Convention: We understand a subset of a topological space to be automatically endowed with the subspace topology and a product of topological spaces to be endowed with the product topology (unless stated otherwise).

**Exercise 1.1** (Locally Euclidean spaces)**.** Show that the following definition of "locally Euclidean space" is equivalent to the one given in the lecture notes:

*Definition.* Let $n \in \mathbb{N} = \{0, 1, \dots\}$. A topological space $M$ is **locally Euclidean** of dimension $n$ if every point $p \in M$ has a neighbourhood that is homeomorphic to $\mathbb{R}^n$.

*Solution.* The new Definition clearly implies the Definition of "locally Euclidean" given in the lecture note as $\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$. Conversely, for every $p \in M$, there are a neighborhood $U$ and a homeomorphism $\varphi$ s.t. $\varphi(U)$ is an open subset of $\mathbb{R}^n$. Then we can find a ball $B(\varphi(p), r) \subseteq \varphi(U) \subseteq \mathbb{R}^n$ for some $r > 0$. Let us consider the map $\psi : B(\varphi(p), r) \to \mathbb{R}^n$ given by $\psi(x) := \frac{x-\varphi(p)}{r-\|x-\varphi(p)\|}$. One can verify that $\psi$ is a homeomorphism with inverse $\psi^{-1}(y) := \varphi(p) + \frac{y}{1+\|y\|}$. Set $U' := \varphi^{-1}(B(\varphi(p), r)) \subseteq M$, which is a neighborhood of $p$ in $M$ and the map $\theta := \psi \circ \varphi : U' \to \mathbb{R}^n$. We showed that $\theta$ is a homeomorphism since $\psi$ and $\varphi$ are both homeomorphisms. □

**Exercise 1.2** (Examples of locally Euclidean spaces)**.** Which of the following spaces are locally Euclidean? Which are (globally) homeomorphic to Euclidean space?

- an open ball in $\mathbb{R}^n, n \in \mathbb{N}$

  *Solution.* $B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}$ is globally homeomorphic to $\mathbb{R}^n$. And the homeomorphism $\varphi(x) = R\frac{x}{1+\|x\|}$ maps $\mathbb{R}^n$ into $B_R(0)$. Observe that $\varphi^{-1}(x) = \frac{x}{R-\|x\|}$. □

- the closed interval $[0, 1] \subset \mathbb{R}$

  *Solution.* The interval $[0, 1]$ is neither locally nor globally homeomorphic to $\mathbb{R}$. Global homeomorphism is excluded since $[0, 1]$ is compact but $\mathbb{R}$ is not. A continuous map will map a compact set to a compact set. Next, suppose, for a contradiction, that $[0, 1]$ is locally homeomorphic to $\mathbb{R}$ and denote by $\varphi$ the homeomorphism. Take one of the extrema (e.g. 0 or 1) of the interval and consider an open neighborhood in the subspace topology: $U = [0, \varepsilon)$ for example. $U$ is connected and open hence $\varphi(U)$ is connected and open as well. Furthermore $(0, \varepsilon)$ is still open and connected but its image through $\varphi$ is not connected because we remove $\varphi(0)$. □

- the circle $S^1 \subset \mathbb{R}^2$

  *Solution.* $S^1$ is locally homeomorphic to $\mathbb{R}$. In fact denote $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and define the nord and south stereographic projections as

$$\begin{aligned} p_\pm : S^1 \setminus \{(0, \pm 1)\} &\to \mathbb{R} \\ (x, y) &\mapsto \frac{x}{1 \mp y} \end{aligned}$$

  It is not difficult to verify that for every point $p \in S^1$ there exists an open set $U$ containing $p$, such that the image of $U$ via one of the two stereographic projections is an open set in $\mathbb{R}$. □

- the zero set of the function $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = xy$

  *Solution.* The set $E = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ is not locally Euclidean because no neighborhood $U$ of the origin in $E$ is homeomorphic to $\mathbb{R}$. To prove this last statement argue by contradiction: suppose that there exist an homeomorphism $\varphi : U \to \mathbb{R}$. Then $U' = U \setminus \{(0,0)\}$ has at least 4 connected components while $\varphi(U')$ has just 2 connected components. The contradiction arises from the fact that a homeomorphism preserves connected components. □

- the "corner" $\{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0, xy = 0\}$.

  *Solution.* The set $E\{(x,y) \in \mathbb{R}^2_+ : xy = 0\}$ is globally homeomorphic to $\mathbb{R}$ via the homeomorphism

$$\begin{aligned} \varphi : E &\to \mathbb{R} \\ (x,y) &\mapsto \begin{cases} x & \text{if } y = 0 \\ -y & \text{if } x = 0 \end{cases} \end{aligned}$$

□

**Exercise 1.3** ("The line with two origins"). Let $X := \{\pm 1\} \times \mathbb{R}$ and let $M$ be the quotient of $X$ by the equivalence relation generated by $(1, x) \sim (-1, x)$ iff $x \in \mathbb{R}\setminus\{0\}$. We endow $M$ with the quotient topology. Show that $M$ is locally Euclidean and second countable, but not Hausdorff.

*Solution.* Denote $\pi : X \to M$ the quotient map $(i, x) \mapsto [(i, x)]$.

The two "origins" are the equivalence classes of the points $(i, 0) \in X$ (for $i = \pm 1$); these classes have just one element each and we denote them $0_i = [(i, 0)] = \{(i, 0)\} \in M$. In contrast, the equivalence class of any other point $(i, x) \in X$ with $x \neq 0$ is the two-point set $\widetilde{x} = [(i, x)] = \{(1, x), (-1, x)\} \in M$. Therefore $M$ is the set of equivalence classes

$$M = {}^{X}/_{\sim} = \{0_+\} \cup \{0_-\} \cup \{\widetilde{x}\}_{x \neq 0}.$$

The space $M$ is locally Euclidean of dimension 1 because it is the union of two open sets $\mathbb{R}_i = \{[(i, x)] \in M : x \in \mathbb{R}\}$ (for $i = \pm 1$), each of which is homeomorphic to $\mathbb{R}$ via the map

$$\mathbb{R} \to \mathbb{R}_i : x \mapsto [(i, x)].$$

To see that the sets $\mathbb{R}_i$ are open in the quotient topology, note that $\pi^{-1}(\mathbb{R}_i) = X \setminus 0_{-i}$, which is open in $X$.

Moreover, $M$ is second countable because it is the union of two second countable open subsets, namely, the sets $\mathbb{R}_i$.

Finally $M$ is not Hausdorff since every pair of open subsets containing $0_-$ and $0_+$ respectively have non-empty intersection. □

**Exercise 1.4** (New manifolds from old). Convince yourself that[1]:

(a) A subset of a Hausdorff (resp. second countable) topological space is a Hausdorff (resp. second countable) space.

*Solution.* Let $S \subset X$ be a subset of a topological space $X$.

If $X$ is Hausdorff, to show that $S$ is Hausdorff as well, take two distinct points $p, q \in S$. Let $U, V$ be disjoint neighborhoods of $p, q$ in $X$. Then the sets $U' = U \cap S$, $V' = V \cap S$ are disjoint open neighborhoods of $p, q$ in $S$.

If $X$ is second countable, let $\{U_i\}_{i \in I}$ be a countable basis. Then $\{U_i \cap S\}_{i \in I}$ is a countable basis for $S$. Thus $S$ is also second-countable. □

(b) An open subset of a topological $n$-manifold is a topological $n$-manifold.

*Solution.* Let $S \subset M$ be an open set. Then $\forall p \in S$, we can find a neighborhood $U \subset M$ that is homeomorphic to an open set $V \subseteq \mathbb{R}^n$. Let $\varphi : U \to V$ be a homeomorphism. Since $S$ is open in $M$, thus $U \cap S$ is also open in $U$. Therefore $\varphi(U \cap S)$ is open in $V$ (and thus, in $\mathbb{R}^n$), and the restricted map $\varphi : U \cap S \to \varphi(U \cap S)$ is a homeomorphism. This shows that $S$ is locally Euclidean. The Hausdorff and second countablity properties of $S$ follow by (a). □

[1] This means that if you find the exercise trivial you don't have to write down a detailed proof.

(c) The product of two Hausdorff (resp. second countable) spaces is Hausdorff (resp. second countable).

*Solution.* Let us show that the product of two Hausdorff spaces $X_0$, $X_1$ is Hausdorff as well. Let $x = (x_0, x_1),\ y = (y_0, y_1)$ be two distinct points in $X_0 \times X_1$. We assume w.l.o.g. that $x_0 \neq y_0$ in $X_0$. Then we can find disjoint open sets $U, V \subset X_0$ s.t. $x_0 \in U$ and $y_0 \in V$. Then the sets $U \times X_1$, $V \times X_1$ are disjoint neighborhoods of $x$ and $y$. These sets are open in the product topology, therefore the product space $X \times Y$ is Hausdorff.

For second-countable spaces $X$ and $Y$, let $\{U_i\}_{i\in I}$, $\{V_j\}_{j\in J}$ be respective countable bases. Then $\{U_i \times V_j\}_{i\in I, j\in J}$ is a countable basis of $X \times Y$. □

(d) The product of two topological manifolds is a topological manifold. What is its dimension?

*Solution.* Let $M$, $N$ be topological manifolds of dimensions $m$, $n$ respectively. Let us show that $M \times N$ is a locally Euclidean of dimension $m + n$. (The Hausdorff and second countability properties follow by (c), thus we conclude that $M \times N$ is a topological $(m + n)$-manifold.)

For every $(p, q) \in M \times N$, we can find open neighborhoods $U \subset M$, $V \subseteq N$ of $p$ and $q$ that are respectively homeomorphic to $\mathbb{R}^m$ and $\mathbb{R}^n$. It follows that the set $U \times V$ (which is an open neighborhood of $(p, q)$ in the product topology) is homeomorphic to $\mathbb{R}^{m+n}$. □

**Exercise 1.5** (Projective space)**.** We define $\mathbb{P}^n$ as the quotient space of $\mathbb{R}^{n+1}\setminus\{0\}$ by the equivalence relation $x \sim y$ iff $x = \lambda y$ for some $\lambda \in \mathbb{R}$.

Show that $\mathbb{P}^n$, endowed with the quotient topology, is a topological manifold. *Hint: For $x \in \mathbb{R}^{n+1}\setminus\{0\}$ let $[x]$ denote its equivalence class in $\mathbb{P}^n$.*

*To show that $\mathbb{P}^n$ is locally Euclidean consider for $i = 0, 1, \dots, n$ the sets*

$$U_i := \{[x] \in \mathbb{P}^n \mid x_i \neq 0\}$$

*and the coordinate chart on $U_i$:*

$$\varphi_i : U_i \to \mathbb{R}^n : [(x_0, \dots, x_n)] \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right).$$

*Solution.* We will prove several facts

(i) *The quotient map $\pi : \mathbb{R}^{n+1}_{\neq 0} \to \mathbb{P}^n$ is open.* Let $U \subseteq \mathbb{R}^{n+1}_{\neq 0}$ be an open set. To see that $\pi(U)$ is open in the quotient topology, we verify that its preimage $\pi^{-1}(\pi(U)) = \bigcup_{\lambda\neq 0} \lambda U$ is open, being a union of open sets $\lambda U$.

(ii) *$\mathbb{P}^n$ is second countable.* Cleary $\mathbb{R}^{n+1}_{\neq 0}$ is second countable, being a subspace of the countable space $\mathbb{R}^{n+1}$. Let $(W_j)_{j\in\mathbb{N}}$ be a countable topological basis for $\mathbb{R}^{n+1}_{\neq 0}$. Then $(\pi(W_j))_{j\in\mathbb{N}}$ is a countable basis for $\mathbb{P}^n$, being the image of a topological basis by a surjective open map.

(iii) *$\mathbb{P}^n$ is locally homeomorphic to $\mathbb{R}^n$.* To see that $U_i$ is open in the quotient topology, we verify that its preimage $\pi^{-1}(U_i)$ is open in $\mathbb{R}^{n+1}_{\neq 0}$. And indeed, its preimage is the set

$$V_i = \{x \in \mathbb{R}^{n+1}_{\neq 0} : x_i \neq 0\},$$

which is open. To see that $\varphi_i$ is a bijection, let's find its inverse function. A direct calculation provides us with the formula

$$\varphi_i^{-1}(x_0, \dots, x_{n-1}) = [x_0, \dots, x_{i-1}, 1, x_i, \dots, x_n].$$

This formula also shows that $\varphi_i^{-1}$ is continuous. Finally, to see that $\varphi_i$ itself is continuous, it suffices to note that the composite map

$$\widetilde{\varphi}_i = \varphi_i \circ \pi|_{V_i}^{U_i} : V_i \to \mathbb{R}^n$$

is continuous. (Here we are using the universal property of the quotient. The map $\pi|_{V_i}^{U_i}$ is a quotient map because it is surjective and open.) And indeed, the map

$$\widetilde{\varphi}_i(x_0, \dots, x_n) = \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)$$

is continuous. Thus $\varphi_i$ is a homeomorphism between the open set $U_i \subseteq \mathbb{P}^n$ and $\mathbb{R}^n$.

(iv) $\mathbb{P}^n$ *is Hausdorff.* Let $[x]$, $[y]$ be two distinct points of $\mathbb{P}^n$. We will show there are disjoint open neighboorhoods $U, V$ of $x, y$ in $\mathbb{R}^{n+1}_{\neq 0}$ that are saturated by the equivalence relation $\sim$. Then it follows that $\pi(U)$, $\pi(V)$ are disjoint respective neighorhoods of $[x]$, $[y]$ in $\mathbb{P}^n$.

We consider two cases. The first case is when both points $x, y$ have a nonzero coordinate at the same place, i.e. $x_i, y_i \neq 0$ for some $i$. Then the points $[x], [y]$ are contained in the open subset $U_i$, which is homeomorphic to $\mathbb{R}^n$, hence Hausdorff. Thus there are disjoint open neighborhoods $V, W$ of $[x], [y]$ in $U_i$, and these sets are also open in $\mathbb{P}^n$.

The remaining case is when there is no $i$ such that $x_i, y_i \neq 0$. In this case let $i$, $j$ such that $x_i \neq 0$ (hence $y_i = 0$) and $y_j \neq 0$ (hence $x_j = 0$). Then we have in $\mathbb{R}^{n+1}_{\neq 0}$ the saturated open sets

$$V = \{z \in \mathbb{R}^{n+1}_{\neq 0} : |z_i| > |z_j|\}$$

$$W = \{z \in \mathbb{R}^{n+1}_{\neq 0} : |z_j| > |z_i|\}$$

which are disjoint neighborhoods of $x$ and $y$ respectively.

□

**Exercise 1.6.** Show that the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, defined as the quotient of $\mathbb{R}^n$ by the equivalence relation

$$x \sim y \iff y - x \in \mathbb{Z}^n,$$

is a topological $n$-manifold.

*Solution.* Let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the quotient map

$$x \mapsto [x] = \{x + z : z \in \mathbb{Z}^n\}.$$

Note that two points $x, y$ of $\mathbb{R}^n$ are in the same equivalence class if and only if the coordinates $x_i, y_i$ coincide modulo 1 for each $i$. (In other words, the real numbers $x_i, y_i$ have the same integer part.)

(1) $\pi$ *is an open map.* Indeed, let $U \subseteq \mathbb{R}^n$ be an open set. To see that $\pi(U)$ is open in the quotient topology, we verify that its preimage $\pi^{-1}(\pi(U)) = \bigcup_{z\in\mathbb{Z}^n} U + \{z\}$ is open, being a union of translate copies of $U$.

(2) $\mathbb{T}^n$ *is second countable.* Indeed, the image of any (countable) topological basis by a surjective open map is a (countable) topological basis.

(3) To prove that $\mathbb{T}^n$ is locally Euclidean, we show that:
*The quotient map* $\pi$ *is locally injective*, i.e., each point $x \in X$ has an open neighborhood $U$ where the quotient map $\pi$ is injective. Indeed, let $U$ be an open neighborhood of $x$ with diameter $< 1$. Then there are no two different points $x', x'' \in U$ such that $x'' - x' \in \mathbb{Z}^n$. Therefore $\pi$ is injective on $U$. Furthermore, the set $\pi(U)$ is open in $\mathbb{T}^n$, and the restricted quotient map $\pi : U \to \pi(U)$ is a homeomorphism because it is bijective and open. This proves that the $\mathbb{T}^n$ is locally Euclidean of dimension $n$.

(4) $\mathbb{T}^n$ *is Hausdorff.* Take two different points $\pi(x), \pi(y) \in \mathbb{T}^n$. Then there is some $i$ such that the coordinates $x_i, y_i$ are different modulo 1. Let $\varepsilon > 0$ be the distance between the numbers $x_i, y_i$ taken modulo 1, that is, $\varepsilon = \min_{z\in\mathbb{Z}} y_i - (x_i + z)$. This number is the least we would have to move $y_i$ so that it coincides with $x_i$ modulo 1. Then the Euclidean open balls $U = B(x, \frac{\varepsilon}{2})$, $V = B(y, \frac{\varepsilon}{2})$ satisfy $\pi(U) \cap \pi(V) = \emptyset$ because for every pair of points in $U$ and $V$, their $i$-th coordinates do not coincide modulo 1. The sets $\pi(U)$, $\pi(V)$

are disjoint open neighborhoods of $\pi(x)$, $\pi(y)$, as needed to show that $\mathbb{T}^n$ is Hausdorff.

□