Exercise 1 *Rotations on the Bloch sphere*

A general vector can be written in the form $\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|\downarrow\rangle$ in the Bloch sphere.

a) The eigenvectors for σ_z basis are $|\uparrow\rangle$ and $|\downarrow\rangle$, corresponding to $(\theta = 0, \phi = 0)$ and $(\theta = \pi, \phi = 0)$, respectively.

The eigenvectors for σ_y basis are $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{i}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{i}{\sqrt{2}}|\downarrow\rangle$, corresponding to $(\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2})$ and $(\theta = \frac{\pi}{2}, \phi = -\frac{\pi}{2})$, respectively.

The eigenvectors for σ_x basis are $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$, corresponding to $(\theta = \frac{\pi}{2}, \phi = 0)$ and $(\theta = \frac{\pi}{2}, \phi = \pi)$, respectively.

The corresponding representation over the Bloch sphere is shown in Figure 1.

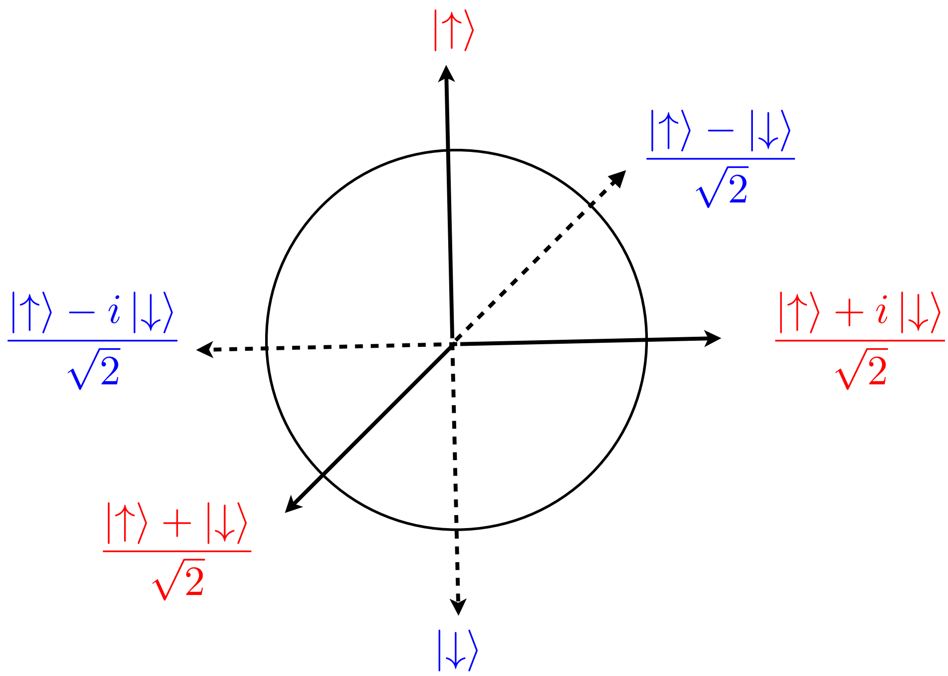

FIGURE 1 – Representation of basis vectors on Bloch Sphere

b) Using the general formula proved in homework 8 :

$$\exp\left(i\frac{\theta}{2}\vec{\sigma}\cdot\vec{n}\right) = \cos\left(\frac{\theta}{2}\right)I + i\vec{\sigma}\cdot\vec{n}\sin\left(\frac{\theta}{2}\right),$$

we obtain

$$
\begin{aligned}
\exp\Big(-i\frac{\alpha}{2}\sigma_x\Big) &= \cos\Big(\frac{\alpha}{2}\Big)I - i\sigma_x(\sin\Big(\frac{\alpha}{2}\Big)) \\
&= \begin{pmatrix} \cos\big(\frac{\alpha}{2}\big) & -i\sin\big(\frac{\alpha}{2}\big) \\ -i\sin\big(\frac{\alpha}{2}\big) & \cos\big(\frac{\alpha}{2}\big) \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\exp\Bigg(-i\frac{\beta}{2}\sigma_y\Bigg) &= \cos\Bigg(\frac{\beta}{2}\Bigg)I - i\sigma_y(\sin\Bigg(\frac{\beta}{2}\Bigg)) \\
&= \begin{pmatrix} \cos\big(\frac{\beta}{2}\big) & -\sin\big(\frac{\beta}{2}\big) \\ \sin\big(\frac{\beta}{2}\big) & \cos\big(\frac{\beta}{2}\big) \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\exp\Big(-i\frac{\gamma}{2}\sigma_z\Big) &= \cos\Big(\frac{\gamma}{2}\Big)I - i\sigma_z(\sin\Big(\frac{\gamma}{2}\Big)) \\
&= \begin{pmatrix} \cos\big(\frac{\gamma}{2}\big) - i\sin\big(\frac{\gamma}{2}\big) & 0 \\ 0 & \cos\big(\frac{\gamma}{2}\big) + i\sin\big(\frac{\gamma}{2}\big) \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}.
\end{aligned}
$$

c) The matrix $\exp\big(-i\frac{\alpha}{2}\sigma_x\big)$ is a rotation matrix of angle α around the X-axis, thus the state vector $\cos\big(\frac{\theta}{2}\big)\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin\big(\frac{\theta}{2}\big)\left|\downarrow\right\rangle$ is transformed to the vector $\cos\big(\frac{\theta-\alpha}{2}\big)\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin\big(\frac{\theta-\alpha}{2}\big)\left|\downarrow\right\rangle$. One can see the transformation geometrically on the Bloch sphere, however one can also show by direct calculation :

$$
\exp\Big(-i\frac{\alpha}{2}\sigma_x\Big)\left(\cos\Big(\frac{\theta}{2}\Big)\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin\Big(\frac{\theta}{2}\Big)\right)\left|\downarrow\right\rangle = \cos\Big(\frac{\theta-\alpha}{2}\Big)\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin\Big(\frac{\theta-\alpha}{2}\Big)\left|\downarrow\right\rangle.
$$

Similarly, one can see that $\exp\big(i\frac{\gamma}{2}\sigma_z\big)$ is a rotation of angle γ around the Z-axis. Therefore,

$$
\exp\Big(-i\frac{\gamma}{2}\sigma_z\Big)\left(\cos\Big(\frac{\theta}{2}\Big)\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin\Big(\frac{\theta}{2}\Big)\right)\left|\downarrow\right\rangle = e^{-i\frac{\gamma}{2}}\left(\cos\Big(\frac{\theta}{2}\Big)\left|\uparrow\right\rangle + e^{i(\frac{\pi}{2}+\gamma)}\sin\Big(\frac{\theta}{2}\Big)\left|\downarrow\right\rangle\right).
$$

Exercise 2 *Entanglement creation by a magnetic interaction*

The final state is (using that $|\uparrow\rangle, |\downarrow\rangle$ are eigenvectors of σ_z with eigenvalues $+1$ et -1).

$$
\begin{aligned}
e^{-\frac{it}{\hbar}\mathcal{H}}|\psi_0\rangle &= e^{-itJ\sigma_1^z\otimes\sigma_2^z}\cdot\frac{1}{2}\left(\left|\uparrow\uparrow\right\rangle - \left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle - \left|\downarrow\downarrow\right\rangle\right) \\
&= \frac{1}{2}\left(e^{-itJ}\left|\uparrow\uparrow\right\rangle - e^{itJ}\left|\uparrow\downarrow\right\rangle + e^{itJ}\left|\downarrow\uparrow\right\rangle - e^{-itJ}\left|\downarrow\downarrow\right\rangle\right) \\
&= \frac{e^{-itJ}}{2}\left(\left|\uparrow\uparrow\right\rangle - e^{2itJ}\left|\uparrow\downarrow\right\rangle + e^{2itJ}\left|\downarrow\uparrow\right\rangle - \left|\downarrow\downarrow\right\rangle\right).
\end{aligned}
$$

a) for $t = \frac{\pi}{4J}$ on a $e^{2itJ} = e^{\frac{i\pi}{2}} = i$

$\Rightarrow |\psi_t\rangle = \frac{e^{-\frac{i\pi}{4}}}{2}\left(\left|\uparrow\uparrow\right\rangle - i\left|\uparrow\downarrow\right\rangle + i\left|\downarrow\uparrow\right\rangle - \left|\downarrow\downarrow\right\rangle\right)$.

b) suppose the state can be written
$(\alpha|\uparrow\rangle + \beta|\downarrow\rangle) \otimes (\gamma|\uparrow\rangle + \delta|\downarrow\rangle) = \alpha\gamma|\uparrow\uparrow\rangle + \alpha\delta|\uparrow\downarrow\rangle + \beta\gamma|\downarrow\uparrow\rangle + \beta\delta|\downarrow\downarrow\rangle$,
then $\alpha\gamma = 1,\ \alpha\delta = -i,\ \beta\gamma = i,\ \beta\delta = -1$.
One can always set $\alpha = 1$ (global phase). Thus $\gamma = 1,\ \delta = -i,\ \beta = i$ et $\delta = i \Rightarrow$ contradiction onδ. You can also take any value for α and show the contradiction appears.

c) At $t = \frac{\pi}{2J}$ with $e^{\pm itJ} = e^{\pm i\frac{\pi}{2}} = \pm i$,

$$
\begin{aligned}
|\psi_t\rangle &= \frac{1}{2}\left(-i|\uparrow\uparrow\rangle - i|\uparrow\downarrow\rangle + i|\downarrow\uparrow\rangle + i|\downarrow\downarrow\rangle\right) \\
&= \frac{-i}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)
\end{aligned}
$$

is a product state. So another $\frac{\pi}{4J}$ time of evolution cancels the entanglement.

d) At $t = \frac{\pi}{J}$ with $e^{\pm itJ} = e^{\pm i\pi} = -1$,

$$
\begin{aligned}
|\psi_t\rangle &= \frac{1}{2}\left(-|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\right) \\
&= \frac{-1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)
\end{aligned}
$$

is also a product state.

Complement on the Hamiltonian in matrix and Dirac notation.

1. **Matrix notation.** In the canonical bases, we have $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Using the tensor product rule one obtains that

$$
\begin{aligned}
\sigma_1^z \otimes \sigma_2^z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\end{aligned}
$$

thus the Hamiltonian is

$$
\mathcal{H} = \begin{pmatrix} \hbar J & 0 & 0 & 0 \\ 0 & -\hbar J & 0 & 0 \\ 0 & 0 & -\hbar J & 0 \\ 0 & 0 & 0 & \hbar J \end{pmatrix}.
$$

2. **Dirac notation.** In the bra-ket formalism one has $\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$, thus

$$
\begin{aligned}
\sigma_1^z \otimes \sigma_2^z &= \left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right) \otimes \left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right) \\
&= |\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|.
\end{aligned}
$$

Therefore, we have

$$
\mathcal{H} = \hbar J(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|).
$$

3. **Connection between matrix and Dirac notations.** Notice that to verify this one can use

$$|\uparrow\rangle\langle\uparrow| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix},$$

which implies that

$$(|\uparrow\rangle\langle\uparrow|)\otimes(|\uparrow\rangle\langle\uparrow|) = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}.$$

Similarly one can show that

$$|\uparrow\downarrow\rangle\langle\uparrow\downarrow| = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix},$$

$$|\downarrow\uparrow\rangle\langle\downarrow\uparrow| = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{pmatrix},$$

$$|\downarrow\downarrow\rangle\langle\downarrow\downarrow| = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}.$$

4. **Eigenvalues and eigenvectors.** One can see that the eigen-values are $\hbar J$ corresponding to the eigenvectors $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$ and $-\hbar J$ corresponding to the eigenvectors $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$.