because of equivalence, we can set $s_m = 0$ with no loss of generality. If, in addition, s_1 were fixed at 1, then each different positional scoring procedure in the extended formulation would be characterized by a unique s.

Plurality voting has $s = (1, 0, \ldots, 0)$; the procedure which assigns 3 points to a first-place candidate, 1 point to a second-place candidate, and zero points thereafter has $s = (3, 1, 0, \ldots, 0)$. Apart from plurality voting, positional scoring procedures are seldom used in practice. The one exception is the use of Borda's method in elections with small numbers of candidates and voters. We note for the Borda procedure with $s = (m-1, m-2, \ldots, 1, 0)$ that $s(x,d)$ is identical to the aggregate number of candidates ranked lower than x on all ballots. We used this fact in describing Black's procedure and Nanson's procedure in the preceding section.

9.2. Axioms

Positional scoring procedures have been axiomatized by Smith (1973) and Young (1975a), and axiomatic characterizations of Borda's method are given in Young (1974), Hansson and Sahlquist (1976), and Nitzan and Rubinstein (1981). We note versions of these and then show how their extended formulation yields axioms for approval voting.

Positional scoring procedures are not subject to the multiple-districts paradox of Section 7.3. To express this axiomatically, we use Young's formulation in which Π is the set of all functions π that map the $m!$ linear orders on X into nonnegative integers with $\pi > 0$ for some order. Each *summary profile* π, which presumes anonymity, tells how many voters have each linear order for a ballot response profile in the D^+ setting. For convenience we let $C(\pi) = F^+(X,d)$ when d generates π. Young's (1975a) axiom that avoids the multiple-districts paradox is referred to as

consistency: $C(\pi) \cap C(\pi') \neq \emptyset \Rightarrow C(\pi + \pi') = C(\pi) \cap C(\pi')$,

where $(\pi + \pi')(\succ) = \pi(\succ) + \pi'(\succ)$. This says that if two disjoint groups of voters have some candidate in common in their social choice sets, then the choice set of the combined groups consists of the common choices of the separate groups.

Young's other axioms for positional scoring procedures are neutrality, monotonicity (to obtain $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_m$), the nonconstancy condition for C (which then gives $s_1 > s_m$), and the following condition of

continuity: If $C(\pi) = \{x\}$ and π' is any other member of Π, then $C(N\pi + \pi') = \{x\}$ for all sufficiently large integers N.

Young shows that consistency, neutrality, and continuity characterize C as a scoring procedure for some $s = (s_1, \ldots, s_m)$; then monotonicity and nonconstancy yield $s_1 \geqslant \cdots \geqslant s_m$ with $s_1 > s_m$.

Theorem 9.1. *Suppose C maps Π into the nonempty subsets of X. Then C is a positional scoring procedure if and only if it is nonconstant, neutral, monotonic, consistent, and continuous.*

Myerson (1995b) generalizes this theorem by not requiring voters to have linear preference orders. He refers to the preceding consistency and continuity conditions as "reinforcement" and "overwhelming majority," respectively. Young (1974) specializes the preceding theorem to Borda's procedure, which is characterized by neutrality, consistency, *faithfulness* [if $n = 1$, $C(d)$ contains only the voter's first place candidate], and a "cancellation property" whose primary function is to ensure that

$$s_1 - s_2 = s_2 - s_3 = \cdots = s_{m-1} - s_m.$$

Other conditions on a C of Theorem 9.1 that imply this equal-successive-difference property for s are noted in Section 9.4.

Young's approach motivated the axiomatization of approval voting in Fishburn (1978a). For the nonranked context, let Π^* be the set of functions π that map the *subsets* of X (approval ballots) into nonnegative integers with $\pi > 0$ for some subset, and let $\pi[x] = \sum\{\pi(A): x \in A\}$, the number of voters whose ballots contain candidate x. The approval voting choice set for $\pi \in \Pi^*$ is the subset of candidates that maximize $\pi[x]$ over X.

Theorem 9.2. *Suppose C maps Π^* into the nonempty subsets of X. Then C is the approval voting procedure if and only if it is neutral, consistent $[C(\pi) \cap C(\pi') = \emptyset \Rightarrow C(\pi + \pi') = C(\pi) \cap C(\pi')]$, and satisfies the disjoint equality property which says that if π consists of exactly two ballots A and B with $A \neq B$ and $A \cap B = \emptyset$, then $C(\pi) = A \cup B$.*

A different characterization of approval voting that features strategyproofness with dichotomous preferences (see Theorem 6.9) is included in Fishburn (1979a).

9.3. Paradoxes

Paradoxes of positional voting arise from the algebraic structure of positional scoring procedures and their sensitivity to perturbations in ballot response profiles and positional scoring vectors [Zwicker (1991)]. They include *choice-set paradoxes*, which focus on F or C, and *ranking paradoxes*, which consider the ways in which positional scoring procedures rank candidates according to values of $s(x,d)$ over X. The most thorough analyses of these and many other paradoxes are included in Saari (1987, 1989, 1992, 1994, 1995a,b, 2000a,b, 2001b), Chapter 25 in Volume 2 of this Handbook, and references cited below. We begin with examples of choice-set paradoxes.

Condorcet's "other paradox" [Condorcet (1785), Fishburn (1974a)] occurs when there is a strict Condorcet candidate and *every* positional scoring procedure would choose another candidate when $s_1 > s_2 > s_3 > \cdots > s_m$. The seven-voter response profile in which

3 voters have *xab*
2 voters have *abx*
1 voter has *axb*
1 voter has *bxa*

yields x as the strict Condorcet candidate. However, $s(a,d) - s(x,d) = s_2 - s_3$, so x is never in the choice set of a positional scoring procedure when $s_2 > s_3$.

Another choice-set paradox occurs when a winner turns into a loser after candidates other than the winner are removed from X. Fishburn (1974b) constructs a profile for any $m \geqslant 3$ candidates with a unique Borda winner x such that, for every $Y \subset X$ with $x \in Y$ and $|Y| \geqslant 2$ (except for one such Y with $|Y| = 2$), x is a Borda loser when the Borda scores are recomputed on the basis of Y.

Removal of a candidate from X can affect the s-order of the remaining candidates in specific or in arbitrary ways [Davidson and Odeh (1972), Fishburn (1974a, 1981b), Saari (1982)]. Consider Borda's procedure applied to the seven-voter profile in which

3 voters have *cbax*
2 voters have *baxc*
2 voters have *axcb*.

The Borda scores for a, b, c and x are 13, 12, 11 and 6, respectively, so the Borda order is $a > b > c > x$. When x is removed and Borda scores are recomputed for the reduced profile, the Borda order is $c > b > a$, a complete inversion from the original. Fishburn (1981b) generalizes this for any $m \geqslant 3$ by considering any $s = (s_1, s_2, \ldots, s_m)$ with $s_1 > s_2 > \cdots > s_m$, and any $t = (t_1, t_2, \ldots, t_{m-1})$ with $t_1 > t_2 > \cdots > t_{m-1}$. Let $X = \{x_1, x_2, \ldots, x_m\}$. Given s and t, there is a profile $\pi \in \Pi$ whose best-to-worst s-order for X is $x_1 x_2 \cdots x_m$, whose t-order for $X \setminus \{x_1\}$ is $x_m x_{m-1} \cdots x_2$, and whose t-order for $X \setminus \{x_m\}$ is $x_{m-1} \cdots x_2 x_1$. Other profiles give a complete inversion of the remaining candidates when an intermediate member of the s-order is removed. Saari (1982) generalizes this by allowing s and t to be any nonconstant vectors, and by prespecifying an s-order, a candidate to be removed, and a t-order on the remainder. Then there is a profile that produces the prespecified orders.

Fishburn (1981b) also considers complete inversions *without* removals. Let s and s' be any two nonequivalent positional scoring vectors for m candidates with $s_1 > \cdots > s_m$ and $s'_1 > \cdots > s'_m$. Then there is a $\pi \in \Pi$ with s-order $x_1x_2 \cdots x_m$ and s'-order $x_m \cdots x_2x_1$. Saari (1984) generalizes this by considering any $h \geqslant 2$ nonconstant and not necessarily monotonic scoring vectors $s^1, \ldots, s^h$ and any prespecified linear orders $l_1, \ldots, l_h$ on X. He proves that if $s^1, \ldots, s^h$ and $(1, \ldots, 1)$ are linearly independent, then there is a $\pi \in \Pi$ whose s^j-order is l_j for $j = 1, \ldots, h$.

9.4. *In praise of Borda, mostly*

Borda's procedure occupies a unique place among all positional scoring procedures by being less susceptible than all other procedures to many unsettling possibilities and paradoxes. For example, all positional scoring procedures are susceptible to strategic manipulation [Nitzan (1985)], but Borda's procedure is least susceptible [Saari (1990a, 2001b)]. The next several paragraphs note other results favorable to Borda.

We begin with deterministic results. Smith (1973) showed for $m \geqslant 3$ that if $d \in D^+$ has a strict Condorcet candidate x then $s_B(x, d) > s_B(y, d)$ for some $y \in X \setminus \{x\}$, where s_B denotes Borda's procedure. However, if s is not Borda's procedure, then there is a $d \in D^+$ with a strict Condorcet candidate x such that $s(y, d) > s(x, d)$ for every $y \in X \setminus \{x\}$.

Saari (1987) generalizes this as follows. Let S denote a function on $\{A \subseteq X\colon |A| \geqslant 2\}$ that assigns a positional scoring procedure $s(A)$ to each such A: if $|A| = j$, $s(A) = (s(A)_1, \ldots, s(A)_j)$, with $s(A)$ necessarily the plurality or simple majority procedure when $|A| = 2$. Let R on $\{A \subseteq X\colon |A| \geqslant 2\}$ assign a weak order $R(A)$ to every such A, and let S_B denote the S composed entirely of Borda procedures. Suppose $m = 3$. If $S = S_B$, then there is an R such that, for every $\pi \in \Pi$, the $s(A)$-order for π on A is not the same as $R(A)$ for at least one A. However, if S is not equivalent to S_B, then for every R there is a $\pi \in \Pi$ such that the $s(A)$-order for π on A equals $R(A)$ for all $A \subseteq X$ with $|A| \geqslant 2$.

Saari (1989) goes further. For $m \geqslant 3$, let $R[S]$ denote the set of all R for which there is a $\pi \in \Pi$ such that the $s(A)$-order for π on A equals $R(A)$ for all $A \subseteq X$ with $|A| \geqslant 2$. Then, for every S that is not equivalent to S_B, $R[S_B]$ is a proper subset of $R[S]$. In other words, if something can happen with the Borda assignment, then it also happens to every other S assignment or, in Saari's words [Saari (1989, p. 454)], "any fault or paradox admitted by Borda's method also must be admitted by all other positional voting methods". For a characterization of $R[S_B]$, see Saari (1990b).

We now turn to probabilistic results under the assumption that every voter independently selects a linear order for d or π at random. Gehrlein and Fishburn (1978b) prove for $m = 3$ and $n \to \infty$ that, among all positional scoring procedures, the Borda procedure uniquely maximizes the probability that s elects a strict Condorcet candidate, given that such a candidate exists. Van Newenhizen (1992) proves the same thing for fixed n. Tataru and Merlin (1997) prove for $m = 3$ and $n \to \infty$ that, among all positional scoring procedures, the Borda procedure uniquely minimizes the probability that the s-order has a strict Condorcet candidate in last place, given that such a candidate exists.

We note two other results for the uniform-distribution probability model and $n \to \infty$. First, if $m \in \{3, 4\}$, then the Borda procedure maximizes the probability that the s-order between any two candidates is the same as the simple majority relation between the two [Gehrlein and Fishburn (1980)]. For the other result, let $P_m(s, t)$ be the probability for $n \to \infty$ that the s-winner with $s = (s_1, \ldots, s_m)$ for X is also the t-winner with $t = (t_1, \ldots, t_{m-1})$ for $X \setminus \{y\}$ after one $y \neq x$ is randomly removed from X. Then $P_m(s, t)$ is uniquely maximized when both s and t are Borda procedures [Gehrlein, Gopinath, Lagarias and Fishburn (1982)].

Despite Borda's pre-eminence among positional scoring procedures, it does have defects illustrated by paradoxes described earlier. Moreover, it is almost certainly more susceptible to manipulation than approval voting. Consider, for example, a preference profile $v = (abc, abc, abc, bca, bca)$. Recognizing the vulnerability of their first choice a, the first three voters might rank the candidates insincerely as acb on their ballots, maximizing the difference between a and its closest competitor b. This would make a the Borda winner.

Recently, Sertel and Yilmaz (1999) and Brams and Kilgour (2001) independently proposed a procedure in which, in the 5-voter example of the preceding paragraph, a would be chosen by sincere voters if the decision rule, or quota q, were simple majority, but b would be chosen if q were unanimity. The procedure works by having voters rank candidates from best to worst. If at least q voters rank a candidate first, that candidate is chosen; if not, then one next asks if there are at least q voters who rank a candidate either first or second – and so on, descending to lower and lower levels in the rankings until there is agreement by at least q voters on a candidate or candidates. Thus, if $q = 3$ (simple majority), there is agreement on a, based only on first choices, making a the "majoritarian compromise" [Sertel and Yilmaz (1999)]. If $q = 5$ (unanimity), there is no agreement without descending to second choices, at which level all 5 voters rank b either first or second, making b the "fallback bargaining" choice [Brams and Kilgour (2001)]. In a voting context, Sertel and Yilmaz (1999) argue that simple majority is sensible, whereas in a bargaining context Brams and Kilgour (2001) argue that unanimity is sensible. Whatever the decision rule, this procedure may not select a Condorcet candidate, but the candidate or candidates chosen by it are always Pareto-optimal – there are no other candidates that all voters prefer – and maximizes the minimum "satisfaction" (based on rankings) of the q most satisfied voters.

Manipulation is quite difficult under this procedure [Brams and Kilgour (2001)], as it is under many other voting procedures. But the Borda procedure is an exception: voters can gain by ranking the most serious rival of their favorite candidate last, which is a relatively easy strategy to effectuate, in order to lower the rival's point total [Ludwig (1978), Dummett (1998)].

10. Point distribution procedures

In this section and the next we consider choose-k social choice functions for $k \geqslant 2$. Two common choose-k procedures for small k are the nonranked procedures that ask voters to vote for exactly k candidates, or for no more than k candidates. The top k vote getters, or more if there is a tie for kth place, are the winners. The same criterion can be used with approval voting, positional scoring procedures, and other procedures used primarily for choose-1 situations. A different criterion, referred to as a cutoff or quota, does not specify k in advance but elects every candidate whose vote count exceeds the cutoff. This is frequently used by groups to elect new members or to bestow an honorific title on present members.

The literature for choose-k procedures is, apart from that for proportional representation, comparatively sparse. Examples include Fishburn (1981a), Brams (1982, 1990), Gehrlein (1985), Staring (1986), Bock, Day and McMorris (1998), Barberà, Sonnenschein and Zhou (1991), Debord (1992) and Brams and Fishburn (1992, 1993). Staring (1986) gives an example of voters with linear preference orders who vote sincerely under the vote-for-exactly-k procedure, which illustrates an *increasing-committee-size paradox*: the winners for $k = 3$ are disjoint from the winners for $k = 2$, and the winners for $k = 4$ are disjoint from those for $k \in \{2, 3\}$. Debord (1992) gives an axiomatic choose-k generalization of Young's (1974) Borda axiomatization.

All voting procedures described previously use nonranked or ranked ballots that do not allow voters to express *intensities* of preference in a more complete manner. *Point distribution procedures* accommodate this possibility by asking each voter to distribute a fixed number of points, say 100, to the candidates in any way he or she please. The k candidates with the most points are the winners. The usual term for such a procedure is *cumulative voting* [Glasser (1959), Brams (1975), Bolger (1983, 1985)]. It has been used by corporations to elect boards of directors, and may be viewed as a method for proportional representation in which minorities can ensure their approximate proportional representation by concentrating their votes on a subset of candidates commensurate with their size in the electorate. Indeed, cumulative voting is one of a class of voting procedures that encourage minority representation [Guinier (1994)] and maximize majority welfare [Chwe (1999)].

To illustrate cumulative voting and the calculation of optimal strategies, suppose there is a single minority position among the electorate favored by one-third of the voters. The other two-thirds favor a majority position. Assume that $n = 300$, six candidates are to be elected ($k = 6$), and each voter has six votes (points) to distribute over the candidates. The minority controls 600 votes, and the majority controls 1200 votes. Hence if the minority divides its votes equally between two minority candidates ($600/2 = 300$ each), it can ensure their election no matter what the majority does. If the two-thirds majority instructs its supporters to distribute their votes equally among five candidates ($1200/5 = 240$), it will not match the vote totals of the two minority candidates but can still ensure the election of four of its five candidates –

and possibly get its fifth candidate elected if the minority splits its votes equally among three minority candidates (600/3 = 200).

Against these majority (support five) and minority (support two) strategies, it is easy to show that neither side can improve its position. To elect five rather than four candidates with 301 votes each, the majority would need 1505 instead of 1200 votes; similarly, to elect three rather than two candidates with 241 votes each, the minority would need 723 instead of 600 votes.

It is evident that the optimal strategy for the leaders of both the majority and minority is to instruct their members to allocate their votes as evenly as possible among a certain number of candidates. The number to support should be proportionally about equal to the number of their supporters in the electorate (if known).

Any deviation from this strategy – for example, by putting up a full slate of candidates and not instructing supporters to vote for only some on this slate – offers the other side an opportunity to capture more than its proportional "share" of the k seats. Patently, good planning and disciplined supporters are needed to carry out an optimal strategy.

Brams (1975) includes a systematic analysis of optimal strategies under cumulative voting. These strategies are compared to strategies actually adopted by the Democratic and Republican parties in elections for the Illinois General Assembly, where cumulative voting was used until 1982. Cumulative voting was adopted by two cities in the United States (Alamogordo, NM, and Peoria, IL) in 1987, and other small cities more recently, to satisfy court requirements of minority representation in municipal elections.

Bolger (1983, 1985) formulates six procedures for cumulative voting in choose-k elections and investigates their susceptibility to several paradoxes. Each procedure allots k points to each of n voters to distribute over the candidates and uses an election quota $q_0 = (nk + 1)/(k + 1)$. Any candidate who receives at least q_0 votes is elected in an initial stage. The procedures differ in their vote distribution rules and in how votes are processed after the initial stage if fewer than k are elected there. In some procedures, a voter votes for $h \leqslant k$ candidates, and each of the h gets k/h votes from the voter; others allow the k points to be distributed in any way among $h \leqslant k$ or among any number of candidates. Vote processing after the initial stage may involve transfers of surplus votes above q_0 from initial electees to others, or elimination of low-ranking candidates. The paradoxes include violations of monotonicity and new voter and no-show paradoxes. The *new-voter paradox* occurs when a new voter who votes only for the original k electees causes one of these for whom he or she votes to become a loser in the augmented profile. The *no-show paradox* occurs when an original electee turns into a loser after a ballot involving only original losers is deleted from the ballot response profile. All six procedures exhibit the latter two paradoxes when $k \geqslant 4$, and all but two do this when $k \geqslant 2$.

11. Proportional representation

Unlike cumulative voting, most choose-k procedures use ballot types discussed earlier.

We have already noted common nonranked procedures for electing committees, and in this section we consider other procedures designed to elect representative legislatures and governing bodies.

11.1. The Hare system of single transferable vote

First proposed by Thomas Hare in England and Carl George Andrae in Denmark in the 1850s, single transferable vote (STV) procedures have been adopted throughout the world. They are used in such countries as Australia, Malta, the Republic of Ireland, and Northern Ireland; in local elections in Cambridge, MA, and formerly in other cities in the United States [Tideman (1995)]. John Stuart Mill (1862) placed STV "among the greatest improvements yet made in the theory and practice of government". Although STV violates some desirable properties of voting procedures [Kelly (1987)], it has strengths as a method of proportional representation. In particular, minorities can elect a number of candidates roughly proportional to their numbers in the electorate. Also, if one's vote does not help elect a first choice, it can still count for lower choices.

To define one version of STV with $|X| = m$, suppose k of the m candidates are to be elected by n ballots which rank from 1 to m candidates. (In practice, voters are encouraged to rank as many candidates as possible.) The point *quota* needed for election is

$$q = \left\lfloor \frac{n}{k+1} \right\rfloor + 1,$$

where $\lfloor z \rfloor$ is the integer part of z. We denote by p_i the points for ballot i. Initially, $p_i = 1$, but p_i can change during the ballot-processing stages because (1) the top candidate not yet removed from ballot i is elected, or (2) no candidate is left on ballot i, or (3) no candidate is left on other ballots. The initial p_i sum is n; after j candidates have been elected, the revised p_i sum is $n - jq$. Whenever points are counted to determine if new candidates reach q, the p_i points of ballot i are awarded to the top-ranked candidate remaining on ballot i.

Let e denote the number of candidates elected thus far, and let A denote the subset of candidates still in contention. The following steps are used to move e from 0 to k.

Step 0: Set $e = 0$, $A = X$, and $p_i = 1$ for all i. Go to step 1.

Step 1: If $e + |A| \leqslant k$, declare all candidates in A as elected, and if $e + |A| < k$, choose $k - (e + |A|)$ of the not yet elected $m - (e + |A|)$ candidates at random, declare them elected also, and stop. If $e + |A| > k$, for each $x \in A$ compute $p(x)$ as the sum of the p_i for all ballots that rank x first, then let $E = \{x \in A: p(x) \geqslant q\}$, and declare the candidates in E as elected. If $e + |E| = k$, stop. Otherwise, change e to $e + |E|$, go to step 2 if $|E| \geqslant 1$, and go to step 3 if $|E| = 0$.

Step 2: For each $x \in E$, let $\lambda_x = q/p(x)$, and for each ballot with x ranked first, replace p_i by $(1 - \lambda_x)p_i$. This removes q points from the process for each newly elected candidate in E. Delete all $x \in E$ from all ballots, change A to $A \setminus E$, and go to step 4.

Step 3: Determine the candidate in A, say y, with the minimum $p(x)$. (If two or more in A have the min $p(x)$ value, choose one at random for y.) Delete y from all ballots, change A to $A \setminus \{y\}$, and go to step 4.

Step 4: Let P be the sum of the p_i for ballots that, because of deletions, have no remaining candidates, set $p_i = 0$ for these ballots and, when n' nonempty ballots remain, increase the p_i of each by adding P/n'. Go to step 1.

Step 4 is used to maintain the current point total when all candidates ranked on a ballot have been elected or deleted. When y is ranked first on a ballot in step 3, its p_i at that point is transferred to the second-ranked candidate if there is one. The *surplus* $p(x) - q$ of points needed for election of a newly elected candidate in step 1 is retained by step 2 for the ballots that rank the elected candidate first, while q points are removed from those ballots, but if the set E of newly elected candidates exhausts all that remain on a ballot, its adjusted points get transferred to other ballots in step 4.

The paradoxes described in Section 7.3 for plurality with a runoff apply to STV when there are three candidates and $k = 1$ [Doron and Kronick (1977), Fishburn and Brams (1983)]. The following examples [Brams (1982), Brams and Fishburn (1984c)] illustrate the mechanics of STV and phenomena associated with truncated rankings.

Example 11.1. Assume that two of four candidates are to be elected, and there are three classes of voters who rank the candidates as follows:

I. 6 voters have *xabc*
II. 6 voters have *xbca*
III. 5 voters have *xcab*.

Then $n = 17$, so $q = \lfloor 17/3 \rfloor + 1 = 6$. The initial point totals are 17 for x and 0 for the others, so x is elected. The surplus of $11 = 17 - 6$ points for x are redistributed in the proportions 6 : 6 : 5 to the classes, so I and II are left with $66/17 \doteq 3.9$ points each, and III is left with $55/17 \doteq 3.2$ points. Candidate x is deleted (step 2) and, since none of the others has q revised points, c is deleted (lowest total, step 3) to give

I. 66/17 points, *ab*
II. 66/17 points, *ba*
III. 55/17 points, *ab*.

Then a (7.1 points) is elected along with x.

Now suppose that two of the six class II voters had ranked only their first choice x. As before, x is elected on the first round. Its deletion, and points reductions of step 2, give

I. 66/17 points, *abc*
II.1. 22/17 points, no remaining candidates
II.2. 44/17 points, *bca*
III. 55/17 points, *cab*.

We now use step 4 with $P = 22/17$ and $n' = 15$:

I. $66/17 + (6/15)(22/17) = 4.4$ points, *abc*
II.2. $44/17 + (4/15)(22/17) = 2.933\cdots$ points, *bca*
III. $55/17 + (5/15)(22/17) = 3.666\cdots$ points, *cab.*

Since none of a, b and c here makes q, the low candidate b is eliminated and c is elected with 6.6 points.

Observe that the two class II voters who ranked only x induced a better second choice (c instead of a) for themselves by submitting truncated ballots. Thus, it may be advantageous not to rank all candidates on one's ballot, contrary to a claim made by a professional society that "there is no tactical advantage to be gained by marking few candidates" [Brams (1982)]. Put another way, one may do better under STV by *not* expressing preferences – at least beyond first choices. ■

Lest one think that an advantage gained by truncation requires allocation of surplus votes, we give a truncation example for $k = 1$. Here STV is similar to plurality with successive elimination (Section 7.1), but with the added feature of ranked ballots.

Example 11.2. Assume that one of four candidates is to be elected by 21 voters:

I. 7 voters have *abcx*
II. 6 voters have *bacx*
III. 5 voters have *cbax*
IV. 3 voters have *xcba.*

Here $q = 11$. No candidate makes q initially, so x is eliminated and a, b and c then have 7, 6 and 8 votes, respectively. Because none of these makes q, b is eliminated and a is elected with $13 = 7 + 6$ votes even though b is the strict Condorcet candidate.

Now suppose the three class IV voters rank only x as their first choice. As before, x is eliminated first, and since the ballots of IV have no other candidates, their 3 points go to the others:

I. $7 + (7/18)3 = 49/6$ points, *abc*
II. $6 + (6/18)3 = 7$ points, *bac*
III. $5 + (5/18)3 = 35/6$ points, *cba.*

Now c is eliminated and b is the winner with $7 + 35/6 \doteq 12.8$ votes. Because the class IV voters prefer b to a, it is in their interest not to rank candidates below x. ■

It is true under STV that a first choice can never be hurt by ranking a second choice, a second choice by ranking a third choice, ..., because higher choices are eliminated before the lower choices can affect them. However, lower choices can affect the *order of elimination* and, hence, transfer of votes. Consequently, a higher choice can influence whether a lower choice is elected.

We do not suggest that voters would routinely make the strategic calculations in Examples 11.1 and 11.2. Such calculations are not only complex but also might be neutralized by counterstrategies of other voters. Rather, the point is that to rank *all* candidates for whom one has preferences is not always rational under STV. Additional discussion of STV's manipulability in this regard, and its relationship to the election of Condorcet candidates, is in Fishburn and Brams (1984).

11.2. Additional-member systems

In most parliamentary democracies, it is not candidates who run for office but political parties that put up lists of candidates. Under *party-list voting*, voters vote for parties, which receive seats in a parliament proportional to the total numbers of votes they receive. There is often a threshold, such as 5% of the total vote, which a party must exceed to gain any seats.

This is a rather straightforward procedure of ensuring proportional representation (PR) of parties that surpass the threshold, though it is not paradox-free with respect to the distribution of seats that take account of the complete preference orders of voters [Van Deemen (1993)]. More interesting are systems in which some legislators are elected from districts, but new members may be added to ensure that parties underrepresented on the basis of their national-vote proportions gain additional seats.

Denmark and Sweden, for example, use votes summed over each party's district candidates as the basis for allocating additional seats. In elections to Germany's Bundestag and Iceland's Parliament, voters vote twice, once for district representatives and once for a party. Half of the Bundestag is chosen from party lists, on the basis of the national party vote, with adjustments to the district results made to ensure approximate PR of parties. Italy, New Zealand, and several Eastern European countries and former Soviet republics have recently adopted similar systems. In Puerto Rico, if the largest party in one house of its bicameral legislature wins more than two-thirds of the seats in district elections, then that house can be increased by as much as one-third to redress underrepresentation of minority parties.

We offer insight into an important strategic feature of additional-member systems by assuming, as in Puerto Rico, that a variable number of additional members can be added to a legislature to adjust for underrepresentation. We consider a procedure, called *adjusted district voting*, or ADV [Brams and Fishburn (1984a,b)], that is characterized by four assumptions:

(1) There is a jurisdiction divided into equal-size districts, each of which elects a single representative to a legislature.
(2) The jurisdiction has two main factions, one majority and one minority, whose sizes can be determined.
(3) The legislature consists of the district winners plus the largest vote-getters among the losers – necessary to achieve PR – if PR is not realized by the district winners. This addition would typically be minority-faction losers in district elections.

(4) The legislature's size is *variable*, with a lower bound equal to the number of districts (if no additions are needed to achieve PR), and an upper bound equal to twice the number of districts (if a nearly 50% minority wins no district election).

To illustrate ADV, suppose the jurisdiction has eight districts with an 80% majority faction and a 20% minority faction. If the minority wins no district election, then its two biggest vote-getters could be given seats in a 10-member parliament that achieves PR exactly.

Now suppose the minority wins one seat, so its initial representation is $\frac{1}{8}$, or about 13%. If it were given an additional seat, its representation would rise to $\frac{2}{9}$ (22%), which is closer than $\frac{1}{8}$ to its 20% proportion in the electorate. Assume, however, that additions can never make its proportion in the legislature exceed its proportion in the electorate, so the addition is not made.

Paradoxically, the minority would benefit by winning no district election. To prevent a minority from benefiting by *losing* in district elections, assume the following *no-benefit constraint*: the allocation of extra seats to the minority can never give it a greater proportion in the legislature than it would obtain had it won more district elections. Because $\frac{1}{9} < \frac{1}{8} < \frac{2}{10}$, this implies that if the minority wins in no district, then it can be given only one rather than two seats for a representation of $\frac{1}{9}$ (11%) rather than $\frac{2}{10}$ (20%).

It can be proved in the general case that the no-benefit constraint may prevent a minority from receiving up to about half of the extra seats it would be entitled to otherwise [Brams and Fishburn (1984a)]. This constraint can be interpreted as a sincere-voting promoter in ADV. It makes it unprofitable for a minority party deliberately to lose district elections in order to do better with extra-seat additions. This comes at a price, however. As our example and its generalization demonstrate, the constraint can severely restrict the ability of ADV to satisfy PR, giving rise to the following dilemma: under ADV, one cannot assure a close correspondence between a party's proportion in the electorate and its representation in the legislature if one insists on the no-benefit constraint; dropping it allows one to approximate PR, but this may give the minority party an incentive purposely to lose in certain district contests in order to do better after the adjustment.

It is worth noting that the "second chance" for minority candidates afforded by ADV would encourage them to run in the first place, because even if most or all lose their district races, their biggest vote-getters would still have a chance at extra seats. But these extra seats might be cut by up to a factor of two from the minority's proportion in the electorate should one want to motivate district elections with the no-benefit constraint. Indeed, [Spafford (1980, p. 393)], anticipating this dilemma, recommended that only an (unspecified) fraction of seats that the minority is entitled to be alloted to it in the adjustment phase to give it "some incentive to take the single-member contests seriously, ..., though that of course would be giving up strict PR".

11.3. Minimizing representational imbalance

We conclude our discussion of PR with a few comments occasioned by Monroe's (1995) proposal to select winning candidates in an election for a legislature by minimizing an aggregate measure of representational imbalance. Such a measure would depend on the ballot type and how ballots are assessed with regard to representativeness, but in any case it is a function of potential winning sets of k candidates.

Let $\mathcal{A} = \{A \subseteq X\colon |A| = k\}$ and for $A \in \mathcal{A}$ let f map $\{1, \ldots, n\}$ into A. If ballots are approval ballots, the misrepresentation score for voter i under f is 0 if $f(i)$ is in voter i's approved set, and is 1 otherwise. If ballots are linear orders, the misrepresentation score for voter i under f is $j - 1$ when $f(i)$ is jth-ranked in voter i's order. The total misrepresentation of assignment f is the sum over i of the voters' misrepresentation scores.

Monroe (1995) suggests that f be restricted so that approximately the same number of voters are assigned to each candidate, or "represented" by each candidate, in A. Subject to this restriction, one then determines the elected set to be an $A \in \mathcal{A}$ for which the minimum total misrepresentation of an f for A is as small as possible. Potthoff and Brams (1998) note that this is the same as a proposal of Chamberlin and Courant (1983) when no restrictions are placed on f (a proposal rejected by Monroe), and that if, in addition, k is unrestricted, a proposal of [Tullock (1967, Chapter 10)] is obtained.

Potthoff and Brams (1998) demonstrate the efficiency of using integer programming to compute a solution for Monroe's procedure as well as for a variety of related procedures. One of these uses an f that maps $\{1, \ldots, n\}$ into h-candidate subsets of A with $1 \leqslant h \leqslant k$, restricted so that each candidate is in the h-candidate subsets of approximately hn/k voters. When $h = k$ with approval ballots, the elected A consists of the k candidates with the greatest approval votes. When $h = k$ with fully ranked ballots, the elected A is the set of k candidates with the most Borda points. Intermediate values of h may be more faithful to the intention of electing a proportionately representative legislature.

12. Conclusions

There is no perfect voting procedure [Niemi and Riker (1976), Fishburn (1984), Nurmi (1986), Amy (2000)], but some procedures are clearly superior to others in satisfying certain criteria.

Among nonranked voting procedures to elect one candidate, approval voting distinguishes itself as more sincere, strategyproof, and likely to elect Condorcet candidates than other procedures, including plurality voting and plurality with a runoff. Its use in earlier centuries in Europe [Cox (1984, 1987a), Lines (1986)], and its recent adoption by a number of professional societies – including the Institute of Management

Sciences [Fishburn and Little (1988)], the Mathematical Association of America [Brams (1988)], the American Statistical Association [Brams and Fishburn (1988)], the Institute of Electrical and Electronics Engineers [Brams and Nagel (1991)], the American Mathematical Society, and the Social Choice and Welfare Society [Brams and Fishburn (2001), Saari (2001a)] – augurs well for its more widespread use, including possible adoption in public elections [Brams (1993), Brams and Herschbach (2001)]. Bills have been introduced in several U.S. state legislatures for its enactment for state primaries, and its consideration has been urged in such countries as Finland [Anckar (1984)] and New Zealand [Nagel (1987)]. Its probable effects in the U.S. presidential elections of 1864 [Tabarrok and Spector (1999)], 1980 [Brams and Fishburn (1983)], 1992 [Brams and Merrill (1994), Tabarrok (2001)], and 2000 [Saari (2001b)], along with the likely effects of other voting procedures such as the Borda count, have been assessed.

Among ranked positional scoring procedures to elect one candidate, Borda's method is superior in many respects, including susceptibility to strategic manipulation, propensity to elect Condorcet candidates, and ability to minimize paradoxical possibilities [Smith (1973), Gehrlein and Fishburn (1978b), Saari (1989, 1990a, 1994, 1995a,b, 2000a,b, 2001b), Chapter 25 in Volume 2 of this Handbook, Van Newenhizen (1992)]. Some Condorcet voting procedures, such as the Schwartz and Kemeny procedures, have a number of attractive properties [Fishburn (1977), Young (1988)], but they have witnessed more theoretical than practical interest. Despite Borda's superiority in many respects, it is easier to manipulate than many other procedures. For example, the strategy of ranking the most serious rival of one's favorite candidate last is a transparent way of diminishing the rival's chances.

While plurality with a runoff, and STV for elections of one or more candidates, are commonly used, they are subject to some of the more noxious paradoxes, including violations of monotonicity which can turn a potential winner into a loser when it rises in the ballot response profile. Additional-member systems, and specifically ADV that results in a variable-size legislature, provide a mechanism for approximating proportional representation in a legislature without the nonmonotonicity of STV or the manipulability of Borda-type procedures. Cumulative voting also offers a means for factions or parties to ensure their proportional representation, but it requires considerable organizational effort on the part of parties. In the face of uncertainty about their level of support in the electorate, party leaders may well make suboptimal choices about how many candidates their supporters should concentrate their votes on, which weakens the argument that cumulative voting can guarantee proportional representation in practice. But the no-benefit constraint on allocation of additional seats to underrepresented parties under ADV – in order to deny them the incentive to throw district races – also vitiates fully satisfying proportional representation, underscoring the difficulties of satisfying a number of desiderata.

An understanding of these difficulties, and possible trade-offs that must be made, facilitates the selection of procedures to meet certain needs. Over the past half century the explosion of results in social choice theory, and the burgeoning decision-theoretic and game-theoretic analyses of different voting procedures, not only enhance one's theoretical understanding of the foundations of social choice but also contribute to the better design of practical voting procedures that satisfy the criteria that one deems important.

References

Abello, J.M. (1981), "Toward a maximum consistent set", Tech. report TRCS 11-81 (University of California, Santa Barbara).

Abello, J.M. (1991), "The weak Bruhat order of S_Σ, consistent sets, and Catalan numbers", SIAM Journal on Discrete Mathematics 4:1–16.

Abello, J.M., and C.R. Johnson (1984), "How large are transitive simple majority domains?", SIAM Journal on Algebraic and Discrete Methods 5:603–618.

Amy, D.J. (2000), Behind the Ballot Box: A Citizen's Guide to Voting Systems (Praeger, Westport, CT).

Anckar, D. (1984), "Presidential elections in Finland: a plea for approval voting", Electoral Studies 3:125–138.

Arrow, K.J. (1950), "A difficulty in the concept of social welfare", Journal of Political Economy 58:328–346.

Arrow, K.J. (1951), Social Choice and Individual Values (Yale University Press, New Haven, CT); 2nd Edition 1963 (Wiley, New York).

Balinski, M.L., and H.P. Young (1982), Fair Representation: Meeting the Ideal of One Man, One Vote (Yale University Press, New Haven, CT).

Balinski, M.L., and H.P. Young (2001), Fair Representation: Meeting the Ideal of One Man, One Vote, 2nd Edition (Brookings Institute, Washington, D.C.).

Banks, J.S. (1985), "Sophisticated voting outcomes and agenda control", Social Choice and Welfare 2:295–306.

Banzhaf III, J.F. (1965), "Weighted voting doesn't work: a mathematical analysis", Rutgers Law Review 19:317–343.

Barberà, S., H. Sonnenschein and L. Zhou (1991), "Voting by committees", Econometrica 59:595–609.

Berga, D. (1998), "Strategy-proofness and single-plateaued preferences", Mathematical Social Sciences 35:105–120.

Black, D. (1958), The Theory of Committees and Elections (Cambridge University Press, Cambridge).

Bock, H.-H., W.H.E. Day and F.R. McMorris (1998), "Consensus rules for committee elections", Mathematical Social Sciences 35:219–232.

Boehm, G.A.W. (1976), "One fervent vote against Wintergreen", Mimeograph.

Bolger, E.M. (1983), "Proportional representation", in: S.J. Brams, W.F. Lucas and P.D. Straffin Jr, eds., Modules in Applied Mathematics, Vol. 2 (Springer, New York) pp. 19–31.

Bolger, E.M. (1985), "Monotonicity and other paradoxes in some proportional representation schemes", SIAM Journal on Algebraic and Discrete Methods 6:283–291.

Borda (J.-C. de Borda) (1781), "Mémoire sur les élections par scrutin", Mémoires de l'Académie Royale des Sciences année 1781, pp. 657–665. Translated in English in 1953 by A. de Grazia: "Mathematical derivation of an election system", Isis 44:42–51.

Bordley, R.F. (1983), "A pragmatic method for evaluating election schemes through simulation", American Political Science Review 77:123–141.

Brams, S.J. (1975), Game Theory and Politics (Free Press, New York).

Brams, S.J. (1977), "When is it advantageous to cast a negative vote?", in: R. Henn and O. Moeschlin, eds., Mathematical Economics and Game Theory: Essays in Honor of Oskar Morgenstern (Springer, Berlin) pp. 564–572.

Brams, S.J. (1978), The Presidential Election Game (Yale University Press, New Haven, CT).

Brams, S.J. (1982), "The AMS nomination procedure is vulnerable to truncation of preferences", Notices of the American Mathematical Society 29:136–138.

Brams, S.J. (1983), "Comparison voting", in: S.J. Brams, W.F. Lucas and P.D. Straffin Jr, eds., Modules in Applied Mathematics, Vol. 2 (Springer, New York) pp. 32–65.

Brams, S.J. (1985), Rational Politics: Decisions, Games, and Strategy (CQ Press, Washington, DC).

Brams, S.J. (1988), "MAA elections produce decisive winners", Focus (Mathematical Association of America) 8:1–2.

Brams, S.J. (1990), "Constrained approval voting: a voting system to elect a governing board", Interfaces 20:65–80.

Brams, S.J. (1993), "Approval voting and the good society", PEGS Newsletter 3:10, 14.

Brams, S.J. (1994), "Voting procedures", in: R.J. Aumann and S. Hart, eds., Handbook of Game Theory, Vol. 2 (Elsevier, Amsterdam) pp. 1055–1089.

Brams, S.J., and M.D. Davis (1973), "Models of resource allocation in presidential campaigning: implications for democratic representation", Annals of the New York Academy of Sciences 219: 105–123.

Brams, S.J., and M.D. Davis (1974), "The 3/2's rule in presidential campaigning", American Political Science Review 68:113–134.

Brams, S.J., and M.D. Davis (1982), "Optimal resource allocation in presidential primaries", Mathematical Social Sciences 3:373–388.

Brams, S.J., and P.C. Fishburn (1978), "Approval voting", American Political Science Review 72: 831–847.

Brams, S.J., and P.C. Fishburn (1981), "Reconstructing voting processes: the 1976 House Majority Leader election under present and alternative rules", Political Methodology 7:95–108.

Brams, S.J., and P.C. Fishburn (1983), Approval Voting (Birkhäuser, Boston).

Brams, S.J., and P.C. Fishburn (1984a), "A note on variable-size legislatures to achieve proportional representation", in: A. Lijphart and B. Grofman, eds., Choosing an Electoral System: Issues and Alternatives (Praeger, New York) pp. 175–177.

Brams, S.J., and P.C. Fishburn (1984b), "Proportional representation in variable-size legislatures", Social Choice and Welfare 1:397–410.

Brams, S.J., and P.C. Fishburn (1984c), "Some logical defects of the single transferable vote", in: A. Lijphart and B. Grofman, eds., Choosing an Electoral System: Issues and Alternatives (Praeger, New York) pp. 147–151.

Brams, S.J., and P.C. Fishburn (1985), "Comment on 'The problem of strategic voting under approval voting' ", American Political Science Review 79:816–818.

Brams, S.J., and P.C. Fishburn (1988), "Does approval voting elect the lowest common denominator?", PS: Political Science & Politics 21:277–284.

Brams, S.J., and P.C. Fishburn (1992), "Coalition voting", in: P.E. Johnson, ed., Mathematical and Computer Modelling 16:15–26.

Brams, S.J., and P.C. Fishburn (1993), "Yes–no voting", Social Choice and Welfare 10:35–50.

Brams, S.J., and P.C. Fishburn (2001), "A nail-biting election", Social Choice and Welfare 18:409–414.

Brams, S.J., and D.R. Herschbach (2001), "The science of elections", Science 292:1449.

Brams, S.J., and D.M. Kilgour (2001), "Fallback bargaining", Group Decision and Negotiation 10: 287–316.

Brams, S.J., and S. Merrill III (1994), "Would Ross Perot have won the 1992 presidential election under approval voting?" PS: Political Science and Politics 27:39–44.

Brams, S.J., and J.H. Nagel (1991), "Approval voting in practice", Public Choice 71:1–17.

Brams, S.J., P.C. Fishburn and S. Merrill III (1988a), "The responsiveness of approval voting: comments on Saari and Van Newenhizen", Public Choice 59:121–131.

Brams, S.J., P.C. Fishburn and S. Merrill III (1988b), "Rejoinder to Saari and Van Newenhizen", Public Choice 59:149.

Brams, S.J., D.M. Kilgour and W.S. Zwicker (1997), "Voting on referenda: the separability problem and possible solutions", Electoral Studies 16:359–377.

Brams, S.J., D.M. Kilgour and W.S. Zwicker (1998), "The paradox of multiple elections", Social Choice and Welfare 15:211–236.

Chamberlin, J.R., and P.N. Courant (1983), "Representative deliberations and representation decisions: proportional representation and the Borda rule", American Political Science Review 77:718–733.

Chamberlin, J.R., and F. Featherston (1986), "Selecting a voting system", Journal of Politics 48:347–369.

Charon, I., O. Hudry and F. Woirgard (1997), "A 16-vertex tournament for which Banks set and Slater set are disjoint", Discrete Applied Mathematics 80:211–215.

Ching, S., and S. Serizawa (1998), "A maximal domain for the existence of strategy-proof rules", Journal of Economic Theory 78:157–166.

Chwe, M.S.-K. (1999), "Minority voting rights can maximize majority welfare", American Political Science Review 93:85–97.

Condorcet (M.J.A.N. de Condorcet) (1785), Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix (Imprimerie Royale, Paris); facsimile published in 1972 by Chelsea Publishing Company, New York.

Copeland, A.H. (1951), "A 'reasonable' social welfare function", Mimeograph (University of Michigan Seminar on Applications of Mathematics to the Social Sciences).

Coughlin, P.J. (1992), Probabilistic Voting Theory (Cambridge University Press, Cambridge).

Cox, G.W. (1984), "Strategic electoral choice in multi-member districts: approval voting in practice", American Journal of Political Science 28:722–738.

Cox, G.W. (1987a), The Efficient Secret: The Cabinet and the Development of Political Parties in Victorian England (Cambridge University Press, New York).

Cox, G.W. (1987b), "Electoral equilibrium under alternative voting institutions", American Journal of Political Science 31: 82–108.

Cox, G.W. (1997), Making Votes Count: Strategic Coordination in the World's Electoral Systems (Cambridge University Press, New York).

Craven, J. (1996), "Majority-consistent preference orderings", Social Choice and Welfare 13:259–267.

Dasgupta, P., P.J. Hammond and E. Maskin (1979), "The implementation of social choice rules: some general results on incentive compatibility", Review of Economic Studies 46:185–216.

Davidson, R.R., and R.E. Odeh (1972), "Some inconsistencies in judging problems", Journal of Combinatorial Theory A 13:162–169.

de Grazia, A. (1953), "Mathematical derivation of an election system" [an English translation of "Mémoire sur les élections par scrutin", by J.-C. de Borda (1781)], Isis 44:42–51.

Debord, B. (1992), "An axiomatic characterization of Borda's k-choice function", Social Choice and Welfare 9:337–343.

DeMeyer, F., and C.R. Plott (1970), "The probability of a cyclical majority", Econometrica 38:345–354.

Dodgson, C.L. (Lewis Carroll) (1876), A Method of Taking Votes on More than Two Issues (Clarendon Press, Oxford). Reprinted, 1958, in: D. Black, ed., The Theory of Committees and Elections (Cambridge University Press, Cambridge) pp. 224–234.

Doron, G. (1979), "The Hare voting system is inconsistent", Political Studies 27:283–286.

Doron, G., and R. Kronick (1977), "Single transferable vote: an example of a perverse social choice function", American Journal of Political Science 21:303–311.

Dubey, P., and L.S. Shapley (1979), "Mathematical properties of the Banzhaf power index", Mathematics of Operations Research 4:99–131.

Dummett, M. (1984), Voting Procedures (Clarendon Press, Oxford).

Dummett, M. (1998), "The Borda count and agenda manipulation", Social Choice and Welfare 15: 287–296.

Dutta, B. (1988), "Covering sets and a new Condorcet choice correspondence", Journal of Economic Theory 44:63–80.

Enelow, J.M., and M.J. Hinich (1984), The Spatial Theory of Election Competition: An Introduction (Cambridge University Press, Cambridge).
Enelow, J.M., and M.J. Hinich, eds. (1990), Advances in the Spatial Theory of Voting (Cambridge University Press, New York).
Epstein, D. (1998), "Uncovering some subtleties of the uncovered set: Social choice theory and distributive politics", Social Choice and Welfare 15:81–93.
Erdős, P., and L. Moser (1964), "On the representation of directed graphs as the unions of orderings", Publications of the Mathematical Institute of the Hungarian Academy of Sciences 9:125–132.
Farquharson, R. (1969), Theory of Voting (Yale University Press, New Haven, CT).
Felsenthal, D.S. (1989), "On combining approval with disapproval voting", Behavioral Science 34: 53–60.
Felsenthal, D.S., and M. Machover (1998), The Measurement of Voting Power (Edward Elgar, Cheltenham, UK).
Felsenthal, D.S., and Z. Maoz (1988), "A comparative analysis of sincere and sophisticated voting under the plurality and approval procedures", Behavioral Science 33:116–130.
Fine, K. (1972), "Some necessary and sufficient conditions for representative decision on two alternatives", Econometrica 40:1083–1090.
Fishburn, P.C. (1970), Utility Theory for Decision Making (Wiley, New York).
Fishburn, P.C. (1971), "The theory of representative majority decision", Econometrica 39:273–284.
Fishburn, P.C. (1973), The Theory of Social Choice (Princeton University Press, Princeton, NJ).
Fishburn, P.C. (1974a), "Paradoxes of voting", American Political Science Review 68:537–546.
Fishburn, P.C. (1974b), "On the sum-of-ranks winner when losers are removed", Discrete Mathematics 8:25–30.
Fishburn, P.C. (1977), "Condorcet social choice functions", SIAM Journal on Applied Mathematics 33:469–489.
Fishburn, P.C. (1978a), "Axioms for approval voting: direct proof", Journal of Economic Theory 19: 180–185; Corrigendum 45 (1988) 212.
Fishburn, P.C. (1978b), "A strategic analysis of nonranked voting systems", SIAM Journal on Applied Mathematics 35:488–495.
Fishburn, P.C. (1979a), "Symmetric and consistent aggregation with dichotomous voting", in: J.-J. Laffont, ed., Aggregation and Revelation of Preferences (North-Holland, Amsterdam) pp. 201–218.
Fishburn, P.C. (1979b), "Heights of representative systems", Discrete Applied Mathematics 1:181–199.
Fishburn, P.C. (1981a), "An analysis of simple voting systems for electing committees", SIAM Journal on Applied Mathematics 41:499–502.
Fishburn, P.C. (1981b), "Inverted orders for monotone scoring rules", Discrete Applied Mathematics 3:27–36.
Fishburn, P.C. (1982), "Monotonicity paradoxes in the theory of elections", Discrete Applied Mathematics 4:119–134.
Fishburn, P.C. (1984), "Discrete Mathematics in voting and group choice", SIAM Journal on Algebraic and Discrete Methods 5:263–275.
Fishburn, P.C. (1987), Interprofile Conditions and Impossibility (Harwood Academic, Chur, Switzerland).
Fishburn, P.C. (1997), "Acyclic sets of linear orders", Social Choice and Welfare 14:113–124.
Fishburn, P.C., and S.J. Brams (1981a), "Approval voting, Condorcet's principle, and runoff elections", Public Choice 36:89–114.
Fishburn, P.C., and S.J. Brams (1981b), "Efficacy, power, and equity under approval voting", Public Choice 37:425–434.
Fishburn, P.C., and S.J. Brams (1981c), "Expected utility and approval voting", Behavioral Science 26:136–142.
Fishburn, P.C., and S.J. Brams (1983), "Paradoxes of preferential voting", Mathematics Magazine 56:207–214.

Fishburn, P.C., and S.J. Brams (1984), "Manipulability of voting by sincere truncation of preferences", Public Choice 44:397–410.

Fishburn, P.C., and W.V. Gehrlein (1976), "An analysis of simple two-stage voting systems", Behavioral Science 21:1–12.

Fishburn, P.C., and W.V. Gehrlein (1977), "An analysis of voting procedures with nonranked voting", Behavioral Science 22:178–185.

Fishburn, P.C., and W.V. Gehrlein (1982), "Majority efficiencies for simple voting procedures: summary and interpretation", Theory and Decision 14:141–153.

Fishburn, P.C., and J.D.C. Little (1988), "An experiment in approval voting", Management Science 34:555–568.

Fishburn, P.C., W.V. Gehrlein and E. Maskin (1979), "Condorcet proportions and Kelly's conjectures", Discrete Applied Mathematics 1:229–252.

Fishburn, P.C., J.F. Fishburn and A. Hagy (1992), "Are there better ways to elect bishops?", Circuit Rider, July–August: 11–12.

Galton, F. (1907), "One vote, one value", Nature 75:414.

Gehrlein, W.V. (1981), "Single stage election procedures for large electorates", Journal of Mathematical Economics 8:263–275.

Gehrlein, W.V. (1982), "Condorcet efficiency and constant scoring rules", Mathematical Social Sciences 2:123–130.

Gehrlein, W.V. (1983), "Condorcet's paradox", Theory and Decision 15:161–197.

Gehrlein, W.V. (1985), "The Condorcet criterion and committee selection", Mathematical Social Sciences 10:199–209.

Gehrlein, W.V. (1993), "Condorcet efficiency of two stage constant scoring rules", Quality and Quantity 27:95–101.

Gehrlein, W.V. (1995), "Condorcet efficiency and social homogeneity", in: C.R. Barnett, H. Moulin, M. Salles and N. Schofield, eds., Social Choice, Welfare and Ethics (Cambridge University Press, Cambridge) pp. 127–143.

Gehrlein, W.V. (1997), "Condorcet's paradox and the Condorcet efficiency of voting rules", Mathematica Japonica 45:173–199.

Gehrlein, W.V. (1999), "Approximating the probability that a Condorcet winner exists", Proceedings of the International Decision Sciences Institute Meeting, Athens, Greece, pp. 626–628.

Gehrlein, W.V., and P.C. Fishburn (1976), "The probability of the paradox of voting: a computable solution", Journal of Economic Theory 13:14–25.

Gehrlein, W.V., and P.C. Fishburn (1978a), "Probabilities of election outcomes for large electorates", Journal of Economic Theory 19:38–49.

Gehrlein, W.V., and P.C. Fishburn (1978b), "Coincidence probabilities for simple majority and positional voting rules", Social Science Research 7:272–283.

Gehrlein, W.V., and P.C. Fishburn (1979), "Proportions of profiles with a majority candidate", Computers and Mathematics with Applications 5:117–124.

Gehrlein, W.V., and P.C. Fishburn (1980), "Robustness of positional scoring over subsets of alternatives", Applied Mathematics and Optimization 6:241–255.

Gehrlein, W.V., B. Gopinath, J.C. Lagarias and P.C. Fishburn (1982), "Optimal pairs of score vectors for positional scoring rules", Applied Mathematics and Optimization 8:309–324.

Gibbard, A.F. (1973), "Manipulation of voting schemes: a general result", Econometrica 41:587–601.

Glasser, G.J. (1959), "Game theory and cumulative voting for corporate directors", Management Science 5:151–156.

Good, I.J. (1971), "A note on Condorcet sets", Public Choice 10:97–101.

Goodman, L.A. (1954), "On methods of amalgamation", in: R.M. Thrall, C.H. Coombs and R.L. Davis, eds., Decision Processes (Wiley, New York) pp. 39–48.

Guilbaud, G.Th. (1952), "Les théories de l'intérêt général et la problème logique de l'agrégation", Économie Appliquee 5:501–584.

Guinier, L. (1994), The Tyranny of the Majority: Fundamental Fairness in Representative Democracy (Free Press, New York).

Hansson, B., and H. Sahlquist (1976), "A proof technique for social choice with variable electorate", Journal of Economic Theory 13:193–200.

Hare, T. (1861), The Election of Representatives, Parliamentary and Municipal: A Treatise (Longman, Green, London).

Henriet, D. (1985), "The Copeland choice function: an axiomatic characterization", Social Choice and Welfare 2:49–63.

Hoag, C.G., and G.H. Hallett (1926), Proportional Representation (Macmillian, New York).

Hoffman, D.T. (1982), "A model of strategic voting", SIAM Journal on Applied Mathematics 42: 751–761.

Hoffman, D.T. (1983), "Relative efficiency of voting systems", SIAM Journal on Applied Mathematics 43:1213–1219.

Holzman, R. (1988–1989), "To vote or not to vote: What is the quota?", Discrete Applied Mathematics 22:133–141.

Inada, K. (1964), "A note on the simple majority decision rule", Econometrica 32:525–531.

Keiding, H. (1984), "Heights of representative systems: a proof of Fishburn's conjecture", Social Choice and Welfare 1:149–158.

Kelly, J.S. (1974), "Voting anomalies, the number of voters, and the number of alternatives", Econometrica 42:239–251.

Kelly, J.S. (1978), Arrow Impossibility Theorems (Academic, New York).

Kelly, J.S. (1987), Social Choice Theory: An Introduction (Springer, New York).

Kemeny, J. (1959), "Mathematics without numbers", Daedalus 88:577–591.

Kim, K.H., and F.W. Roush (1980), Introduction to Mathematical Consensus Theory (Marcel Decker, New York).

Laffond, G., and J.-F. Laslier (1991), "Slater's winners of a tournament may not be in the Banks set", Social Choice and Welfare 8:365–369.

Lake, M. (1979), "A new campaign resource allocation model", in: S.J. Brams, A. Schotter and G. Schwödiauer, eds., Applied Game Theory: Proceedings of a Conference at the Institute for Advanced Studies, Vienna, June 13–16, 1978 (Physica-Verlag, Würzburg) pp. 118–132.

Laslier, J.-F. (1997), Tournament Solutions and Majority Voting (Springer, Berlin).

Le Breton, M., and M. Truchon (1997), "A Borda measure for social choice functions", Mathematical Social Sciences 34:249–272.

Levin, J., and B. Nalebuff (1995), "An introduction to vote-counting schemes", Journal of Economic Perspectives 9:3–26.

Lines, M. (1986), "Approval voting and strategy analysis: a Venetian example", Theory and Decision 20:155–172.

Ludwig, W.G. (1978), "Strategic voting and the Borda method", Public Choice 33:85–90.

May, K.O. (1952), "A set of independent necessary and sufficient conditions for simple majority decision", Econometrica 20:680–684.

May, R.M. (1971), "Some mathematical results on the paradox of voting", Behavioral Science 16: 143–151.

McGarvey, D.C. (1953), "A theorem on the construction of voting paradoxes", Econometrica 21:608–610.

McLean, I. (1996), "E.J. Nanson, social choice and electoral reform", Australian Journal of Political Science 31:369–385.

McLean, I., and A.B. Urken, eds. (1995), Classics of Social Choice (University of Michigan Press, Ann Arbor).

Merrill, S. (1979), "Approval voting: a 'best buy' method for multicandidate elections", Mathematics Magazine 52:98–102.

Merrill, S. (1981), "Strategic decision under one-stage multicandidate voting systems", Public Choice 36:115–134.

Merrill, S. (1982), "Strategic voting in multicandidate elections under uncertainty and under risk", in: M. Holler, ed., Power, Voting, and Voting Power (Physica-Verlag, Würzburg) pp. 179–187.

Merrill III, S. (1984), "A comparison of efficiency of multicandidate electoral systems", American Journal of Political Science 28:23–48.

Merrill III, S. (1985), "A statistical model for Condorcet efficiency using simulation under spatial model assumptions", Public Choice 47:389–403.

Merrill III, S. (1988), Making Multicandidate Elections More Democratic (Princeton University Press, Princeton, NJ).

Merrill III, S., and J. Nagel (1987), "The effect of approval balloting on strategic voting under alternative decision rules", American Political Science Review 81:509–524.

Mill, J.S. (1862), Considerations on Representative Government (Harper, New York).

Miller, N. (1980), "A new solution set for tournaments and majority voting: further graph-theoretical approaches to the theory of voting", American Journal of Political Science 24:68–96.

Miller, N.R. (1995), Committees, Agendas, and Voting (Harwood Academic, Chur).

Miller, N.R., B. Grofman and S.L. Feld (1990), "The structure of the Banks set", Public Choice 66:243–251.

Monroe, B.L. (1995), "Fully proportional representation", American Political Science Review 89:925–940.

Moulin, H. (1980), "On strategy-proofness and single peakedness", Public Choice 35:437–455.

Moulin, H. (1988a), Axioms of Cooperative Decision Making (Cambridge University Press, Cambridge).

Moulin, H. (1988b), "Condorcet's principle implies the no show paradox", Journal of Economic Theory 45:53–64.

Moulin, H. (1994), "Social choice", in: R.J. Aumann and S. Hart, eds., Handbook of Game Theory, Vol. 2 (North-Holland, Amsterdam) pp. 1091–1125.

Murakami, Y. (1966), "Formal structure of majority decision", Econometrica 34:709–718.

Murakami, Y. (1968), Logic and Social Choice (Routledge and Kegan Paul, London).

Myerson, R.B. (1995a), "Analysis of democratic institutions: structure, conduct and performance", Journal of Economic Perspectives 9:77–90.

Myerson, R.B. (1995b), "Axiomatic derivation of scoring rules without the ordering assumption", Social Choice and Welfare 12:59–74.

Nagel, J. (1987), "The approval ballot as a possible component of electoral reform in New Zealand", Political Science 39:70–79.

Nanson, E.J. (1883), "Methods of elections", Transactions and Proceedings of the Royal Society of Victoria 19:197–240.

Nanson, E.J. (1907), Methods of Elections (British Government Blue Book Miscellaneous No. 3).

Niemi, R.G. (1984), "The problem of strategic voting under approval voting", American Political Science Review 78:952–958.

Niemi, R.G., and W.H. Riker (1976), "The choice of voting systems", Scientific American 234:21–27.

Niemi, R.G., and H.F. Weisberg (1968), "A mathematical solution to the problem of the paradox of voting", Behavioral Science 13:317–323.

Nitzan, S. (1985), "The vulnerability of point-voting schemes to preference variation and strategic manipulation", Public Choice 47:349–370.

Nitzan, S., and A. Rubinstein (1981), "A further characterization of Borda ranking method", Public Choice 36:153–158.

Nurmi, H. (1986), "Mathematical models of elections and their relevance for institutional design", Electoral Studies 5:167–182.

Nurmi, H. (1987), Comparing Voting Systems (Reidel, Dordrecht).

Nurmi, H. (1988), "Discrepancies in the outcomes resulting from different voting schemes", Theory and Decision 25:193–208.

Nurmi, H. (1989), "On Nanson's method", in: J. Paastela, ed., Democracy in the Modern World: Essays for Tatu Vanhanen, Acta Univ. Tamperensis A 260:199–210.

Nurmi, H. (1998a), Rational Behavior and the Design of Institutions: Concepts, Theories and Models (Edward Elgar, Cheltenham).

Nurmi, H. (1998b), "Voting paradoxes and referenda", Social Choice and Welfare 15:333–350.

Nurmi, H. (1999), Voting Paradoxes and How to Deal with Them (Springer, Berlin).

Nurmi, H., and Y. Uusi-Heikkilä (1985), "Computer simulations of approval and plurality voting: the frequency of weak Pareto violations and Condorcet loser choices in impartial cultures", European Journal of Political Economy 2(1):47–59.

Peleg, B. (1984), Game-Theoretical Analysis of Voting in Committees (Cambridge University Press, Cambridge).

Potthoff, R.F., and S.J. Brams (1998), "Proportional representation: broadening the options", Journal of Theoretical Politics 10:147–178.

Rapoport, A., and D.S. Felsenthal (1990), "Efficacy in small electorates under plurality and approval voting", Public Choice 64:57–71.

Ray, D. (1986), "On the practical possibility of a 'no show paradox' under the single transferable vote", Mathematical Social Sciences 11:183–189.

Raynaud, H. (1982), "The individual freedom allowed by the value restriction conditions", Tech. report 360 (Center for Research in Organizational Efficiency, Stanford University).

Raz, R. (2000), "VC-dimension of sets of permutations", Combinatorica 20:1–15.

Regenwetter, M., and B. Grofman (1998), "Approval voting, Borda winners, and Condorcet winners: evidence from seven elections", Management Science 44:520–533.

Richelson, J.T. (1978), "A characterization result for the plurality rule", Journal of Economic Theory 19:548–550.

Richelson, J.T. (1979), "A comparative analysis of social choice functions I, II, III: a summary", Behavioral Science 24:355.

Riker, W.H. (1958), "The paradox of voting and Congressional rules for voting on amendments", American Political Science Review 52:349–366.

Riker, W.H. (1982), Liberalism Against Populism: A Confrontation Between the Theory of Democracy and the Theory of Social Choice (Freeman, San Francisco).

Riker, W.H. (1986), The Art of Political Manipulation (Yale University Press, New Haven).

Riker, W.H. (1996), The Strategy of Rhetoric: Campaigning for the American Constitution (Yale University Press, New Haven).

Roberts, F.S. (1991), "Characterization of the plurality function", Mathematical Social Sciences 21: 101–127.

Saari, D.G. (1982), "Inconsistencies of weighted summation voting systems", Mathematics of Operations Research 7:479–490.

Saari, D.G. (1984), "The ultimate of chaos resulting from weighted voting systems", Advances in Applied Mathematics 5:286–308.

Saari, D.G. (1987), "The source of some paradoxes from social choice and probability", Journal of Economic Theory 41:1–22.

Saari, D.G. (1989), "A dictionary for voting paradoxes", Journal of Economic Theory 48:443–475.

Saari, D.G. (1990a), "Susceptibility to manipulation", Public Choice 64:21–41.

Saari, D.G. (1990b), "The Borda dictionary", Social Choice and Welfare 7:279–317.

Saari, D.G. (1992), "Millions of election rankings from a single profile", Social Choice and Welfare 9:277–306.

Saari, D.G. (1994), Geometry of Voting (Springer, New York).

Saari, D.G. (1995a), Basic Geometry of Voting (Springer, New York).

Saari, D.G. (1995b), "A chaotic exploration of voting paradoxes", SIAM Review 37:37–52.

Saari, D.G. (2000a), "Mathematical structure of voting paradoxes I: pairwise vote", Economic Theory 15:1–53.

Saari, D.G. (2000b), "Mathematical structure of voting paradoxes II: positional voting", Economic Theory 15:55–101.

Saari, D.G. (2001a), "Analyzing a nail-biting election", Social Choice and Welfare 18:415–430.

Saari, D.G. (2001b), Chaotic Elections! A Mathematician Looks at Voting (American Mathematical Society, Providence).

Saari, D.G., and J. Van Newenhizen (1988a), "The problem of indeterminacy in approval, multiple, and truncated voting systems", Public Choice 59:101–120.

Saari, D.G., and J. Van Newenhizen (1988b), "Is approval voting an 'unmitigated evil'?: a response to Brams, Fishburn, and Merrill", Public Choice 59:133–147.

Satterthwaite, M.A. (1975), "Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions", Journal of Economic Theory 10: 187–218.

Scarsini, M. (1998), "A strong paradox of multiple elections", Social Choice and Welfare 15:237–238.

Schwartz, T. (1972), "Rationality and the myth of the maximum", Noûs 6:97–117.

Schwartz, T. (1974), "Serial collective choice", Mimeograph (Carnegie-Mellon University).

Schwartz, T. (1986), The Logic of Collective Choice (Columbia University Press, New York).

Sen, A.K., and P.K. Pattanaik (1969), "Necessary and sufficient conditions for rational choice under majority decision", Journal of Economic Theory 1:178–202.

Sertel, M.R. (1978), "Characterizing approval voting", Journal of Economic Theory 45:207–211.

Sertel, M.R., and B. Yilmaz (1999), "The majoritarian compromise is majoritarian-optimal and subgame-perfect implementable", Social Choice and Welfare 16:615–627.

Shapley, L.S., and M. Shubik (1954), "A method for evaluating the distribution of power in a committee system", American Political Science Review 48:787–792.

Shepsle, K.A., and B.R. Weingast (1984), "Uncovered sets and sophisticated voting outcomes with implications for agenda control", American Journal of Political Science 28:49–74.

Slater, P. (1961), "Inconsistencies in a schedule of paired comparisons", Biometrica 48:303–312.

Smith, J.H. (1973), "Aggregation of preferences with variable electorate", Econometrica 41:1027–1041.

Snyder, J.M. (1989), "Election goals and the allocation of political resources", Econometrica 57:637–660.

Spafford, D. (1980), "Book review", Canadian Journal of Political Science 11:392–393.

Staring, M. (1986), "Two paradoxes of committee elections", Mathematics Magazine 59:158–159.

Stavely, E.S. (1972), Greek and Roman Voting and Elections (Cornell University Press, Ithaca, NY).

Stearns, R. (1959), "The voting problem", American Mathematical Monthly 66:761–763.

Straffin Jr, P.D. (1980), Topics in the Theory of Voting (Birkhäuser, Boston).

Tabarrok, A. (2001), "President Perot or fundamentals of voting theory illustrated with the 1992 election", Public Choice 106:275–297.

Tabarrok, A., and L. Spector (1999), "Would the Borda count have avoided the Civil War?", Journal of Theoretical Politics 11:261–288.

Tataru, M., and V. Merlin (1997), "On the relationship of the Condorcet winner and positional voting rules", Mathematical Social Sciences 34:81–90.

Tideman, T.N. (1987), "Independence of clones as a criterion for voting rules", Social Choice and Welfare 4:185–206.

Tideman, T.N. (1995), "The single transferable vote", Journal of Economic Perspectives 9:27–38.

Tullock, G. (1967), Toward a Mathematics of Politics (University of Michigan Press, Ann Arbor, MI).

Van Deemen, A. (1993), "Paradoxes of voting in list systems of proportional representation", Electoral Studies 12:234–241.

Van Newenhizen, J. (1992), "The Borda method is most likely to respect the Condorcet principle", Economic Theory 2:69–83.

Ward, B. (1965), "Majority voting and alternative forms of public enterprise", in: J. Margolis, ed., The Public Economy of Urban Communities (Johns Hopkins University Press, Baltimore, MD) pp. 112–126.

Weber, R.J. (1977), "Comparison of voting systems", Discussion Paper 498A (Cowles Foundation, Yale University).

Weber, R.J. (1995), "Approval voting", Journal of Economic Perspectives 9:37–49.

Young, H.P. (1974), "An axiomatization of Borda's rule", Journal of Economic Theory 9:43–52.

Young, H.P. (1975a), "Social choice scoring functions", SIAM Journal on Applied Mathematics 28: 824–838.

Young, H.P. (1975b), "Extending Condorcet's rule", Mimeograph (The City University of New York).

Young, H.P. (1988), "Condorcet's theory of voting", American Political Science Review 82:1231–1244.

Young, H.P. (1995), "Optimal voting rules", Journal of Economic Perspectives 9:51–64.

Young, H.P., and A. Levenglick (1978), "A consistent extension of Condorcet's election principle", SIAM Journal on Applied Mathematics 35:285–300.

Zwicker, W.S. (1991), "The voters' paradox, spin, and the Borda count", Mathematical Social Sciences 22:187–227.