Homework 6: Solution
Traitement Quantique de l'Information

Exercise 1 *Bell states*

1) One has to show that $\langle B_{x,y}|B_{x',y'}\rangle = \delta_{x,x'}\delta_{y,y'}$. We show it explicitly for two cases:

$$\begin{aligned}\langle B_{00}|B_{00}\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle + |11\rangle) \\ &= \frac{1}{2}(\langle 00|00\rangle + \langle 00|11\rangle + \langle 11|00\rangle + \langle 11|11\rangle).\end{aligned}$$

Now we have

$$\begin{aligned}\langle 00|00\rangle = \langle 0|0\rangle\langle 0|0\rangle = 1, \langle 00|11\rangle = \langle 0|1\rangle\langle 0|1\rangle = 0, \\ \langle 11|00\rangle = \langle 1|0\rangle\langle 1|0\rangle = 0, \langle 11|11\rangle = \langle 1|1\rangle\langle 1|1\rangle = 1.\end{aligned}$$

Thus we get that $\langle B_{00}|B_{00}\rangle = \frac{1}{2}(1+0+0+1) = 1$. Now let us consider

$$\begin{aligned}\langle B_{00}|B_{01}\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)(|01\rangle + |10\rangle) \\ &= \frac{1}{2}(\langle 00|01\rangle + \langle 00|10\rangle + \langle 11|01\rangle + \langle 11|10\rangle) \\ &= \frac{1}{2}(0+0+0+0) = 0.\end{aligned}$$

2) The proof is by contradiction. Suppose there exists a_1, b_1 and a_2, b_2 such that

$$|B_{00}\rangle = (a_1|0\rangle + b_1|1\rangle) \otimes (a_2|0\rangle + b_2|1\rangle).$$

Then we have

$$\frac{1}{2}(|00\rangle + |11\rangle) = a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle + a_2b_2|11\rangle.$$

Comparing the coefficients of the orthornormal basis, one has

$$\frac{1}{2} = a_1a_2,\ \frac{1}{2} = b_1b_2,\ a_1b_2 = 0,\ b_1a_2 = 0.$$

The third equality indicates that either $a_1 = 0$ or $b_2 = 0$ (or both). If $a_1 = 0$ we get a contradiction with the first equation. If on the other hand $b_2 = 0$, we get a contradiction with the second one. Therefore, there does not exist $|\psi_1\rangle$ and $|\psi_2\rangle$ such that $|B_{00}\rangle$ can be written as $|\psi_1\rangle \otimes |\psi_2\rangle$. Therefore, B_{00} is entangled.

3) We have

$$\begin{aligned}|\gamma\rangle \otimes |\gamma\rangle &= (\cos(\gamma)|0\rangle + \sin(\gamma)|1\rangle) \otimes (\cos(\gamma)|0\rangle + \sin(\gamma)|1\rangle)\\ &= \cos^2(\gamma)|00\rangle + \cos(\gamma)\sin(\gamma)|01\rangle + \sin(\gamma)\cos(\gamma)|10\rangle + \sin^2(\gamma)|11\rangle .\end{aligned}$$

Similarly, we have

$$|\gamma_\perp\rangle \otimes |\gamma_\perp\rangle = = \sin^2(\gamma)|00\rangle - \cos(\gamma)\sin(\gamma)|01\rangle - \sin(\gamma)\cos(\gamma)|10\rangle + \cos^2(\gamma)|11\rangle .$$

Combining the two terms, we find that

$$\begin{aligned}|\gamma\rangle \otimes |\gamma\rangle + |\gamma_\perp\rangle \otimes |\gamma_\perp\rangle &= (\cos^2(\gamma) + \sin^2(\gamma))|00\rangle + (\sin^2(\gamma) + \cos^2(\gamma))|11\rangle\\ &= |00\rangle + |11\rangle\end{aligned}$$

and thus

$$\frac{1}{\sqrt{2}}(|\gamma\rangle \otimes |\gamma\rangle + |\gamma_\perp\rangle \otimes |\gamma_\perp\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |B_{00}\rangle .$$

4) From the rule of the tensor product

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ \\ b\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix},$$

we obtain the basis states as

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, we have

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |B_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

$$|B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \qquad |B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Exercise 2 *Entanglement*

1) Our conventions for tensor products are

$$|0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |0\rangle \otimes |1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|1\rangle \otimes |0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |1\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Therefore we have

$$|\Psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix}1\\0\\i\\1\end{pmatrix}$$

2) Suppose $|\Psi\rangle$ admits a product form $(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$. We are going to show that it would lead to contradiction. By expanding the tensor product, we have

$$\alpha\gamma|0,0\rangle + \alpha\delta|0,1\rangle + \beta\gamma|1,0\rangle + \beta\delta|1,1\rangle = \frac{1}{\sqrt{3}}|0,0\rangle + \frac{i}{\sqrt{3}}|1,0\rangle + \frac{1}{\sqrt{3}}|1,1\rangle .$$

Thus

$$\alpha\gamma = \frac{1}{\sqrt{3}} \tag{1}$$

$$\alpha\delta = 0 \tag{2}$$

$$\beta\gamma = \frac{i}{\sqrt{3}}$$

$$\beta\delta = \frac{1}{\sqrt{3}} \tag{3}$$

Eq. (1) implies that both α and γ are non-zero. Then by (2) δ has to be 0. But this contradicts with (3).

Exercise 3 *Copying or unitary attack from Eve in BB84*

1) Given Alice sent $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$, by linearity, the state of the two photons in the lab of Eve just after she made the copying operation is

$$\begin{aligned} |\Psi\rangle &= U_Z \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |b\rangle \right) \\ &= U_Z \frac{|0\rangle \otimes |b\rangle}{\sqrt{2}} + U_Z \frac{|1\rangle \otimes |b\rangle}{\sqrt{2}} \\ &= \frac{|0\rangle \otimes |0\rangle}{\sqrt{2}} + \frac{|1\rangle \otimes |1\rangle}{\sqrt{2}} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \end{aligned}$$

2) In Bob's lab the outcome is $\frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$ with probabilities $p_\pm = \langle\Psi| \Pi_\pm |\Psi\rangle$, where $|\Psi\rangle$ is given in Solution 3.1.

Following the hint, we have

$$\begin{aligned}
\Pi_\pm &= (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes \left(\frac{|0\rangle \pm |1\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0| \pm \langle 1|}{\sqrt{2}}\right) \\
&= (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes \left(\frac{|0\rangle\langle 0| \pm |0\rangle\langle 1| \pm |1\rangle\langle 0| + |1\rangle\langle 1|}{2}\right) \\
&= \frac{1}{2}\big(|00\rangle\langle 00| \pm |00\rangle\langle 01| \pm |01\rangle\langle 00| + |01\rangle\langle 01| \\
&\qquad + |10\rangle\langle 10| \pm |10\rangle\langle 11| \pm |11\rangle\langle 10| + |11\rangle\langle 11| \big).
\end{aligned}$$

The rest of the calculation is

$$\begin{aligned}
\Pi_\pm |\Psi\rangle &= \frac{1}{2\sqrt{2}}\left(|00\rangle \pm |01\rangle \pm |10\rangle + |11\rangle\right), \\
p_\pm = \langle\Psi| \Pi_\pm |\Psi\rangle &= \frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{2}}\left(\langle 00| + \langle 11|\right)\left(|00\rangle \pm |01\rangle \pm |10\rangle + |11\rangle\right) \\
&= \frac{1}{4}\left(\langle 00|00\rangle + \langle 11|11\rangle\right) \\
&= \frac{1}{2}.
\end{aligned}$$