# Potential Theory I

# Outlines

Newtonian Mechanics:

- refreshing memory

Potential Theory : general results

- Gravitational field force, gravitational potential
- Poisson Equation
- Gauss Theorem
- Total potential energy

Spherical systems:

- Newton's Theorems
- Circular speed, circular velocity, circular frequency, escape speed, potential energy

## Refreshing memory...

# Newtonian mechanics

# Newtonian mechanics : a very short remainder

**point mass** :

| | | |
|---|---|---|
| mass | $m$ | |
| position | $\vec{x}$ | |
| velocity | $\vec{v} = \dfrac{d\vec{x}}{dt}$ | |
| momentum | $\vec{p} = m\vec{v} = m\dfrac{d\vec{x}}{dt}$ | |

## Newtonian mechanics : a very short remainder

**point mass** :

mass $m$

position $\vec{x}$

velocity $\vec{v} = \frac{d\vec{x}}{dt}$

momentum $\vec{p} = m\vec{v} = m\frac{d\vec{x}}{dt}$

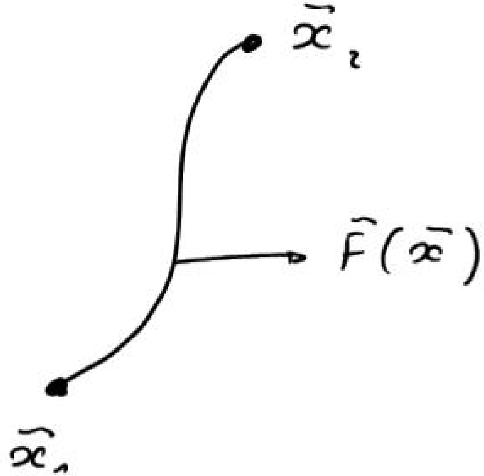
## Newton second law

$$\boxed{\frac{d}{dt}(\vec{p}) = m\frac{d^2\vec{x}}{dt^2} = \vec{F}}$$

- $\vec{F}$ : a force

$\vec{p}$ is constant in absence of a force

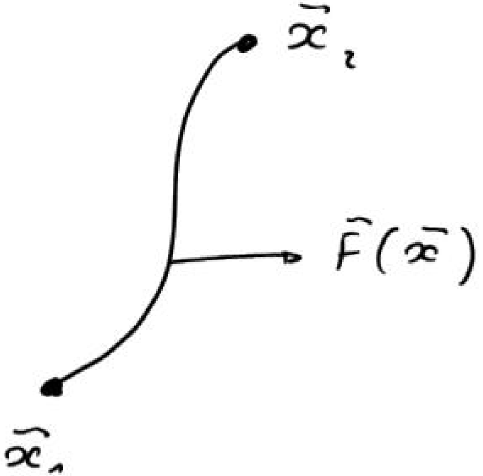
<u>Work</u> : work done by a force in moving the particle from $\vec{x}_1$ to $\vec{x}_2$




$$W_{12} = - \int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x}$$

$$[W_{12}] = g \frac{cm^2}{s^2} = erg$$

**Work** : work done by a force in moving the particle from $\vec{x}_1$ to $\vec{x}_2$




$$W_{12} = -\int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x}$$

$$[W_{12}] = g\,\frac{cm^2}{s^2} = \text{erg}$$

**Power of a force** ( energy rate ) $\quad \frac{\text{erg}}{s}$

with $\frac{\partial \mathfrak{F}(\vec{x})}{\partial x_i} = F_i(\vec{x})$

$$\frac{d}{dt} W_{12}(x(t)) = -\frac{d}{dt}\int_{\vec{x}_1}^{\vec{x}(t) = \vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x} = -\frac{d}{dt}\Big( \mathfrak{F}(\vec{x}(t)) - \mathfrak{F}(\vec{x}_1) \Big)$$

$$= -\vec{\nabla}_{\vec{x}}\, \mathfrak{F}(\vec{x}) \cdot \frac{d}{dt}(\vec{x}(t)) = -\vec{F}(\vec{x})\,\vec{v}(\vec{x})$$

## Kinetic energy

$\vec{x} = \vec{x}(t)$

$d\vec{x} = \frac{d\vec{x}}{dt}\,dt$

Newton 2nd law

$$W_{12} = -\int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x})\cdot d\vec{x} = -m\int_{\vec{x}_1}^{\vec{x}_2} \frac{d\vec{v}}{dt}\cdot\frac{d\vec{x}}{dt}\,dt$$

$$= -m\int_{\vec{x}_1}^{\vec{x}_2} \frac{d\vec{v}}{dt}\cdot\vec{v}\,dt$$

integration by part gives

$$= -m\left[\vec{v}^2\Big|_{\vec{x}_1}^{\vec{x}_2} - \int_{\vec{x}_1}^{\vec{x}_2}\vec{v}\,\frac{d\vec{v}}{dt}\,dt\right] = -m\vec{v}_2^{\,2} + m\vec{v}_1^{\,2} + \underbrace{m\int_{\vec{x}_1}^{\vec{x}_2}\vec{v}\,\frac{d\vec{v}}{dt}\,dt}_{-W_{12}}$$

**Thus** $\quad W_{12} = \frac{1}{2}m\vec{v}_1^{\,2} - \frac{1}{2}m\vec{v}_2^{\,2}$

$$\boxed{W_{12} = K_1 - K_2}$$

$K = \frac{1}{2}m\vec{v}^{\,2}$ : Kinetic energy

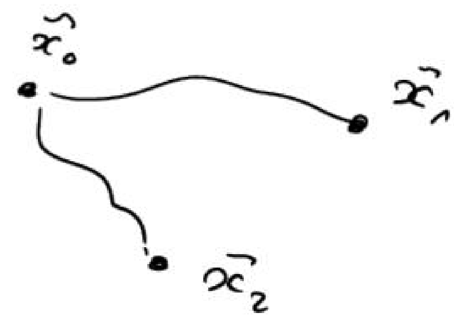
# Potential energy and conservative forces

A force $\vec{F}(\vec{x})$ is called conservative if the work done by this force in moving the particle from $\vec{x}_1$ to $\vec{x}_2$ is independent of the path.


Then, for any given point $\vec{x}_0$ we can define the function (potential) $V_0(\vec{x})$

$$V_0(\vec{x}) := V_{0x} = -\int_{\vec{x}_0}^{\vec{x}} \vec{F}(\vec{x}') \, d\vec{x}'$$

Then $W_{12} = W_{10} + W_{02}$




$$\boxed{W_{12} = V(\vec{x}_2) - V(\vec{x}_1)}$$

Useful convention : $\vec{x}_0 \to \infty$ (far away from all interacting bodies)

# Gradient of the potential

with $\dfrac{\partial \mathcal{F}(\vec{x})}{\partial x_i} = F_i(\vec{x})$

$$\vec{\nabla}_{\vec{x}} \cdot V(\vec{x}) = -\vec{\nabla}_{\vec{x}} \cdot \left[ \int_{\vec{x}_0}^{\vec{x}} \vec{F}(\vec{x}')\, d\vec{x}' \right] = -\vec{\nabla}_{\vec{x}} \Big( \mathcal{F}(\vec{x}) - \underbrace{\mathcal{F}(\vec{x}_0)}_{ct \Rightarrow 0} \Big) = -\vec{F}(\vec{x})$$

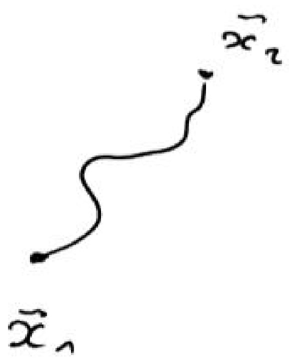
$$\boxed{\vec{\nabla}_{\vec{x}} \cdot V(\vec{x}) = -\vec{F}(\vec{x})}$$

We can represent a conservative force field by its potential

**Total Energy $E$**

$$E := K + V = \frac{1}{2} m \vec{v}^{\,2} + V(\vec{x})$$

**Theorem** The energy $E$ of a system evolving under conservative forces $\vec{F}(\vec{x})$ (associated to a potential $V(\vec{x})$) is constant.




$$E_1 = E(\vec{x}_1) = \frac{1}{2} m v_1^2 + V(\vec{x}_1)$$

$$E_2 = E(\vec{x}_2) = \frac{1}{2} m v_2^2 + V(\vec{x}_2)$$

$$E_1 - E_2 = \underbrace{K_1 - K_2}_{W_{12}} + \underbrace{V(\vec{x}_1) - V(\vec{x}_2)}_{-W_{12}}$$

$$= 0$$

#

# Angular momentum and Tork

Angular momentum $\vec{L} = \vec{x} \times \vec{p}$

Tork $\vec{N} = \vec{x} \times \vec{F}$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{x}}{dt} \times \vec{p} + \vec{x} \times \frac{d\vec{p}}{dt}$$

$$= \underbrace{\vec{v} \times \vec{p}}_{=0} + \underbrace{\vec{x} \times \vec{F}}_{\vec{N}}$$

$$\boxed{\frac{d\vec{L}}{dt} = \vec{N}}$$

# Potential Theory

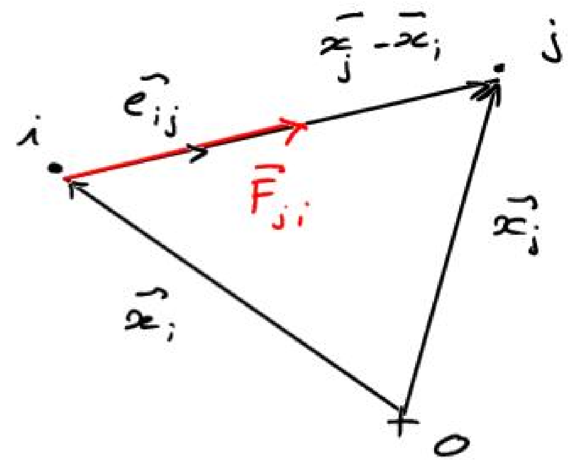
## Potential theory : general results

Goal : compute the gravitational potential / forces due to a large number of stars (point masses)

$N \sim 10^{11}$ for a Milky Way like galaxy

As the relaxation time of such system is very large ( $\gg$ the age of the Universe) we can describe the system with a smooth analytical potential / density.

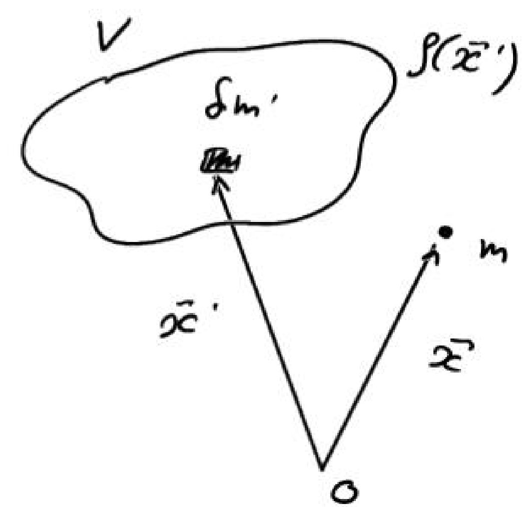
## Newton Law




$$\vec{F}_{ji} = \frac{G m_i m_j}{|\vec{x}_j - \vec{x}_i|^2} \vec{e}_{ij} = \frac{G m_i m_j}{|\vec{x}_{ij}|^3} \vec{x}_{ij}$$

$$\vec{x}_{ij} = \vec{x}_j - \vec{x}_j$$

Force on a particle of mass $m$ in $\vec{x}$ due to a distribution of mass $\rho(\vec{x})$




$$\delta \vec{F}(\vec{x}) = \frac{G m \, \delta m'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})$$

$$= \frac{G m \, \rho(\vec{x}') \, d^3\vec{x}'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})$$

So, the total force writes :

$$\boxed{\vec{F}(\vec{x}) = \int_V \frac{G\, m\, \rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})\, d^3\vec{x}'}$$

$$= m \underbrace{G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})\, d^3\vec{x}'}$$

$\vec{g}(\vec{x})$ : gravitational field

$$[\vec{g}] = \frac{cm}{s^2} = \frac{erg}{g} \frac{1}{cm}$$

# Gravitational Potential

It is easy to see that the function

$$\delta V(\vec{x}) = - \frac{G m \delta m}{|\vec{x}' - \vec{x}|} \qquad \text{is such that}$$

$$\vec{\nabla} \delta V(\vec{x}) = - \frac{G m \delta m}{|\vec{x}' - \vec{x}|^2} \frac{(\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|} = - \delta \vec{F}(\vec{x})$$

so, by defining

$$\boxed{V(\vec{x}) = - G \int_V \frac{m \rho(\vec{x}')}{|\vec{x}' - \vec{x}|} d^3\vec{x}'}$$

we ensure that $\qquad \vec{\nabla} V(\vec{x}) = - \vec{F}(\vec{x})$

We define the specific potential $\phi(\bar{x}) = \dfrac{V(\bar{x})}{m}$

which writes

$$\boxed{\phi(\bar{x}) = -G \int_V \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|} \, d^3\bar{x}'}$$

$[\phi] = \dfrac{erg}{g}$

$\equiv$ specific energy

The gravitational field writes :

$$\boxed{\bar{g}(\bar{x}) = - \bar{\nabla} \phi(\bar{x})}$$

<u>Notes</u>

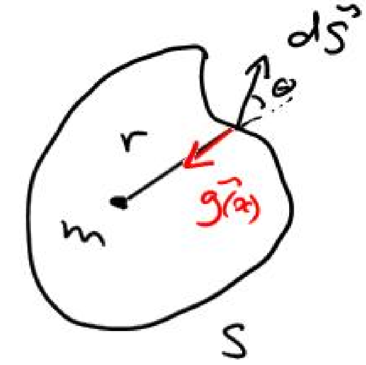
- The gravity is a conservative force
- $\phi(\bar{x})$ : scalar field
  $\vec{g}(\bar{x})$ : vector field
  $\}$ contain the same information
- we will always use "specific" quantities

$$V(\bar{x}) \rightarrow \phi(\bar{x})$$

$$K = \frac{1}{2} m \hat{v}^2 \rightarrow \frac{1}{2} \hat{v}^2$$

$$\frac{1}{2} v^2 + \phi(\bar{x}) = \text{specific energy (conserved quantity)}$$

# The Gauss's Law




<u>Consider</u>: • a single point mass $m$

• a surface $S$ around this point

• a point $\vec{x}$ on the surface at a distance $r$

• $\vec{g}(\vec{x})$ the gravitational field

• $d\vec{S}$, the normal at the surface

• $\theta$ the angle between $\vec{g}(\vec{x})$ and $d\vec{S}$

$$\vec{g}(\vec{x}) \cdot d\vec{S} = -\vec{g}(\vec{x}) \cdot |d\vec{S}| \cos\theta$$

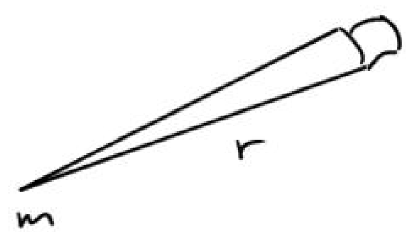
<u>But</u> $\quad |d\vec{S}| \cos\theta = r^2 \, d\Omega$

$$|\vec{g}(x)| = \frac{Gm}{r^2}$$

$|d\vec{S}| \cos\theta$

$\equiv$

$d\Omega \, r^2$



$$\boxed{\vec{g}(\vec{x}) \cdot d\vec{S} = - G m \, d\Omega}$$

integrating over any surface

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = \begin{cases} -4\pi G m & \text{if } m \text{ inside } S \\ 0 & \text{instead} \end{cases}$$

For multiple masses $m_i$

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \sum_{i \text{ in } S} m_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$\boxed{\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \int_V \rho(\vec{x}) \, d\vec{x} = -4\pi G M}$$

Gauss's Law

# Divergeance of the specific force

(A) $\vec{\nabla}_x \cdot \vec{g}(\vec{x})$

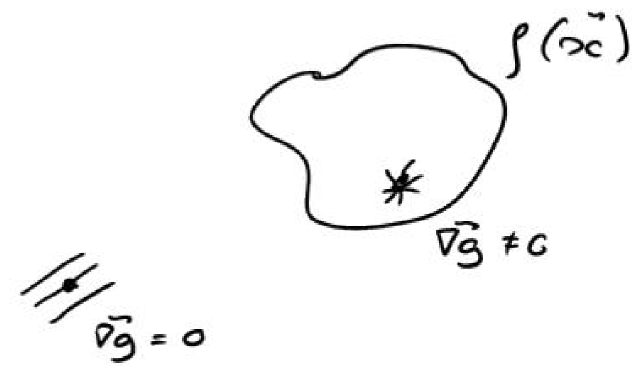
div. theorem

$$\int_V \vec{\nabla} \cdot \vec{g}(\vec{x}) \, d^3\vec{x} = \int_S \vec{g}(\vec{x}) \, d\vec{S}$$

Gauss's Law

$$= -4\pi G \int_V \rho(\vec{x}) \, d\vec{x}$$

$$\boxed{\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})}$$



# Divergeance of the specific force

B $\quad \vec{\nabla}_x \cdot \vec{g}(\vec{x})$

$$\vec{g}(\vec{x}) = G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \, d^3\vec{x}'$$

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_x \cdot \left( \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3\vec{x}'$$

- $$\vec{\nabla}_x \cdot \left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) = \frac{d}{dx_1}\left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_2}\left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_3}\left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right)$$

$$= -\frac{3}{|\vec{x}' - \vec{x}|^3} + \frac{3(\vec{x}' - \vec{x}) \cdot (\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|^5}$$

$$= 0 \quad \text{if} \quad \vec{x}' \neq \vec{x}$$

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_x \cdot \left( \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3\vec{x}'$$


$\rho(\vec{x}) = \text{cte}$

$$= G \rho(\vec{x}) \int_{|\vec{x}' - \vec{x}| \le h} \vec{\nabla}_x \cdot \left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) d^3\vec{x}'$$

variable exchange

$\vec{\nabla}_x f(\vec{x} - \vec{x}') = - \vec{\nabla}_{x'} f(x - x')$

$$= - G \rho(\vec{x}) \int_{|\vec{x}' - \vec{x}| \le h} \vec{\nabla}_{x'} \cdot \left( \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) d^3\vec{x}'$$

divergence theorem

$$= - G \rho(\vec{x}) \underbrace{\int_{|\vec{x}' - \vec{x}| = h} \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} d^2\vec{S}'}_{4\pi h^2 \cdot \left. \frac{1}{r^2} \right|_{h=r} = 4\pi}$$

$r = |\vec{x}' - \vec{x}| = h$

$$\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} = \frac{1}{r^2}$$

$$\underline{\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = - 4\pi G \rho(\vec{x})}$$

# The Poisson Equation

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})$$

with : $\vec{\nabla}_x \phi(\vec{x}) = -\vec{g}(\vec{x})$ $\qquad\qquad$ $\vec{\nabla}_x \cdot (\vec{\nabla}_x) = \vec{\nabla}_x^2$

$$\boxed{\vec{\nabla}_x^2 \phi(\vec{x}) = 4\pi G \rho(\vec{x})}$$

Poisson Equation

<u>Note</u> : To ensure a unique solution, boundary conditions are necessary $\qquad$ ($2^{nd}$ order diff. eqn.)

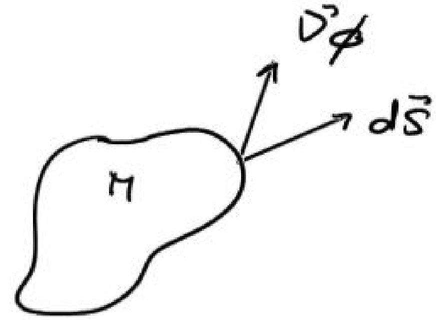
<u>ex</u> : $\phi(\infty) = 0$

$\vec{\nabla}\phi(\infty) = \vec{g}(x) = 0$

## Gauss theorem

integrate the Poisson equation over a volume V that contains a mass M

$$\int_V \vec{\nabla}^2 \phi(\vec{x})\, d^3\vec{x} = \underbrace{\int_V 4\pi G \rho(\vec{x})\, d^3\vec{x}}$$




div. theorem

$$\boxed{\int_S d^2\vec{S} \cdot \vec{\nabla}\phi = 4\pi G M}$$

Gauss theorem

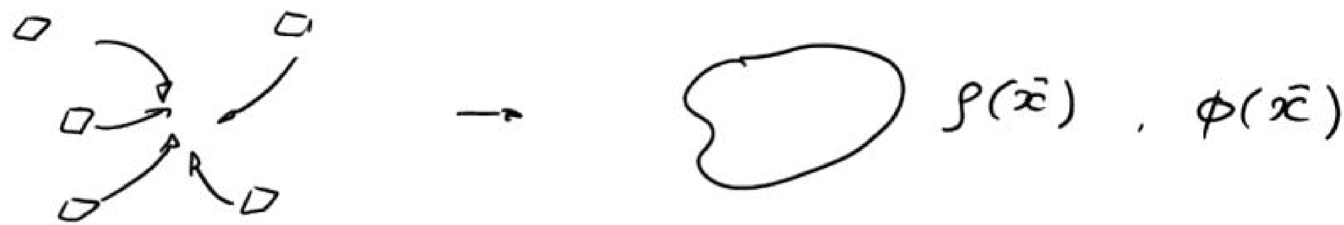
Equivalently :

$$\boxed{\int_S d^2\vec{S} \cdot \vec{g}(\vec{x}) = -4\pi G M}$$

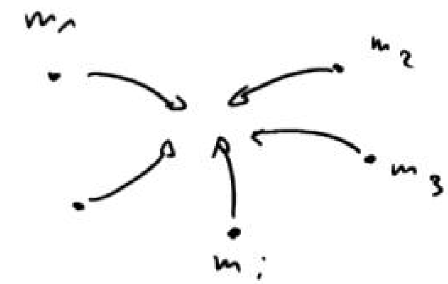
Gauss's Law

# Total potential energy (A)

Total work needed to assemble a density distribution $\rho(\bar{x})$




Assume a set of discrete points




- The work to bring the 1st point from $\infty$ to $\vec{x}_1$ is $0$
- The work to bring the 2nd point from $\infty$ to $\vec{x}_2$ is $-\dfrac{G m_1 m_2}{r_{12}}$
- The work to bring the 3rd point from $\infty$ to $\vec{x}_3$ is $-\dfrac{G m_1 m_3}{r_{13}} - \dfrac{G m_2 m_3}{r_{23}}$

The total work is thus

$$W = -\frac{G m_1 m_2}{r_{12}} - \frac{G m_1 m_3}{r_{13}} - \frac{G m_2 m_3}{r_{23}} - \dots - \sum_{j=1}^{N-1} \frac{G m_{jN}}{r_{jN}}$$

$$= -\sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{G m_i m_j}{r_{ij}} = -\frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{G m_i m_j}{r_{ij}}$$

With $\phi_i = -\sum_{\substack{j=1 \\ j \neq i}} \frac{G m_j}{r_{ij}}$

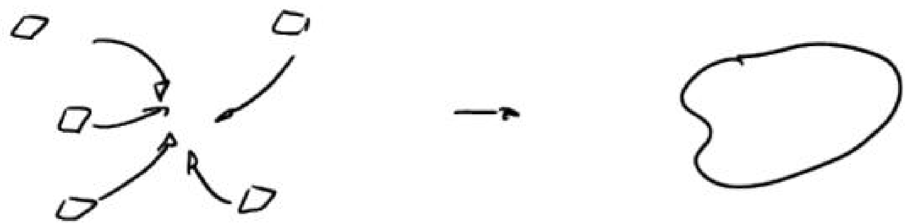
$$W = \frac{1}{2} \sum_{i=1}^{N} m_i \phi_i \equiv \frac{1}{2} \sum_{i=1}^{N} V_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$\boxed{W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) \, d^3\vec{x}}$$

# Total potential energy B

Total work needed to assemble a density distribution $\rho(\vec{x})$

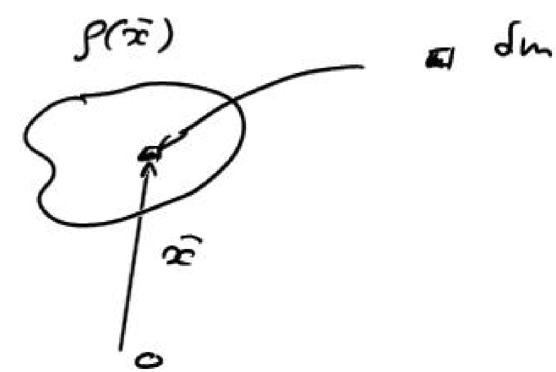


$\rho(\vec{x})$ , $\phi(\vec{x})$

① Work done to assemble a piece of mass $\delta m = \delta\rho \, d\vec{x}^3$ from $\infty$ to $\vec{x}$ assuming an existing mass distribution $\rho(\vec{x})$, $\phi(\vec{x})$

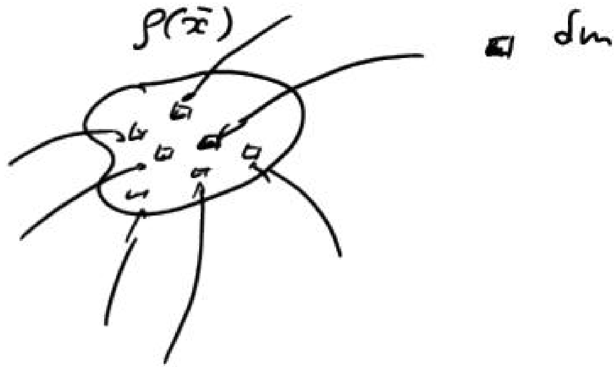
$$\delta W_{\vec{x}} = V(\vec{x}) - \underbrace{V(\infty)}_{=0}$$

$$= \delta m \, \phi(\vec{x}) = \delta\rho(\vec{x}) \, d^3\vec{x} \; \phi(\vec{x})$$



To increase everywhere the mass distribution by $\delta\rho$

$$\rho(\bar{x}) \to \rho(\bar{x}) + \delta\rho(\bar{x})$$




$$\delta W = \int \delta\rho(\bar{x})\, d^3\bar{x}\ \phi(\bar{x})$$

Poisson: $\delta\rho(\bar{x}) = \frac{1}{4\pi G} \vec{\nabla}^2 \delta\phi(x)$

$$\delta W = \frac{1}{4\pi G} \int \vec{\nabla}^2 \delta\phi(x)\ \phi(\bar{x})\ d^3\bar{x}$$

divergence theorem

$$\int_V d^3x\, g\, \vec{\nabla}\cdot\vec{F} = \int_S g\, \vec{F}\cdot d^2\vec{s} - \int_V d^3x\, \vec{F}\cdot\vec{\nabla} g$$

$$= \underbrace{\frac{1}{4\pi G} \int_{S \text{ at } \infty} \phi(\bar{x})\, \vec{\nabla}\delta\phi(\bar{x})}_{=0} - \frac{1}{4\pi G} \int \vec{\nabla}\phi(\bar{x}) \cdot \vec{\nabla}(\delta\phi(\bar{x}))\ d^3\bar{x}$$

as $\phi(\infty) = 0$

$\vec{\nabla}\delta\phi(\infty) = \delta g(\infty) = 0$

$$\delta W = -\frac{1}{4\pi G}\int \vec{\nabla}\phi(\vec{x})\cdot\vec{\nabla}(\delta\phi(\vec{x}))\, d^3\vec{x}$$

with

$$\frac{1}{2}\,\delta|\vec{\nabla}\phi(\vec{x})|^2 = \delta\vec{\nabla}\phi(\vec{x})\cdot\vec{\nabla}\phi(\vec{x}) = \vec{\nabla}(\delta\phi(\vec{x}))\cdot\vec{\nabla}\phi(\vec{x})$$

$$\delta W = -\frac{1}{8\pi G}\int \delta|\vec{\nabla}\phi|^2\, d^3x \equiv -\frac{1}{8\pi G}\,\delta\int |\vec{\nabla}\phi|^2\, d^3x$$

② Contribution of all $\delta W$ to $W$

$$\boxed{W = -\frac{1}{8\pi G}\int |\vec{\nabla}\phi|^2\, d^3x}$$

$$\boxed{W = -\frac{1}{8\pi G}\int |\vec{\nabla}\phi|^2 \, d^3x}$$

<u>Other expression</u> using the divergeance theorem

$$\int_V d^3\vec{x}\, \vec{F}\cdot\vec{\nabla} g = \int_S g \cdot \vec{F}\cdot d^2\vec{S} - \int d^3\vec{x}\, g\, \vec{\nabla}\cdot\vec{F}$$

$$\int |\vec{\nabla}\phi|^2 d^3x = \int_V d^3\vec{x}\, \vec{\nabla}\phi\cdot\vec{\nabla}\phi = \int_S \underbrace{\phi\, \vec{\nabla}\phi\, d^2\vec{S}}_{=0\ \vec{x}\to\infty} - \int_V d^3\vec{x}\, \phi\, \underbrace{\vec{\nabla}\cdot(\vec{\nabla}\phi)}_{=\vec{\nabla}^2\phi = 4\pi G\rho}$$

Poisson

$$W = -\frac{1}{8\pi G} \cdot \int -4\pi G\, \phi \cdot \rho \, d^3\vec{x}$$

$$\boxed{W = \frac{1}{2}\int \rho(\vec{x})\, \phi(\vec{x})\, d^3\vec{x}}$$

Other useful expression

$$\boxed{W = -\int \rho(\vec{x})\, \vec{x}\cdot\vec{\nabla}\phi(\vec{x})\, d^3\vec{x}}$$

# Spherical Systems

$$\rho(\vec{x}) = \rho(r)$$

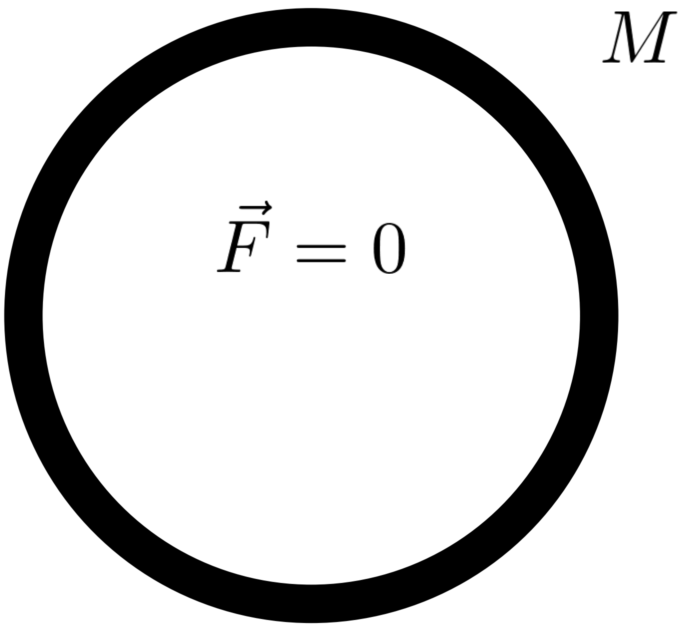
$$r = \sqrt{x^2 + y^2 + z^2}$$

# Newton's Theorems

Newton (1642-1727)

First theorem:

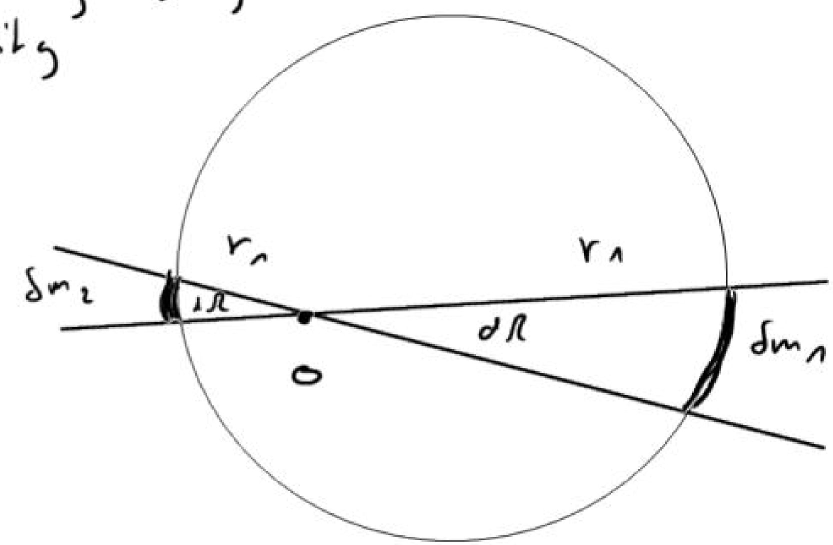
A body that is inside a spherical shell of matter experiences no net gravitational force from the shell.



# First Newton theorem

A body that is inside a spherical shell of matter experiences no net gravitational force from that shell

a shell of constant density $\rho(\vec{x}) = \rho$




$$\begin{cases} \delta m_1 = \rho(r_1) \cdot r_1^2 \, d\Omega \, dr \\ \delta m_2 = \rho(r_2) \cdot r_2^2 \, d\Omega \, dr \end{cases}$$

thus : $$\frac{\delta m_1}{\delta m_2} = \frac{r_1^2}{r_2^2}$$

and $$\frac{\delta m_1}{r_1^2} = \frac{\delta m_2}{r_2^2}$$

<u>consequently</u> : $\delta \vec{F_1} = - \delta \vec{F_2}$

by integrating over the entire shell ($d\Omega$)

all forces cancel out ! #

Corollary The gravitational potential $\phi(\bar{x})$ is constant inside the sphere.

As $\vec{\nabla}_x \phi(\bar{x}) = \bar{g} = 0$ $\qquad \phi(\bar{x}) = ct$ #

What is the value of $\phi(\bar{x})$ ?

$$\phi(\bar{x}) = -\int_V \frac{G\,\rho(\bar{x}')}{|\bar{x}' - \bar{x}|}\, d^3\bar{x}'$$

Spherical coordinates $\qquad d^3\bar{x} = r^2\, dr\, d\Omega = 4\pi\, r^2 dr$

At the center $\bar{x} = 0$

$$\phi(0) = -4\pi G \int_0^\infty \frac{\rho(r')}{r}\, r^2 dr = -4\pi G \int_0^\infty \rho(r')\, r\, dr$$

Density of a shell of mass $M$, radius $R$ : $\rho(r) = \dfrac{M}{4\pi r^2}\,\delta(R-r)$

$$\left( \text{as} \quad 4\pi \int_0^\infty \frac{M}{4\pi r^2}\,\delta(R-r)\, r^2\, dr = M \right)$$

$$\phi(r) = -GM \int_0^\infty \frac{\delta(R-r)}{r^2}\, r\, dr = -\frac{GM}{R}$$

As the potential is constant for $r < R$

$$\boxed{\phi(\bar{x}) = -\frac{GM}{R} \qquad \bar{x} \in \text{sphere}}$$

# Newton’s Theorems

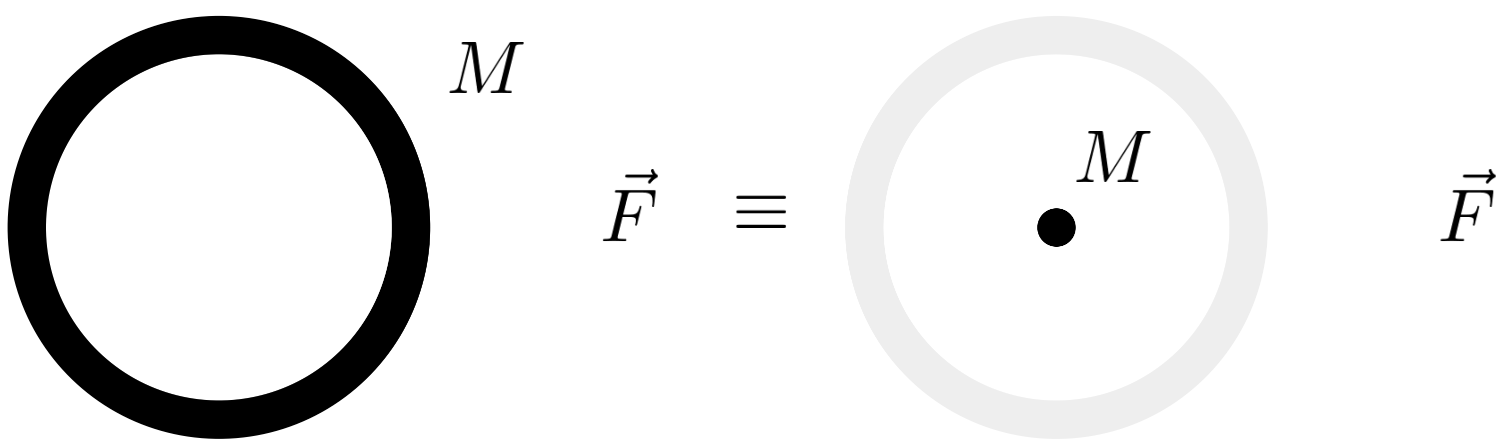
Newton (1642-1727)

First theorem:

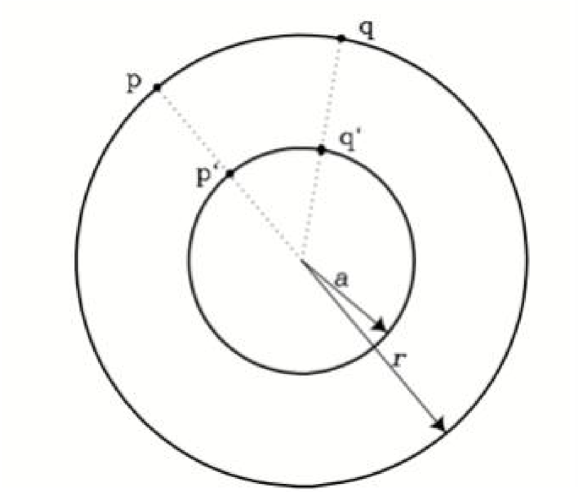
A body that is inside a spherical shell of matter experiences no net gravitational force from the shell.

Second theorem:

The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell’s matter where concentrated into a point at its centre.



## Second Newton theorem



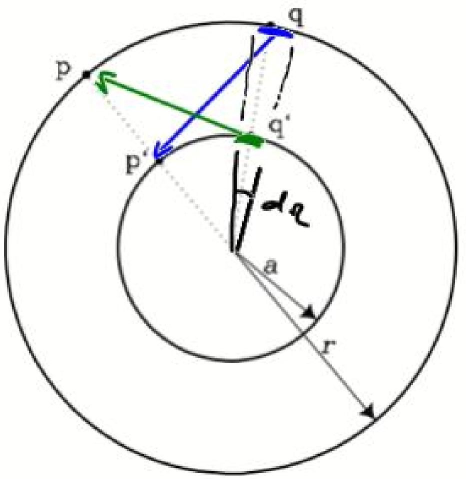

The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center

### Consider two shells

1. inner, with radius $a$ and mass $M$
2. outer, with radius $r$ and mass $M$

Compute 1. $\phi_p = \phi_i(r)$

2. $\phi_{p'} = \phi_o(a) = -\dfrac{GM}{r}$




1. contribution of shell$_i$, in $q'$ with solid angle $\delta\Omega$

- $$\delta\phi_i(p) = -\frac{G\,\delta m_{q'}}{|p-q'|} = -\frac{GM}{|p-q'|}\frac{\delta\Omega}{4\pi}$$

mass inside the solid angle

$$\delta m = \frac{M\,\delta\Omega}{4\pi}$$

2. contribution of shell$_o$, in $q$ with solid angle $\delta\Omega$

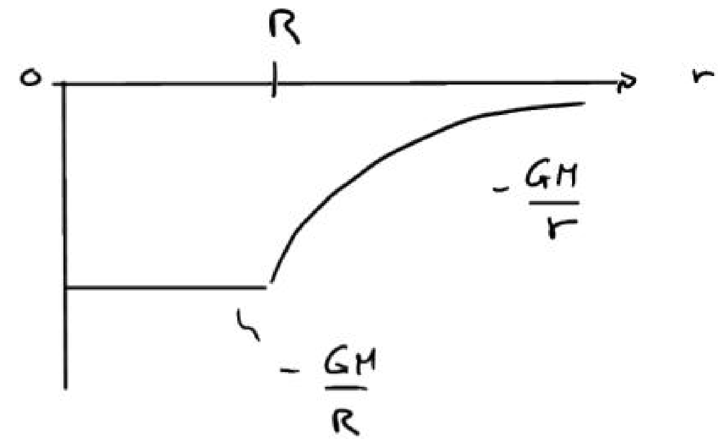
- $$\delta\phi_o(p') = -\frac{G\,\delta m_q}{|p'-q|} = -\frac{GM}{|p'-q|}\frac{\delta\Omega}{4\pi} = \delta\phi_i(p)$$

Summing over all $q'$ $\equiv$ Summing over all $q$

$$\phi_i(p) = \phi_o(p') = -\frac{GM}{r}$$ #

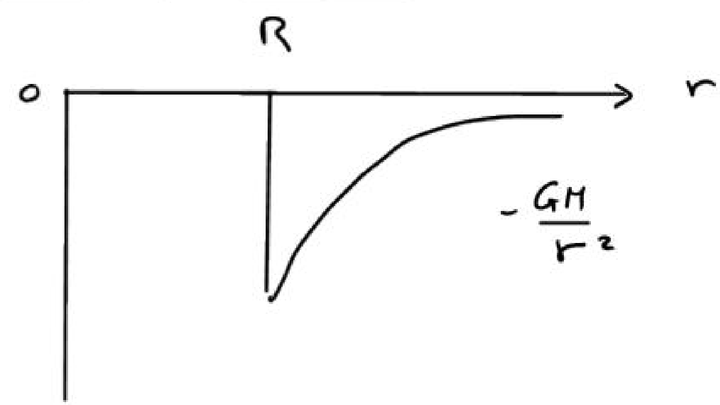
## Total potential of a shell of mass $M$, radius $R$

$$\phi(r) = \begin{cases} -\dfrac{GM}{R} & r < R \\[2ex] -\dfrac{GM}{r} & r \geq R \end{cases}$$




## Total gravitational field of a shell of mass $M$, radius $R$

$$\vec{g}(r) = \begin{cases} 0 & r < R \\[2ex] -\dfrac{GM}{r^2}\,\vec{e}_r & r \geq R \end{cases}$$



# Spherical Systems
# general distribution of mass

$$\rho(\vec{x}) = \rho(r)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

## Gravitational field of a spherical model $\rho(r)$

### Sum of shells

$$g(r) = \int_0^\infty \delta g_{r'}(r) \qquad \delta g_{r'}(r) = \text{force due to the shell of radius } r'$$

$$= \underbrace{\int_0^r \delta g_{r'}(r)}_{\text{inner shells}} + \underbrace{\int_r^\infty \delta g_{r'}(r)}_{\text{outer shells}} \quad = 0 \text{ as we are inside}$$

mass of a shell

$$\delta M(r') = 4\pi r'^2 dr' \rho(r') \qquad \delta g_{r'}(r) = -\frac{G\,\delta M(r')}{r^2} = -4\pi\rho(r')\frac{r'^2}{r^2}dr'$$

$$g(r) = -\frac{G}{r^2}\underbrace{4\pi\int_0^r \rho(r')\,r'^2\,dr'}_{M(r)} = -\frac{G M(r)}{r^2}$$

## Potential of a spherical system $\rho(r)$

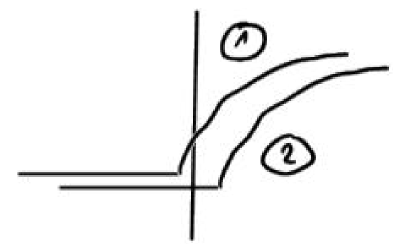
### Sum of shells

$$\phi(r) = \int_0^\infty \delta\phi_{r'}(r)$$

$\delta\phi_{r'}(r)$ = potential on r due to a shell in r'

$$= \int_0^r \delta\phi_{r'}(r) + \int_r^\infty \delta\phi_{r'}(r)$$

$$= -\int_0^r \frac{G\,\delta M(r')}{r} - \int_r^\infty \frac{G\,\delta M(r')}{r'}$$




1 shells of radius $r' < r$

2 shells of radius $r' > r$
constant potential

with $\delta M(r') = 4\pi r'^2 dr' \rho(r')$

$$\phi(r) = -\frac{G}{r} \underbrace{4\pi \int_0^r \rho(r') r'^2 dr'}_{M(r)} - 4\pi G \int_r^{\infty} \rho(r') r' dr'$$

$$\boxed{\phi(r) = \underbrace{-\frac{GM(r)}{r}}_{\substack{\text{contribution} \\ \text{of the mass} \\ \text{inside } r}} \underbrace{- 4\pi G \int_r^{\infty} \rho(r') r' dr'}_{\substack{\text{contribution} \\ \text{of the mass} \\ \text{outside } r}}}$$

Summary : for any spherical mass distribution $\rho(r)$

$$\boxed{g(r) = -\frac{GM(r)}{r^2}}$$

$$M(r) = 4\pi \int_0^\infty \rho(r')\, r'^2\, dr'$$

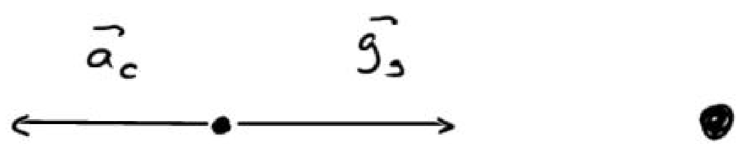
$$\boxed{\phi(r) = -\frac{GM(r)}{r} - 4\pi G \int_r^\infty \rho(r')\, r'\, dr'}$$

Note $\quad g(r) = -\dfrac{\partial \phi}{\partial r}$ $\quad$ as expected from

$$\vec{g}(\vec{x}) = \vec{\nabla} \phi(\vec{x})$$

# Spherical systems : circular speed, circular velocity

Speed of a test particle in a cirular orbit in the potential $\phi(r)$ at a radius $r$ :




$\vec{a}_c$ : centripetal acceleration $\qquad \dfrac{v_c^2}{r}$

$\vec{g}_s$ : gravity acceleration ( spec force ) $\qquad -\dfrac{GM(r)}{r^2} = -\dfrac{\partial\phi}{\partial r}$

$$\boxed{v_c^2 = \frac{GM(r)}{r}}$$

$$\boxed{v_c^2 = r\,\frac{\partial\phi}{\partial r}}$$

$$\boxed{GM(r) = r^2\,\frac{\partial\phi}{\partial r}}$$

$[v_c^2]$ : $\dfrac{erg}{g}$

as $\phi$

$\equiv$ specific energy

## Velocity composition

Note : $V_0^2$ scale with the mass ($M(r)$) : it is thus the "important" quantity ( spec. energy )

Multi-components system : ex : bulge + stellar halo + DM halo

$$\begin{cases} \rho_B(r) \;,\; M_B(r) \;,\; \phi_B(r) \;\rightarrow\; V_{c,B}(r) \\ \rho_H(r) \;,\; M_H(r) \;,\; \phi_H(r) \;\rightarrow\; V_{c,H}(r) \\ \rho_{DM}(r) \;,\; M_{DM}(r) \;,\; \phi_{DM}(r) \;\rightarrow\; V_{c,DM}(r) \end{cases}$$

$$V_{c,tot}^2 = \frac{G M_{tot}(r)}{r} = \frac{G}{r} \sum_i M(r)$$

$$\boxed{V_{c,tot}^2 = \sum_i V_{c,i}^2}$$

$V_c^2 \sim$ energy : extensive quantity

## Period of the circular orbit

$$T(r) = \frac{2\pi r}{V_c(r)} = 2\pi\sqrt{\frac{r^3}{GM(r)}} = 2\pi\sqrt{\frac{r}{\frac{\partial\phi}{\partial r}}}$$

## Circular frequency (angular frequency)

$$\Omega(r) = \frac{2\pi}{T(r)} = \sqrt{\frac{GM(r)}{r^3}} = \sqrt{\frac{1}{r}\frac{\partial\phi}{\partial r}}$$

## Escape speed $V_e$

if $\frac{1}{2}V_e^2 > \phi(r) = E > 0$

the partcle may escape the system

$$V_e(r) = \sqrt{2|\phi(r)|}$$

## Potential energy

from $W = - \int \rho(\vec{x})\, \vec{x} \cdot \vec{\nabla} \phi(\vec{x})\, d^3\vec{x}$

$$\boxed{W = -4\pi G \int_0^{\infty} \rho(r)\, M(r)\, r\, dr}$$

## Gravitational radius

radius at which $\frac{GM^2}{r} = W$

(estimation of the system size)

$$\boxed{r_g = \frac{GM^2}{|W|}}$$

# Spherical systems : useful relations

| | $\rho(r)$ | $\Phi(r)$ | $M(r)$ | $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ |
|---|---|---|---|---|
| $\rho(r)$ | $\rho(r)$ | $\frac{1}{4\,\pi\,G}\,\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$ | $\frac{1}{4\pi r^2}\,\frac{\mathrm{d}M(r)}{\mathrm{d}r}$ | $\frac{1}{4\,\pi\,G}\,\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$ |
| $\Phi(r)$ | $-\frac{G\,M(r)}{r} - 4\pi G\,\int_r^\infty \mathrm{d}r'\,r'\,\rho(r')$ | $\Phi(r)$ | $-G\,\int_r^\infty \mathrm{d}r'\,\frac{M(r')}{r'^2}$ | $-\int_r^\infty \mathrm{d}r'\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ |
| $M(r)$ | $4\pi\,\int_0^r \mathrm{d}r'\,r'^2\,\rho(r')$ | $\frac{r^2}{G}\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ | $M(r)$ | $\frac{r^2}{G}\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ |
| $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ | $\frac{4\pi G}{r^2}\,\int_0^r \mathrm{d}r'\,r'^2\,\rho(r')$ | $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ | $\frac{G\,M(r)}{r^2}$ | $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$ |

Poisson in spherical coordinates

$$\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\,\pi\,G\,\rho(r)$$

Potential in spherical coordinates

$$\Phi(r) = -\frac{G\,M(r)}{r} - 4\pi G\int_r^\infty \rho(r')r'\mathrm{d}r'$$

Mass inside a radius r

$$M(r) = 4\pi\,\int_0^r dr'\,r'^2\,\rho(r')$$

Gradient of the potential in spherical coordinates

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{G\,M(r)}{r^2}$$

# The End