Homework 4 Solution
Traitement Quantique de l'Information

Exercise 1 *Polarization observable and measurement principle*

1) In Homework 3, we have checked that $\langle\alpha|\alpha\rangle = \langle\alpha_\perp|\alpha_\perp\rangle = 1$ and $\langle\alpha|\alpha_\perp\rangle = \langle\alpha_\perp|\alpha\rangle = 0$. Therefore, we have

$$\begin{aligned}
\Pi_\alpha^2 &= |\alpha\rangle\langle\alpha|\alpha\rangle\langle\alpha| = |\alpha\rangle\langle\alpha| = \Pi_\alpha \\
\Pi_{\alpha_\perp}^2 &= |\alpha_\perp\rangle\langle\alpha_\perp|\alpha_\perp\rangle\langle\alpha_\perp| = |\alpha_\perp\rangle\langle\alpha_\perp| = \Pi_{\alpha_\perp} \\
\Pi_\alpha\Pi_{\alpha_\perp} &= |\alpha\rangle\langle\alpha|\alpha_\perp\rangle\langle\alpha_\perp| = 0 \\
\Pi_{\alpha_\perp}\Pi_\alpha &= |\alpha_\perp\rangle\langle\alpha_\perp|\alpha\rangle\langle\alpha| = 0
\end{aligned}$$

2)

$$\begin{aligned}
|\langle\theta|\alpha\rangle|^2 &= \langle\theta|\alpha\rangle\langle\theta|\alpha\rangle^* = \langle\theta|\alpha\rangle\langle\alpha|\theta\rangle = \langle\theta|\Pi_\alpha|\theta\rangle, \\
|\langle\theta|\alpha_\perp\rangle|^2 &= \langle\theta|\alpha_\perp\rangle\langle\theta|\alpha_\perp\rangle^* = \langle\theta|\alpha_\perp\rangle\langle\alpha_\perp|\theta\rangle = \langle\theta|\Pi_{\alpha_\perp}|\theta\rangle
\end{aligned}$$

3) The probabilities are

$$\begin{aligned}
\mathrm{Prob}(p=+1) &= |\langle\alpha|\theta\rangle|^2 = |\cos\alpha\cos\theta + \sin\alpha\sin\theta|^2 = (\cos(\theta-\alpha))^2 \\
\mathrm{Prob}(p=-1) &= |\langle\alpha_\perp|\theta\rangle|^2 = |-\sin\alpha\cos\theta + \cos\alpha\sin\theta|^2 = (\sin(\theta-\alpha))^2
\end{aligned}$$

and they sum to 1,

$$\mathrm{Prob}(p=+1) + \mathrm{Prob}(p=-1) = (\cos(\theta-\alpha))^2 + (\sin(\theta-\alpha))^2 = 1$$

4) The expectation is

$$\begin{aligned}
\mathrm{E}[p] &= (+1)\mathrm{Prob}(p=+1) + (-1)\mathrm{Prob}(p=-1) \\
&= (\cos(\theta-\alpha))^2 - (\sin(\theta-\alpha))^2 \\
&= \cos\big(2(\theta-\alpha)\big)
\end{aligned}$$

and the variance is

$$\begin{aligned}
\mathrm{Var}(p) &= \mathrm{E}[p^2] - (\mathrm{E}[p])^2 \\
&= 1 - (\mathrm{E}[p])^2 \\
&= (\cos(\theta-\alpha))^2 - (\sin(\theta-\alpha))^2 \\
&= 1 - \big(\cos(2(\theta-\alpha))\big)^2 \\
&= \big(\sin(2(\theta-\alpha))\big)^2
\end{aligned}$$

In fact they should match with the computation in Dirac notation because

$$\begin{aligned}\langle\theta| P_\alpha |\theta\rangle &= \langle\theta| \big((+1)\Pi_\alpha + (-1)\Pi_{\alpha_\perp}\big) |\theta\rangle \\ &= (+1)\langle\theta| \Pi_\alpha |\theta\rangle + (-1)\langle\theta| \Pi_{\alpha_\perp} |\theta\rangle \\ &= (+1)\mathrm{Prob}(p=+1) + (-1)\mathrm{Prob}(p=-1) \\ &= \mathrm{E}[p]\end{aligned}$$

and

$$\begin{aligned}\langle\theta| P_\alpha^2 |\theta\rangle &= \langle\theta| \big((+1)\Pi_\alpha + (-1)\Pi_{\alpha_\perp}\big)^2 |\theta\rangle \\ &= \langle\theta| \big(\Pi_\alpha^2 - \Pi_\alpha\Pi_{\alpha_\perp} - \Pi_{\alpha_\perp}\Pi_\alpha + \Pi_{\alpha_\perp}^2\big) |\theta\rangle \\ &= \langle\theta| \big(\Pi_\alpha + \Pi_{\alpha_\perp}\big) |\theta\rangle \\ &= (+1)^2\langle\theta| \Pi_\alpha |\theta\rangle + (-1)^2\langle\theta| \Pi_{\alpha_\perp} |\theta\rangle \\ &= (+1)^2\mathrm{Prob}(p=+1) + (-1)^2\mathrm{Prob}(p=-1) \\ &= \mathrm{E}[p^2]\end{aligned}$$

thereby giving $\mathrm{E}[p] = \langle\theta| P_\alpha |\theta\rangle$ and $\mathrm{Var}(p) = \mathrm{E}[p^2] - (\mathrm{E}[p])^2 = \langle\theta| P_\alpha^2 |\theta\rangle - \langle\theta| P_\alpha |\theta\rangle^2$.

Exercise 2 *Interferometer with an atom on the ray*

1) The matrices in Dirac notation are

$$\begin{aligned}S &= \frac{1}{\sqrt{2}}|H\rangle\langle H| + \frac{1}{\sqrt{2}}|H\rangle\langle V| + \frac{1}{\sqrt{2}}|V\rangle\langle H| - \frac{1}{\sqrt{2}}|V\rangle\langle V| + |\mathrm{abs}\rangle\langle\mathrm{abs}| \\ R &= |H\rangle\langle V| + |V\rangle\langle H| + |\mathrm{abs}\rangle\langle\mathrm{abs}|.\end{aligned}$$

To find $U = SARS$ we proceed by steps:

$$\begin{aligned}RS &= \frac{1}{\sqrt{2}}|H\rangle\langle H| - \frac{1}{\sqrt{2}}|H\rangle\langle V| + \frac{1}{\sqrt{2}}|V\rangle\langle H| + \frac{1}{\sqrt{2}}|V\rangle\langle V| + |\mathrm{abs}\rangle\langle\mathrm{abs}|, \\ ARS &= |H\rangle\langle\mathrm{abs}| + \frac{1}{\sqrt{2}}|V\rangle\langle H| + \frac{1}{\sqrt{2}}|V\rangle\langle V| + \frac{1}{\sqrt{2}}|\mathrm{abs}\rangle\langle H| - \frac{1}{\sqrt{2}}|\mathrm{abs}\rangle\langle V|\end{aligned}$$

and finally

$$\begin{aligned}U = SARS = &\frac{1}{2}|H\rangle\langle H| + \frac{1}{2}|H\rangle\langle V| + \frac{1}{\sqrt{2}}|H\rangle\langle\mathrm{abs}| \\ &- \frac{1}{2}|V\rangle\langle H| - \frac{1}{2}|V\rangle\langle V| + \frac{1}{\sqrt{2}}|V\rangle\langle\mathrm{abs}| \\ &+ \frac{1}{\sqrt{2}}|\mathrm{abs}\rangle\langle H| - \frac{1}{\sqrt{2}}|\mathrm{abs}\rangle\langle V|.\end{aligned}$$

2) As $SARS|H\rangle = \frac{1}{2}|H\rangle - \frac{1}{2}|V\rangle + \frac{1}{\sqrt{2}}|\mathrm{abs}\rangle$, the probabilities of the three events are

$$\begin{aligned}\mathrm{Prob}(D_1) &= |\langle V| SARS |H\rangle|^2 = \frac{1}{4}, \\ \mathrm{Prob}(D_2) &= |\langle H| SARS |H\rangle|^2 = \frac{1}{4}, \\ \mathrm{Prob}(\mathrm{abs}) &= |\langle\mathrm{abs}| SARS |H\rangle|^2 = \frac{1}{2},\end{aligned}$$

which sum to 1.

3) A legitimate matrix has to be unitary. The first matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is not unitary because

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq I.$$

The second matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

is unitary because

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Thus the second matrix may model the absorption and reemission of the photon. Note also that this matrix acts like a Hadamard matrix on the subspace $\{|H\rangle, |\text{abs}\rangle\}$.