Exercise 4.1 (A little heads-up regarding coordinate vectors)**.** Let φ and ψ be smooth charts on a smooth manifold M defined on the same domain U. If the first coordinate functions φ^0 and ψ^0 agree ($\varphi^0 \equiv \psi^0$ on U), this does *not* imply $\frac{\partial}{\partial \varphi^0}|_p = \frac{\partial}{\partial \psi^0}|_p$ for $p \in U$.

Work out a simple example of this fact e.g. on $M = \mathbb{R}^2$ by considering on the one hand the Cartesian coordinates (x, y) and on the other hand the chart (u, v) given by $u = x$, $v = x + y$.
This shows that $\frac{\partial}{\partial \varphi^i}|_p$ depends on the whole system $(\varphi^0, \ldots, \varphi^{n-1})$, not only on φ^i.

Exercise 4.2 (The tangent space of a vector space)**.** Let V be an n-dimensional vector space.

(a) Let $\mathcal{A}$ be the set of linear isomorphisms $\varphi : V \to \mathbb{R}^n$. Show that there is a topology on V such that all $\varphi \in \mathcal{A}$ are homeomorphisms. Show that $\mathcal{A}$ is a smooth atlas on V. *In other words, any vector space has a natural smooth structure.*

(b) Fix $a \in V$. To every $v \in V$ we associate the curve passing through a

$$\gamma_v : \mathbb{R} \to V : t \mapsto a + tv$$

Show that the map $\Phi_a : V \to T_a V : v \mapsto \gamma_v'(0)$ is an isomorphism of vector spaces. *Hence we can identify a vector space with its tangent space in a canonical way.*

(c) Let $f : V \to W$ be a *linear* map between vector spaces V, W. Consider the differential $D_a f : T_a V \to T_{F(a)} W$ at any point $a \in V$. Identifying $T_a V \cong V$ and $T_{f(a)} W \cong W$ via the isomorphisms Φ_a, $\Phi_{f(a)}$, show that $D_a f$ is identified with f. That is, show that the following diagram commutes:

$$
\begin{array}{ccc}
T_a V & \xrightarrow{\ D_a f\ } & T_{F(a)} W \\
{\scriptstyle \Phi_a} \big\uparrow & & \big\uparrow {\scriptstyle \Phi_{F(a)}} \\
V & \xrightarrow{\ \ F\ \ } & W
\end{array}
$$

Exercise 4.3 (Differential of the determinant function)**.** Consider the determinant function $\det : M_n(\mathbb{R}) \to \mathbb{R}$, where $M_n(\mathbb{R}) \simeq \mathbb{R}^{n \times n}$ is the vector space of real $n \times n$, with its natural smooth structure. We want to compute its differential transformation $D_A \det$ at any matrix $A \in \mathrm{GL}_n(\mathbb{R})$ (i.e. at any invertible matrix),

$$D_A \det : T_A M_n(\mathbb{R}) \to T_{\det(A)} \mathbb{R}$$

(Note that we may identify $T_A M_n(\mathbb{R})$ with $M_n(\mathbb{R})$ and $T_{\det(A)} \mathbb{R}$ with $\mathbb{R}$.)

(a) Verify that det is a smooth function.
   Hint: Write the determinant as a sum over all n-permutations.

(b) Show that the differential of det at the identity matrix $I \in M_n(\mathbb{R})$ is

$$D_I \det(B) = \mathrm{tr}(B).$$

where tr denotes the trace.

(c) Show that for arbitrary $A \in \mathrm{GL}_n(\mathbb{R}), B \in M_n(\mathbb{R})$.

$$D_A \det(B) = (\det A)\, \mathrm{tr}(A^{-1} B)$$

   Hint: Write $\det(A + tB) = (\det A)(\det(I + tA^{-1}B))$.

(d) Show that $D_A \det$ is the null linear transformation if $A = 0$ and $n \geq 2$.

Exercise 4.4 (Diffeomorphic manifolds have the same dimension)**.** Let M and N be nonempty diffeomorphic manifolds. Show that $\dim M = \dim N$.

Exercise 4.5 (Tangent vectors as derivations). Let M be a $\mathcal{C}^k$ manifold, $k \geq 1$, and let $p \in M$. Show that the map $\nu_p : X \in T_pM \mapsto D_X \in Der_pM$ defined by

$$D_X : f \in C^k(M, \mathbb{R}) \mapsto D_pf(X) \in T_{f(p)}\mathbb{R} \cong \mathbb{R}$$

is linear and injective.

Hint: To prove injectivity, take a chart φ that is defined at p. Any vector $X \in T_pM$ can be written as $X = \sum_i X^i \frac{\partial}{\partial \varphi^i}|_p$. Show that $D_X(\varphi^j) = X^j$.