Introduction to Differentiable Manifolds	
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Exercise series 3	**2021–10–05**

Exercise 3.1. Prove that for any open cover $\mathcal{U} = \{U_j\}_{j \in J}$ of a $\mathcal{C}^k$ manifold M there exists a partition of unity $(\xi_j)_{j \in J}$ such that $\operatorname{supp}(\xi_j) \subseteq U_j$ for all j.

Exercise 3.2. A continuous map $f : X \to Y$ is called *proper* if $f^{-1}(K)$ is compact for every compact set $K \subseteq Y$. Show that for every $\mathcal{C}^k$ manifold M there exists a $\mathcal{C}^k$ map $f : M \to [0, +\infty)$ that is proper.

Hint: Note that f must be unbounded unless M is compact. Use a function of the form $f = \sum_{i \in \mathbb{N}} c_i f_i$, where $(f_i)_{i \in \mathbb{N}}$ is a partition of unity and the c_i's are real numbers.

Exercise 3.3. Let M be a $\mathcal{C}^k$ manifold and let U be an open neighborhood of the set $M \times \{0\}$ in the space $M \times [0, +\infty)$. Show that there exists a $\mathcal{C}^k$ function $f : M \to (0, +\infty)$ whose graph is contained in U.

Exercise 3.4. Let M be a $\mathcal{C}^k$ manifold with $k \geq 1$. Show that:

(1)
$$(p, \varphi, v) \sim (\widetilde{p}, \widetilde{\varphi}, \widetilde{v}) \quad \iff \quad \widetilde{p} = p \quad \text{and} \quad \widetilde{v} = \mathrm{D}_{\varphi(p)}(\widetilde{\varphi}\,\varphi^{-1})(v)$$
is an equivalence relation between coordinatized tangent vectors.

(2) Fixed a point $p \in M$ and a $\mathcal{C}^k$ chart φ defined on p, the function $\mathbb{R}^n \to T_pM$ sending $v \mapsto [p, \varphi, v]$ is a bijection.

(3) Vector addition and scalar multiplication are well defined and make T_pM a real vector space of dimension n.

(4) The differential of a $\mathcal{C}^k$ map $f : M \to N$ at a point $p \in M$ is a well-defined linear map $D_pf : T_pM \to T_pN$.

(5) *Chain rule*: for $\mathcal{C}^k$ maps $f : M \to N$, $g : N \to L$ and a point $p \in M$,
$$D_p(g \circ f) = D_{f(p)}g \circ D_pf.$$
In particular, if f is a diffeo, then D_pf has inverse $(D_pf)^{-1} = D_{f(p)}(f^{-1})$.

(6) *Change of coordinates*: Let $X \in T_pM$ be a tangent vector and let φ, $\widetilde{\varphi}$ be $\mathcal{C}^k$ charts of M defined at a p. Let $(X^i)_i$ be coordinate tuple of X with respect to the basis $\left(\frac{\partial}{\partial \varphi^i}\big|_p\right)_i$, and let $(\widetilde{X}^j)_j$ be the coordinate tuple of X with respect the basis $\left(\frac{\partial}{\partial \widetilde{\varphi}^j}\big|_p\right)_j$, so that
$$X = \sum_i X^i \frac{\partial}{\partial \varphi^i}\Big|_p = \sum_j \widetilde{X}^j \frac{\partial}{\partial \widetilde{\varphi}^j}\Big|_p.$$
Show that
$$\widetilde{X}^j = \sum_i X^i \frac{\partial \widetilde{\varphi}^j}{\partial \varphi^i}\Big|_{\varphi(p)},$$
where $\frac{\partial \widetilde{\varphi}^j}{\partial \varphi^i}\Big|_{\varphi(p)}$ is the coefficient (j, i) of the matrix expression of the linear transformation $D_{\varphi(p)}(\widetilde{\varphi} \circ \varphi) : \mathbb{R}^n \to \mathbb{R}^n$.

Exercise 3.5 (Velocity vectors of curves)**.** Let M be a $\mathcal{C}^k$ differentiable manifold. The *velocity vector* of a differentiable curve $\gamma : I \subseteq \mathbb{R} \to M$ at an instant $t \in I$ is the vector $g'(t) := D_tg(1) \in T_{\gamma(t)}M$. Show that for any tangent vector $X \in TM$ there exists a $\mathcal{C}^k$ curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma'(0) = X$.

Exercise 3.6 (Spherical coordinates on $\mathbb{R}^3$)**.** Consider the following map defined for $(r, \varphi, \theta) \in W := \mathbb{R}^+ \times (0, 2\pi) \times (0, \pi)$:
$$\Psi(r, \varphi, \theta) = (r \cos\varphi \sin\theta, r \sin\varphi \sin\theta, r \cos\theta) \in \mathbb{R}^3.$$

Check that Ψ is a diffeomorphism[1] onto its image $\Psi(W) =: U$. We can therefore consider Ψ^{-1} as a smooth chart on $\mathbb{R}^3$ and it is common to call the component functions of Ψ^{-1} the **spherical coordinates** (r, φ, θ).

Express the coordinate vectors of this chart

$$\frac{\partial}{\partial r}\Big|_p, \frac{\partial}{\partial \varphi}\Big|_p, \frac{\partial}{\partial \theta}\Big|_p$$

at some point $p \in U$ in terms of the standard coordinate vectors $\frac{\partial}{\partial x}\big|_p, \frac{\partial}{\partial y}\big|_p, \frac{\partial}{\partial z}\big|_p$.

Exercise 3.7 (The tangent plane of the sphere)**.** Consider the inclusion $\iota : S^2 \to \mathbb{R}^3$, where we endow both spaces with the standard smooth structure. Let $p \in S^2$. What is the image of $D_p\iota : T_pS^2 \to T_p\mathbb{R}^3$? (Identify $T_p\mathbb{R}^3$ with $\mathbb{R}^3$ in the standard way. So the result should be the equation for a plane in $\mathbb{R}^3$.)

Hint: Use Exercise 6 on spherical coordinates.

[1] Here "diffeomorphism" is meant in the standard sense of maps between open subsets of $\mathbb{R}^3$. The inverse function theorem can be useful here.