Homework 2 Solution
Traitement Quantique de l'Information

1) The "Ket" and the associated Dirac or usual vector notations are:

- $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\langle H| = \begin{pmatrix} 1 & 0 \end{pmatrix}$
- $|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\langle H| = \begin{pmatrix} 0 & 1 \end{pmatrix}$
- $\alpha |H\rangle + \beta |V\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $\alpha^* \langle H| + \beta^* \langle V| = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$

2) In Dirac notation:

$$\begin{aligned} &(\gamma^* \langle H| + \delta^* \langle V|)(\alpha |H\rangle + \beta |V\rangle) \\ &= \gamma^*\alpha \langle H|H\rangle + \gamma^*\beta \langle H|V\rangle + \delta^*\alpha \langle V|H\rangle + \delta^*\beta \langle V|V\rangle \\ &= \gamma^*\alpha + \delta^*\beta \end{aligned}$$

because $\langle H|V\rangle = \langle V|H\rangle = 0$ and $\langle H|H\rangle = \langle V|V\rangle = 1$.
The equivalent vector notation is

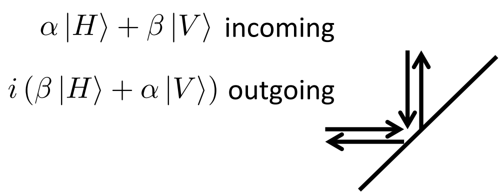
$$\begin{pmatrix} \gamma^* & \delta^* \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \gamma^*\alpha + \delta^*\beta.$$

3) We have $R^\top = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $R^\dagger = R^{\top,*} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$. Thus

$$\begin{aligned} RR^\dagger &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ R^\dagger R &= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Matrices satisfying $MM^\dagger = M^\dagger M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are called unitary matrices.
Let us compute $R(\alpha |H\rangle + \beta |V\rangle)$ in Dirac notation. By linearity of matrix operations,

$$\begin{aligned} R(\alpha |H\rangle + \beta |V\rangle) &= \alpha R |H\rangle + \beta R |V\rangle \\ &= \alpha i |V\rangle + \beta i |H\rangle \\ &= i(\alpha |V\rangle + \beta |H\rangle). \end{aligned}$$

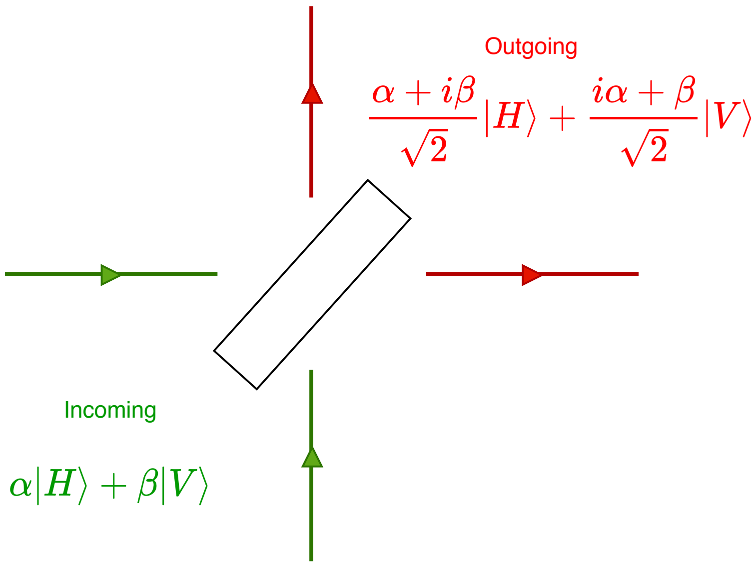


4) We have

$$S\,|H\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\left(|H\rangle + i\,|V\rangle\right),$$

$$S\,|V\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\1\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\left(i\,|H\rangle + |V\rangle\right),$$

$$\begin{aligned}S\left(\alpha\,|H\rangle + \beta\,|V\rangle\right) &= \alpha S\,|H\rangle + \beta S\,|V\rangle\\ &= \frac{\alpha}{\sqrt{2}}\left(|H\rangle + i\,|V\rangle\right) + \frac{\beta}{\sqrt{2}}\left(i\,|H\rangle + |V\rangle\right)\\ &= \frac{\alpha + i\beta}{\sqrt{2}}\,|H\rangle + \frac{i\alpha + \beta}{\sqrt{2}}\,|V\rangle\,.\end{aligned}$$



5) The semi-transparent mirror leaves photons in state $S\big(\alpha\,|H\rangle + \beta\,|V\rangle\big)$. The state is then measured by the detector D which detects photons in state $|V\rangle$. Therefore, the probability of finding a photon in D is the probability of finding a photon in state $|V\rangle$ given that

photons in state $S\big(\alpha\,|H\rangle + \beta\,|V\rangle\big)$ are produced. By the measurement postulate (which will be formally introduced in Chapter 3 of the lecture note) we have

$$\mathrm{Prob}(D) = \big|\langle V|\, S\big(\alpha\,|H\rangle + \beta\,|V\rangle\big)\big|^2 .$$

From the previous question we have

$$S\left(\alpha\,|H\rangle + \beta\,|V\rangle\right) = \frac{\alpha + i\beta}{\sqrt{2}}\,|H\rangle + \frac{i\alpha + \beta}{\sqrt{2}}\,|V\rangle\,,$$

$$\begin{aligned}\langle V|\, S\left(\alpha\,|H\rangle + \beta\,|V\rangle\right) &= \frac{\alpha + i\beta}{\sqrt{2}}\,\langle V|H\rangle + \frac{i\alpha + \beta}{\sqrt{2}}\,\langle V|V\rangle \\ &= \frac{i\alpha + \beta}{\sqrt{2}}.\end{aligned}$$

So we find

$$\mathrm{Prob}(D) = \left|\frac{i\alpha + \beta}{\sqrt{2}}\right|^2 = \frac{1}{2}|i\alpha + \beta|^2 = \frac{1}{2}\big(\alpha^2 + \beta^2\big) = \frac{1}{2}.$$

6) The state after S is

$$S\,|H\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle + i\,|V\rangle\right)$$

The state after R is

$$\begin{aligned}RS\,|H\rangle &= \frac{1}{\sqrt{2}}\left(R\,|H\rangle + iR\,|V\rangle\right) \\ &= \frac{i}{\sqrt{2}}\left(|V\rangle + i\,|H\rangle\right).\end{aligned}$$

The state after the second S is

$$\begin{aligned}SRS\,|H\rangle &= \frac{i}{\sqrt{2}}\left(S\,|V\rangle + iS\,|H\rangle\right) \\ &= \frac{i}{\sqrt{2}}\left(\frac{i\,|H\rangle + |V\rangle}{\sqrt{2}} + i\cdot\frac{|H\rangle + i\,|V\rangle}{\sqrt{2}}\right) \\ &= -\,|H\rangle\,.\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{Prob}(D_1) &= |\langle V|H\rangle|^2 = 0 \\ \mathrm{Prob}(D_2) &= |\langle H|H\rangle|^2 = 1.\end{aligned}$$

All photons are detected in D_2! For "classical balls" we would expect a split between D_1 and D_2. For example, if S act as half–half splitters we would expect $\mathrm{Prob}(D_1) = \mathrm{Prob}(D_2) = 1/2$. **The quantum behavior is completely different**!