Introduction to Differentiable Manifolds
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Exercise series 2 **2021–09–28**

Exercise 2.1 (Stereographic projection.)**.** Let $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ be the *north pole* and $S = -N$ the *south pole* of the sphere $\mathbb{S}^n$. Define stereographic projection $\sigma : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x_0, \dots, x_n) = \frac{1}{1 - x_n}(x_0, \dots, x_{n-1}).$$

Let $\widetilde{\sigma}(x) = \sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

(a) Show that σ is bijective, and

$$\sigma^{-1}(u_0, \dots, u_{n-1}) = \frac{1}{|u|^2 + 1}(2u_0, \dots, 2u_{n-1}, |u|^2 - 1).$$

(b) Compute the transition map $\widetilde{\sigma} \circ \sigma^{-1}$ and verify that $\{\sigma, \widetilde{\sigma}\}$ is a smooth atlas for $\mathbb{S}^n$.

(c) Show that the smooth structure defined by the atlas $\{\sigma, \widetilde{\sigma}\}$ is the same as the one defined via graph coordinates in the lecture.

Exercise 2.2. Show that $\mathbb{P}^n$ is a smooth manifold. (Use exercise from series 1.)

Exercise 2.3 (Open submanifolds)**.** Let N be an open subset of a $\mathcal{C}^k$ manifold M, and let $\mathcal{A}$ be the set of charts of M whose domain is contained in N. Prove that:

(1) $\mathcal{A}$ is a $\mathcal{C}^k$ structure for N, making N into a $\mathcal{C}^k$ manifold. We call the smooth manifold $(N, \mathcal{A})$ an *open submanifold* of M.

(2) The inclusion map $\iota : N \hookrightarrow M$ is $\mathcal{C}^k$.

(3) A function $f : L \to N$ (where L is a $\mathcal{C}^k$ manifold) is $\mathcal{C}^k$ if and only if the composite $\iota \circ f$ is $\mathcal{C}^k$.

Use this to show that the general linear group $GL(n, \mathbb{R})$, i.e. the set consisting of invertible $n \times n$ matrices is naturally a smooth manifold.

Exercise 2.4 (Some basic properties of $\mathcal{C}^k$ manifolds)**.** Let M be a topological manifold and let $\mathcal{A}$ be a $\mathcal{C}^k$ atlas, with $k \geq 1$. Show that:

(1) Let $\mathcal{A}'$ be another smooth atlas on M. Then $\mathcal{A}$ and $\mathcal{A}'$ determine the same smooth structure on M if and only if their union is a smooth atlas.

(2) Every $\mathcal{C}^k$ chart $\varphi : U \to V$ of M is a $\mathcal{C}^k$ diffeomorphism.

(3) The composite $g \circ f$ of two $\mathcal{C}^k$ maps $f : M \to N$, $g : N \to P$ is a $\mathcal{C}^k$ map.

(4) Let $\mathcal{A}_0, \mathcal{A}_1$ be two $\mathcal{C}^k$ atlases on M, defining two $\mathcal{C}^k$ manifolds $M_i = (M, \overline{\mathcal{A}_i})$. Then the two atlases $\mathcal{A}_i$ are equivalent iff the following holds:
For every function $f : N \to M$ (where N is a $\mathcal{C}^k$ manifold), the function f is $\mathcal{C}^k$ as a map $N \to M_0$ if and only if it is $\mathcal{C}^k$ as a map $N \to M_1$.

Exercise 2.5 (Smooth structures on $\mathbb{R}$)**.** On the real line $\mathbb{R}$ (with the standard topology) we define two atlases $\mathcal{A} = \{\mathrm{id}_{\mathbb{R}}\}$, $\mathcal{B} = \{\varphi\}$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is given by $\varphi(x) = x^3$.

- Convince yourself that $\mathcal{B}$ defines a smooth structure on $\mathbb{R}$.
- Show that $\mathcal{A}$ and $\mathcal{B}$ define distinct smooth structures.
- Find a diffeomorphism $(\mathbb{R}, \overline{\mathcal{A}}) \to (\mathbb{R}, \overline{\mathcal{B}})$.

Exercise 2.6 (Product manifolds)**.** Let M_0 and M_1 be $\mathcal{C}^k$ manifolds of dimension m_0 and m_1 respectively. Recall that $M_0 \times M_1$ (with the product topology) is a topological manifold of dimension $m_0 + m_1$.

- Find a natural $\mathcal{C}^k$ structure on $M_0 \times M_1$ such that two projections $\pi_i : M_0 \times M_1 \to M_i$ are $\mathcal{C}^k$ maps.
- Show that a map $f : N \to M_0 \times M_1$ (where N is another $\mathcal{C}^k$ manifold) is $\mathcal{C}^k$ if and only if the two composite maps $\pi_i \circ f$ are $\mathcal{C}^k$.