Homework 3 Solution
Traitement Quantique de l'Information

Exercise 1 *Orthonormal basis and measurement principle*

1) It involves the following checking:

$$\begin{aligned}
\langle\alpha|\alpha\rangle &= \big(\cos\alpha\,\langle x| + \sin\alpha\,\langle y|\,\big)\big(\cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle\,\big)\\
&= \cos^2\alpha + \sin^2\alpha = 1\\
\langle\alpha_\perp|\alpha_\perp\rangle &= \big(-\sin\alpha\,\langle x| + \cos\alpha\,\langle y|\,\big)\big(-\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle\,\big)\\
&= \cos^2\alpha + \sin^2\alpha = 1\\
\langle\alpha_\perp|\alpha\rangle &= \big(-\sin\alpha\,\langle x| + \cos\alpha\,\langle y|\,\big)\big(\cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle\,\big)\\
&= -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha = 0
\end{aligned}$$

$$\begin{aligned}
\langle R|R\rangle &= \frac{1}{2}\big(\,\langle x| - i\,\langle y|\,\big)\big(\,|x\rangle + i\,|y\rangle\,\big) = \frac{1}{2}(1^2 + (-i)i) = 1\\
\langle L|L\rangle &= \frac{1}{2}\big(\,\langle x| + i\,\langle y|\,\big)\big(\,|x\rangle - i\,|y\rangle\,\big) = \frac{1}{2}(1^2 + i(-i)) = 1\\
\langle R|L\rangle &= \frac{1}{2}\big(\,\langle x| - i\,\langle y|\,\big)\big(\,|x\rangle - i\,|y\rangle\,\big) = \frac{1}{2}(1^2 + (-i)(-i)) = 0
\end{aligned}$$

2) For each experiment, the possible states just after the measurement would be the corresponding measurement basis with the following probabilities:

$$\begin{aligned}
\text{Prob}(|x\rangle) &= |\langle x|\psi\rangle|^2 = \cos^2\theta\\
\text{Prob}(|y\rangle) &= |\langle y|\psi\rangle|^2 = |(\sin\theta)e^{i\varphi}|^2 = \sin^2\theta
\end{aligned}$$

where we use $e^{i\varphi} = \cos\varphi + i\sin\varphi$ so that $|e^{i\varphi}|^2 = \cos^2\varphi + \sin^2\varphi = 1$. For the other probabilities we have:

$$\begin{aligned}
\text{Prob}(|R\rangle) &= |\langle R|\psi\rangle|^2\\
&= \left|\frac{1}{\sqrt{2}}\left(\langle x| - i\,\langle y|\right)\left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle\right)\right|^2\\
&= \left|\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}i\sin\theta e^{i\varphi}\right|^2\\
&= \left|\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\sin\varphi - \frac{1}{\sqrt{2}}i\sin\theta\cos\varphi\right|^2\\
&= \left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\sin\varphi\right)^2 + \left(\frac{1}{\sqrt{2}}\sin\theta\cos\varphi\right)^2\\
&= \frac{1}{2} + \cos\theta\sin\theta\sin\varphi
\end{aligned}$$

$$
\begin{aligned}
\text{Prob}(|L\rangle) &= |\langle L|\psi\rangle|^2 \\
&= \left|\frac{1}{\sqrt{2}}\left(\langle x| + i\langle y|\right)\left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}} i\sin\theta e^{i\varphi}\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\sin\varphi + \frac{1}{\sqrt{2}} i\sin\theta\cos\varphi\right|^2 \\
&= \left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\sin\varphi\right)^2 + \left(\frac{1}{\sqrt{2}}\sin\theta\cos\varphi\right)^2 \\
&= \frac{1}{2} - \cos\theta\sin\theta\sin\varphi
\end{aligned}
$$

$$
\begin{aligned}
\text{Prob}(|\alpha\rangle) &= |\langle\alpha|\psi\rangle|^2 \\
&= \left|(\cos\alpha\,\langle x| + \sin\alpha\,\langle y|)\left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle\right)\right|^2 \\
&= \left|\cos\alpha\cos\theta + \sin\alpha\sin\theta e^{i\varphi}\right|^2 \\
&= |\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\varphi + i\sin\alpha\sin\theta\sin\varphi|^2 \\
&= (\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\varphi)^2 + (\sin\alpha\sin\theta\sin\varphi)^2 \\
&= \cos^2\alpha\cos^2\theta + 2\cos\alpha\sin\alpha\cos\theta\sin\theta\cos\varphi + \sin^2\alpha\sin^2\theta \\
\text{Prob}(|\alpha_\perp\rangle) &= |\langle\alpha_\perp|\psi\rangle|^2 \\
&= \left|(\sin\alpha\,\langle x| + \cos\alpha\,\langle y|)\left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle\right)\right|^2 \\
&= \left|-\sin\alpha\cos\theta + \cos\alpha\sin\theta e^{i\varphi}\right|^2 \\
&= |-\sin\alpha\cos\theta + \cos\alpha\sin\theta\cos\varphi + i\cos\alpha\sin\theta\sin\varphi|^2 \\
&= (\sin\alpha\cos\theta - \cos\alpha\sin\theta\cos\varphi)^2 + (\cos\alpha\sin\theta\sin\varphi)^2 \\
&= \sin^2\alpha\cos^2\theta - 2\cos\alpha\sin\alpha\cos\theta\sin\theta\cos\varphi + \cos^2\alpha\sin^2\theta
\end{aligned}
$$

One can verify that these probabilities are normalized to one,

$$\text{Prob}(|x\rangle) + \text{Prob}(|y\rangle) = \text{Prob}(|R\rangle) + \text{Prob}(|L\rangle) = \text{Prob}(|\alpha\rangle) + \text{Prob}(|\alpha_\perp\rangle) = 1.$$

Exercise 2 *Interferometer revisited*

1) Using Exercise 2.3, we have

$$
\begin{aligned}
S &= |H\rangle\langle H| + i\,|H\rangle\langle V| + i\,|V\rangle\langle H| + |V\rangle\langle V| \\
R &= i\,|H\rangle\langle V| + i\,|V\rangle\langle H|\,.
\end{aligned}
$$

2) The computation in Dirac's notation is

$$
\begin{aligned}
RS &= \frac{1}{\sqrt{2}}\left(-\,|H\rangle\langle H| + i\,|H\rangle\langle V| + i\,|V\rangle\langle H| - |V\rangle\langle V|\right), \\
SRS &= -\,|H\rangle\langle H| - |V\rangle\langle V|\,.
\end{aligned}
$$

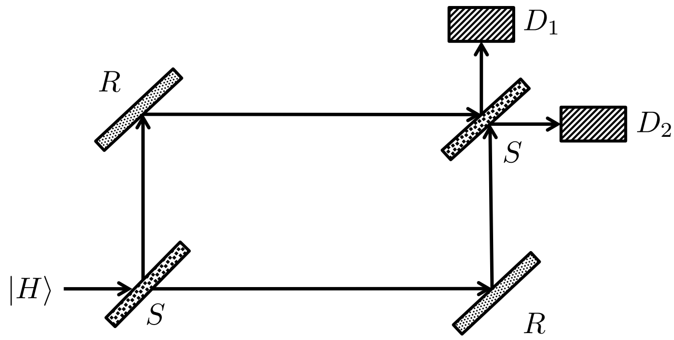
The computation in usual matrix notation is

$$SRS = \frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}-1 & i\\ i & -1\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}.$$

We have

$$\begin{aligned} SRS\,|H\rangle &= \big(-|H\rangle\langle H| - |V\rangle\langle V|\big)\,|H\rangle = -\,|H\rangle \\ |\langle H|\,SRS\,|H\rangle|^2 &= |-\langle H|H\rangle|^2 = 1 \\ |\langle V|\,SRS\,|H\rangle|^2 &= |-\langle V|H\rangle|^2 = 0 \end{aligned}$$

The experimental set-up:



3) We have

$$SRDS = \frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}e^{i\varphi_1} & 0\\ 0 & e^{i\varphi_2}\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-e^{i\varphi_1} - e^{i\varphi_2} & -ie^{i\varphi_1} + ie^{i\varphi_2}\\ ie^{i\varphi_1} - ie^{i\varphi_2} & -e^{i\varphi_1} - e^{i\varphi_2}\end{pmatrix}$$

and

$$SRDS\,|H\rangle = \frac{1}{2}\begin{pmatrix}-e^{i\varphi_1} - e^{i\varphi_2}\\ ie^{i\varphi_1} - ie^{i\varphi_2}\end{pmatrix},$$

which in Dirac notation is

$$SRDS\,|H\rangle = -\frac{e^{i\varphi_1} + e^{i\varphi_2}}{2}\,|H\rangle + \frac{ie^{i\varphi_1} - ie^{i\varphi_2}}{2}\,|V\rangle.$$

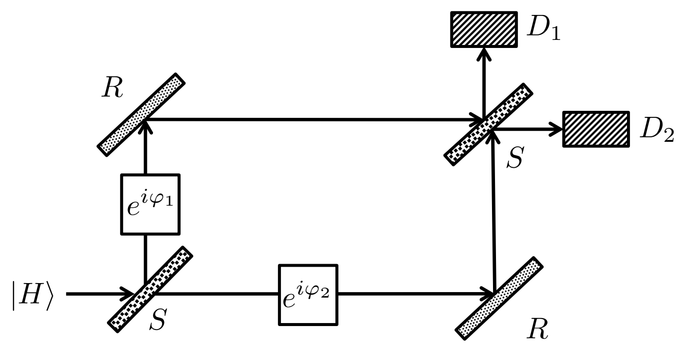
Then we have

$$\begin{aligned} |\langle H|\,SRDS\,|H\rangle|^2 &= \left|\frac{e^{i\varphi_1} + e^{i\varphi_2}}{2}\right|^2 \\ &= \frac{1}{4}\left|\cos\varphi_1 + i\sin\varphi_1 + \cos\varphi_1 + i\sin\varphi_2\right|^2 \\ &= \frac{1}{4}\left((\cos\varphi_1 + \cos\varphi_2)^2 + (\sin\varphi_1 + \sin\varphi_2)^2\right) \\ &= \frac{1}{4}\left(2 + 2\cos\varphi_1\cos\varphi_2 + 2\sin\varphi_1\sin\varphi_2\right) \\ &= \frac{1}{2}(1 + \cos(\varphi_1 - \varphi_2)) \\ &= \cos^2\left(\frac{\varphi_1 - \varphi_2}{2}\right) \end{aligned}$$

and

$$
\begin{aligned}
|\langle V|\,SRDS\,|H\rangle|^2 &= \left|\frac{ie^{i\varphi_1} - ie^{i\varphi_2}}{2}\right|^2 \\
&= \frac{1}{4}\left|-\sin\varphi_1 + i\cos\varphi_1 + \sin\varphi_1 - i\cos\varphi_2\right|^2 \\
&= \frac{1}{4}\left((\sin\varphi_1 - \sin\varphi_2)^2 + (\cos\varphi_1 - \cos\varphi_2)^2\right) \\
&= \frac{1}{4}\left(2 - 2\cos\varphi_1\cos\varphi_2 - 2\sin\varphi_1\sin\varphi_2\right) \\
&= \frac{1}{2}(1 - \cos(\varphi_1 - \varphi_2)) \\
&= \sin^2\left(\frac{\varphi_1 - \varphi_2}{2}\right)
\end{aligned}
$$

The experimental set-up:



A proof that $SRDS$ is unitary: Recall the notation $A^\dagger = A^{\top,*}$. We have checked that $SS^\dagger = S^\dagger S = I$ and $RR^\dagger = R^\dagger R = I$ in Homework 2. It is also easy to check $DD^\dagger = D^\dagger D = I$. The product of unitary matrices is unitary, indeed

$$(U_1U_2)(U_1U_2)^\dagger = U_1U_2U_2^\dagger U_1^\dagger = U_1U_1^\dagger = I.$$

Remarks: The matrix elements are $\begin{pmatrix} \langle H|\,SRDS\,|H\rangle & \langle H|\,SRDS\,|V\rangle \\ \langle V|\,SRDS\,|H\rangle & \langle V|\,SRDS\,|V\rangle \end{pmatrix}$.

- We saw that for a unitary matrix the sum of the modulus squares of rows or columns equal 1. For example for the first column we have $|\langle H|\,SRDS\,|H\rangle|^2 + |\langle V|\,SRDS\,|H\rangle|^2 = 1$. This expresses the fact that the two probabilities of finding the photon in state $\langle H|$ or $\langle V|$ after the measurement is 1.
- Similarly if we would do an experiment with a photon coming in state $|V\rangle$ when it enters the interferometer, the probabilities of finding it in state $\langle H|$ or $\langle V|$ at the detectors should sum to 1. This means $|\langle H|\,SRDS\,|V\rangle|^2 + |\langle V|\,SRDS\,|V\rangle|^2 = 1$ which is the sum of the modulus squares of the second column.
- In fact for each of the sum of columns (or rows) that sums to 1, there is an experimental interpretation.