Markov Chains and Algorithmic Applications: WEEK 1

1 Markov chains: basic definitions

Definitions 1.1. A **time-homogeneous Markov chain** is a discrete-time stochastic process (X_n, $n \geq 0$) with values in a finite or countable set S (the state space) such that:

$$\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) \underset{\substack{\uparrow \\ \text{Markov property}}}{=} \mathbb{P}(X_{n+1} = j | X_n = i) \underset{\substack{\uparrow \\ \text{time-homogeneity}}}{=} p_{ij} \text{ (independent of n)}$$

for every $n \geq 0$ and $j, i, i_{n-1}, \ldots, i_1, i_o \in S$.

The **transition matrix** of the chain is the matrix $P = (p_{ij})_{i,j \in S}$ defined as $p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i)$. It satisfies the following properties:

$$0 \leq p_{i,j} \leq 1 \quad \forall i, j \in S \quad \text{and} \quad \sum_{j \in S} p_{i,j} = \sum_{j \in S} \mathbb{P}(X_{n+1} = j | X_n = i) = 1 \quad \forall i \in S$$

Note however that for a given $j \in S$, $\sum_{i \in S} p_{i,j}$ can be anything.

The **transition graph** of the chain is the oriented graph where vertices are states and an arrow from i to j exists if and only if $p_{ij} > 0$, taking value p_{ij} when it exists.

The **distribution of the Markov chain** at time $n \geq 0$ is given by:

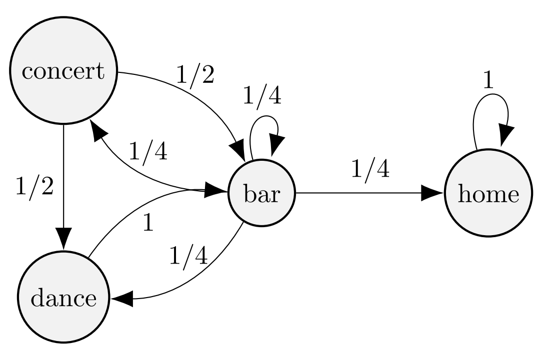
$$\pi_i^{(n)} = \mathbb{P}(X_n = i) \quad i \in S$$

and its **initial distribution** is given by:

$$\pi_i^{(0)} = \mathbb{P}(X_0 = i) \quad i \in S$$

For every $n \geq 0$, we have $\sum_{i \in S} \pi_i^{(n)} = 1$.

Example 1.2. Music festival



The state space is here $S = \{\text{concert, dance, bar, home}\}$ and the transition matrix is given by (with this ordering of the states):

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example 1.3. Simple symmetric random walk.

State space : $\mathbb{Z}$. Let $(X_n, n \geq 1)$ be a sequence of i.i.d. random variables taking values 1 or -1 with probability 1/2. Then the process $(S_n,\, n \geq 0)$ defined as $S_0 = 0$ and $S_n = X_1 + ... + X_n$ for $n \geq 1$ is a Markov chain. Indeed:

$$\mathbb{P}(S_{n+1} = j | S_n = i, \ldots, S_0 = i_0) = \mathbb{P}(S_n + X_{n+1} = j | S_n = i, \ldots, S_0 = i_0)$$
$$= \mathbb{P}(X_{n+1} = j - i | S_n = i, \ldots, S_0 = i_0) = \mathbb{P}(X_{n+1} = j - i) = \begin{cases} 1/2 \text{ if } |j-i| = 1 \\ 0 \text{ otherwise} \end{cases}$$

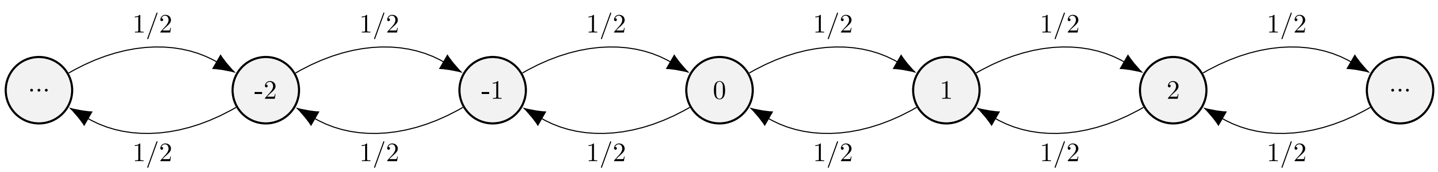
Similarly,

$$\mathbb{P}(S_{n+1} = j | S_n = i) = \mathbb{P}(S_n + X_{n+1} = j | S_n = i)$$
$$= \mathbb{P}(X_{n+1} = j - i | S_n = i) = \mathbb{P}(X_{n+1} = j - i) = \begin{cases} 1/2 \text{ if } |j-i| = 1 \\ 0 \text{ otherwise} \end{cases}$$

which proves the claim.

The transition "matrix" here is actually an operator, as the state space is infinite, but we can simply write that $p_{ij} = 1/2$ if $|j - i| = 1$, 0 otherwise.

The transition graph is given by:



Here are now **two main questions** that will retain our attention for the first part of the course:

A. When does $\pi^{(n)}$ (the distribution at time n) converge as $n \to \infty$ to some limiting distribution π?

B. When it converges, at what rate does is converge? (is $\pi^{(n)}$ any close to π for a given value of n?)

Definition 1.4. m-step transition probabilities For $m \geq 1$ and $i, j \in S$, we define:

$$p_{ij}^{(m)} = \mathbb{P}(X_{n+m} = j | X_n = i) = \mathbb{P}(X_m = j | X_0 = i)$$

where the second equality comes from the time-homogeneity property. We also define by convention:

$$p_{ij}^{(0)} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

How to compute these probabilities? Using the **Chapman-Kolmogorov equations**. For $m = 2$, these read:

$$p_{ij}^{(2)} = \sum_{k \in S} p_{ik}\, p_{kj} = (P \cdot P)_{ij} = (P^2)_{ij}$$

Indeed, we check that

$$p_{ij}^{(2)} = \mathbb{P}(X_2 = j | X_0 = i) = \sum_{k \in S} \mathbb{P}(X_2 = j, X_1 = k | X_0 = i)$$
$$= \sum_{k \in S} \mathbb{P}(X_2 = j | X_1 = k, X_0 = i)\, \mathbb{P}(X_1 = k | X_0 = i) = \sum_{k \in S} p_{ik}\, p_{kj}$$

where we used the Markov property in the last equality.

For higher values of m and $0 \leq \ell \leq m$, Chapman-Kolmogorov equations read:

$$p_{ij}^{(m)} = \sum_{k \in S} p_{ik}^{(\ell)} \, p_{kj}^{(m-\ell)} = (P^\ell \cdot P^{m-\ell})_{ij} = (P^m)_{ij}$$

and the proof goes along the same lines.

Example 1.5. Simple symmetric random walk:

$$p_{00}^{(2n)} = \binom{2n}{n} \frac{1}{2^{2n}}, \quad n \geq 1$$

2 Classification of states

Definitions 2.1. Two states $i, j \in S$ **communicate** ("$i \longleftrightarrow j$") if $\exists n, m \geq 0$ such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$.

The "communicate" relation is an **equivalence relation**: reflexive, symmetric and transitive. The first two are obvious, and the transitivity can be checked by using the above Chapman-Kolmogorov equations:

If $\exists n, m$ s.t. $p_{ij}^{(n)} > 0$ and $p_{jk}^{(m)} > 0$, then $p_{ik}^{(n+m)} = \sum_{l \in S} p_{il}^{(n)} \, p_{lk}^{(m)} \geq p_{ij}^{(n)} \, p_{jk}^{(m)} > 0$

The state space S can be therefore be partitioned into disjoint **equivalence classes**.

A chain is said to **irreducible** if all states communicate (a single class).

A state i is said to be **absorbing** if $p_{ii} = 1$.

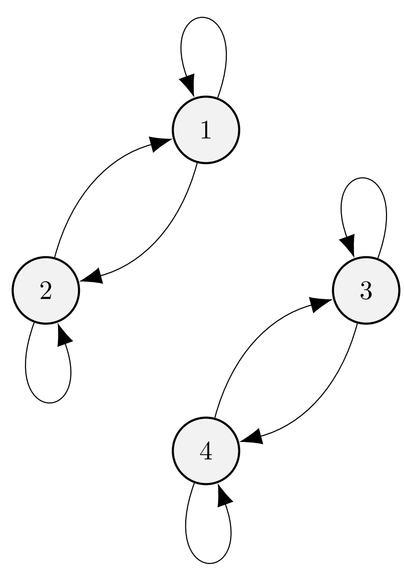


Figure 1: Nodes 1 and 2 are in the same class, while nodes 3 and 4 are in another class.

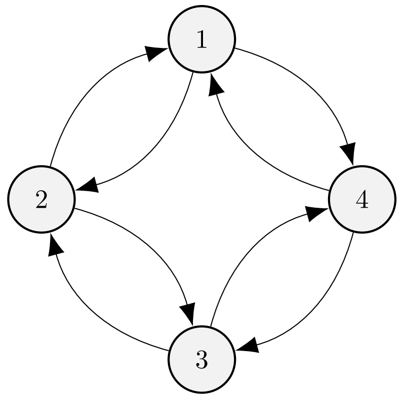
Definition 2.2. Periodicity. For a state $i \in S$, define $d_i = \gcd(\{n \geq 1 \,:\, p_{ii}^{(n)} > 0\})$. If $d_i = 1$, we say that state i is aperiodic. Else if $d_i > 1$, we say that state i is periodic with period d_i.

Facts.

In a given equivalence class, all states have the same period $d_i = d$.

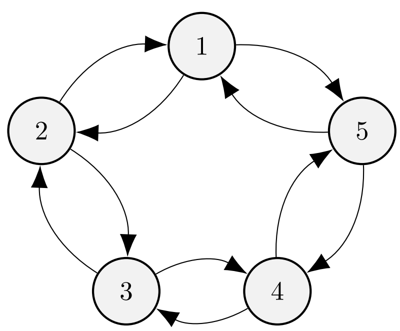
If there is at least on self-loop in the class ($\exists i \in S$ s.t.$p_{ii} \neq 0$), then all states in the class are aperiodic.

Example 2.3. Periodic and aperiodic chains



$$p_{11}^{(1)} = 0$$
$$p_{11}^{(2)} > 0$$
$$p_{11}^{(3)} = 0$$
$$p_{11}^{(4)} > 0$$

$d_1 = 2 = d$ so it is **periodic**



$$p_{11}^{(1)} = 0$$
$$p_{11}^{(2)} > 0$$
$$p_{11}^{(3)} = 0$$
$$p_{11}^{(4)} > 0$$
$$p_{11}^{(5)} > 0$$

$d_1 = 1 = d$ so it is **aperiodic**