Introduction to Differentiable Manifolds

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Contents

Main references

[Lee10] John M. Lee. *Introduction to Topological Manifolds*. Springer, 2010.

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second edition. Springer, 2013.

[Tu11] Loring W Tu. *An Introduction to Manifolds*. Springer, 2011.

0.1 Practical remarks about the course

0.1.1 Content for the exam

Studying the content under a heading marked by an asterisk is not mandatory by itself. You should read it only if you find it interesting or helpful for understanding the rest.

1 Manifolds

The goal of this course is to extend differential and integral calculus from Euclidean space $\mathbb{R}^n$ to all *differentiable manifolds* such as the n-sphere, the n-torus, etc. Roughly speaking, a differentiable manifold is a space that

- is endowed with a certain topology,
- has, in addition, a *differentiable structure* that allows us to distinguish whether a map is differentiable or not, rather than just continuous, and
- locally looks like Euclidean space $\mathbb{R}^n$.

1.1 Topological manifolds

[Lee13], Chapter 1 and [Lee10], Chapter 2

Let us pospone the question of differentiability and focus on topology. As said, we want to study spaces that "locally look like" Euclidean space $\mathbb{R}^n$.

Definition 1.1.1 (Locally Euclidean space)**.** Let $n \in \mathbb{N} = \{0, 1, \dots\}$. A topological space M is **locally Euclidean** of dimension n at a point $p \in M$ if the point p has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. If this holds for all points $p \in M$, we say that M is locally Euclidean of dimension n.

A typical example is the circle: it is locally Euclidean of dimension 1 but not globally homeomorphic to any subset of $\mathbb{R}$.

Example 1.1.2. The **circle** $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ (with the subspace topology) is locally Euclidean of dimension 1: let $(x_0, y_0) \in \mathbb{S}^1$, wlog $y_0 > 0$, then $U := (\mathbb{R} \times \mathbb{R}^+) \cap \mathbb{S}^1$ is an open subset of $\mathbb{S}^1$ containing (x_0, y_0) and homeomorphic to $(-1, 1)$ via the map $U \to (-1, 1)$ that sends $(x, y) \mapsto x$.

We will see more examples later on (see e.g. Examples 1.1.11 below). Let us make some general comments.

Remark 1.1.3. If a space M is locally Euclidean of dimension 0, then every point has a neighborhood homeomorphic to $\mathbb{R}^0 = \{0\}$, i.e. a point. In other words, M is a discrete topological space.

Remark 1.1.4. In the definition of locally Euclidean space, we could have replaced "...homeomorphic to an open subset of $\mathbb{R}^n$" by "...homeomorphic to $\mathbb{R}^n$". (Exercise.)

Remark 1.1.5. Brouwer's theorem of *invariance of domain* implies[1] that if two nonempty open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$. It follows that the dimension of a locally Euclidean space at each point can be defined unambiguously. Furthermore, it is easy to prove that the dimension is constant throughtout each connected component. Thus the only way to get a locally Euclidean space of mixed dimensions is to make a disjoint union of components of different dimensions. Anyway, in the definition of topological manifold (see below) we will not admit this kind of spaces.

[1] The actual theorem says that if $U \subseteq \mathbb{R}^m$ is an open set, then any injective continuous map $f : U \to \mathbb{R}^m$ is an open map.

For the definition of *topological manifold* we demand some further topological properties that ensure that the space is topologically "well-behaved". (For instance, we want the limit of every sequence to be unique.)

Definition 1.1.6. A **topological manifold of dimension** n, or **topological** n**-manifold**, is a topological space M that is locally Euclidean of dimension n, Hausdorff[2] and second countable.[3]

A **topological manifold** is a topological space that is a topological n-manifold for some n.

Side note: Make sure you are familiar with some basic definitions from topology such as Hausdorff, second countable, connected and compact spaces, and the construction of subspace, product, coproduct and quotient topologies. Chapters 2 and 3 in [Lee10] provides a succinct overview of everything we need.

Remark 1.1.7. The conditions of Hausdorff resp. second countable in Definition 1.1.6 are not redundant. For example, the *line with two origins* (see Exercises) is a locally Euclidean, second countable space that is not Hausdorff. The *long line* and the *Prüfer surface* (see Wikipedia if interested) are locally Euclidean of dimension 1 and 2 respectively, Hausdorff, and connected, but not second countable.

The homeomorphisms that locally identify a topological manifold with Euclidean space are called *charts*:

Definition 1.1.8 (Coordinate charts)**.** Let M be a topological n-manifold. A **chart** (or **coordinate chart**) for M is a homeomorphism $\varphi : U \to V$, where $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open sets. Its inverse φ^{-1} is a **local parametrization** of M. An **atlas** for M is a collection of charts whose domains cover M.

For the moment, we can see an atlas simply as a way of showing that a space is locally Euclidean.

Remark 1.1.9. Some authors define a chart for M as a *pair* (U, φ) or even a *triple* (U, V, φ) where $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open sets and $\varphi : U \to V$ is a homeomorphism. Here, instead, we consider the sets U and V as part of the function φ, namely, its domain $\mathrm{Dom}(\varphi)$ and codomain $\mathrm{Cod}(\varphi)$, thus there's no need not specify them separately. [4] From this point of view, the letters "U", "V" are just shorter names for the sets $\mathrm{Dom}(\varphi)$, $\mathrm{Cod}(\varphi)$. We may use the notation (U, φ) when we want to give the name "U" to the domain of the chart φ.

Convention: When talking about subsets (resp. quotients, products, disjoints unions) of topological spaces we'll assume that they are endowed with the subspace (resp. quotient, product, coproduct) topology unless otherwise stated.

Using this convention, let us mention some easy ways to construct new topological manifolds from old ones.

Proposition 1.1.10 (New manifolds from old)**.** *The properties of being Hausdorff or second countable are preserved by taking subspaces, finite products and countable coproducts. In consequence:*

[2] Recall that a topological space X is **Hausdorff** if every two different points $x, y \in X$ have disjoint neighborhoods.

[3] Recall that a topological space X is **second countable** if its topology admits a countable base. A **base** for a topology is a family $\mathcal{B}$ of open sets such that every open set is a union of some sets of $\mathcal{B}$.

[4] Formally, a function f is a triple $f = (X, Y, \Gamma)$ where X, Y are sets (called the **domain** and **codomain** of f, and denoted $\mathrm{Dom}(f)$ and $\mathrm{Cod}(f)$), and Γ is a subset of $X \times Y$ (called the **graph** of f, denoted $\mathrm{Gra}(f)$), such that for each $x \in X$ there is a unique $y \in Y$ (called the **image** of x by f, denoted $f(x)$) such that $(x, y) \in \Gamma$.

- *An open subset of a topological n-manifold is a topological n-manifold.*
- *A disjoint union $M = \coprod_i M_i$ of countably many topological n-manifolds M_i is a topological n-manifold.*
- *A product $M = \prod_i M_i$ of finitely many topological manifolds M_i is a topological manifold of dimension* $\dim(M) = \sum_i \dim(M_i)$.

Proof. Exercise. $\square$

Example 1.1.11 (Examples of topological manifolds)**.**

(a) Of course any open subset of $\mathbb{R}^n$ is a topological manifold.

(b) An example of topological n-manifold is the **graph**

$$\Gamma_f := \{(x, f(x)) \mid x \in U\} \subseteq U \times \mathbb{R}^m$$

of a continuous function $f : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is an open set. Indeed, it is homeomorphic to U via the **graph parametrization**

$$\begin{array}{ccc} U & \to & \Gamma_f \\ x & \mapsto & (x, f(x)), \end{array} \quad \text{whose inverse is the projection} \quad \begin{array}{ccc} \Gamma_f & \to & U \\ (x, y) & \mapsto & x. \end{array}$$

(c) The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a topological manifold. Being a subset of $\mathbb{R}^{n+1}$ it is Hausdorff and second countable. A possible choice of atlas is given by the so-called **graph coordinates**: Cover $\mathbb{S}^n$ by the $2(n+1)$ open sets $U_i^\pm := \{x \in \mathbb{R}^n \mid \pm x_i > 0\}$, then $\mathbb{S}^n \cap U_i^\pm$ is homemorphic to the open unit n-ball $\mathbb{B}^n$ via the projection[5]

$$\begin{array}{cccc} \varphi_i^\pm : & \mathbb{S}^n \cap U_i^\pm & \to & \mathbb{B}^n \\ & (x_0, \ldots, x_n) & \to & (x_0, \ldots, \widehat{x_i}, \ldots, x_n). \end{array}$$

The maps $\varphi_i^\pm$ are coordinate charts for $\mathbb{S}^n$; we call them *graph coordinates.* Locally this is a special case of the previous item (b): each set $\mathbb{S}^n \cap U_i^\pm$ is (up to permutation of coordinates) the graph of the continuous function on the unit n-ball $\mathbb{B}^n(0)$:

$$\mathbb{B}^n \to \mathbb{R} : y \mapsto \pm\sqrt{1 - \textstyle\sum_i y_i^2}.$$

(d) Real projective space $\mathbb{P}^n$ is a topological n-manifold (exercise).

(e) The torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a topological n-manifold (exercise).

(f) More generally, If M is a topological n-manifold and G is a group of homeomorphisms of M that acts properly discontinuously and without fixed points, then the quotient space M/G is a topological n-manifold.

(g) If M is a topological n-manifold and $\pi : N \to M$ is a covering map (where both M and N are connected), then N is a topological n-manifold. In particular, the universal covering space of any connected topological manifold is a topological manifold.

We will not prove the following result (although it can be done elementarily).

Theorem 1.1.12 (Classification of topological 1-manifolds)**.** *Every connected topological 1-manifold is homeomorphic to either $\mathbb{S}^1$ (if it is compact) or to $\mathbb{R}$ (if it is not compact).*

[5]The hat on $\widehat{x_i}$ means that we omit the respective coordinate x_i.

1.2 Differentiable manifolds

Our next goal is to define a kind of spaces and maps called *differentiable manifolds* and *differentiable maps* (or, more precisely $\mathcal{C}^k$ manifolds and maps) with which we can actually do differential calculus. Topological manifolds do not have enough structure because a topology does not allow us to determine whether a function is differentiable or not; it only distinguishes continuous functions. Differentiable manifolds should be locally equivalent to Euclidean open sets (where we already have a well defined notion of $\mathcal{C}^k$ maps; see below), but at the global level they should be allowed to have a more interesting topology. In particular, the sphere, torus, projective space, etc. should become differentiable manifolds.

Before defining $\mathcal{C}^k$ manifolds, let us set up some terminology for $\mathcal{C}^k$ maps in $\mathbb{R}^n$.

Definition 1.2.1 (Euclidean open sets and Euclidean $\mathcal{C}^k$ maps)**.** A **Euclidean open set** is an open subset of some Euclidean space $\mathbb{R}^n$.
Let $k \in \{0, 1, \ldots, \infty\}$. A function $f : U \to V$ between Euclidean open sets is $\mathcal{C}^k$ at a point $p \in U$ if its partial derivatives of order $\leq k$ are defined in a neighborhood of p and continuous at p. We say that f is $\mathcal{C}^k$ (and we call it a **Euclidean $\mathcal{C}^k$ map**) if it is $\mathcal{C}^k$ at all points $p \in U$.
An **Euclidean $\mathcal{C}^k$ isomorphism** is an Euclidean $\mathcal{C}^k$ map that has an Euclidean $\mathcal{C}^k$ inverse.

Note that every Euclidean $\mathcal{C}^k$ map is continuous because the function f itself is a partial derivative (of order 0) of f. In fact, a $\mathcal{C}^0$ map is the same thing as a continuous map.

We are now ready to define $\mathcal{C}^k$ manifolds. The key to turn a topological manifold into a $\mathcal{C}^k$ manifold is to choose an appropriate atlas.

Definition 1.2.2 ($\mathcal{C}^k$ manifolds)**.** Let M be a topological n-manifold and $k = 0, \ldots, \infty$. Two charts φ, ψ for M, with respective domains $U, V \subseteq M$, are **$\mathcal{C}^k$-compatible** if the **transition map** from φ to ψ, that is, the homeomorphism

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V),$$

is a $\mathcal{C}^k$ isomorphism (i.e. itself and its inverse are both Euclidean $\mathcal{C}^k$ maps).

A **$\mathcal{C}^k$-consistent atlas** (or **$\mathcal{C}^k$ atlas**, for short) is an atlas for M whose charts are $\mathcal{C}^k$ compatible with each other. Two $\mathcal{C}^k$ atlases for M are **$\mathcal{C}^k$-equivalent** if their union is $\mathcal{C}^k$-consistent. A $\mathcal{C}^k$ **structure** on M is a maximal $\mathcal{C}^k$ atlas, i.e., a $\mathcal{C}^k$ atlas that is not contained in any other strictly larger $\mathcal{C}^k$ atlas. A $\mathcal{C}^k$ **manifold** is a topological manifold M endowed with a $\mathcal{C}^k$ structure $\mathcal{A}$. (More formally, the $\mathcal{C}^k$ manifold is the pair $(M, \mathcal{A})$.)

Note that a $\mathcal{C}^0$ manifold is the same thing as a topological manifold. A $\mathcal{C}^k$ manifold with $k \geq 1$ is called a **$\mathcal{C}^k$-differentiable manifold**. A **smooth manifold** is a $\mathcal{C}^\infty$ manifold.

Remark 1.2.3 (Domains and codomains of functions)**.** To be precise, the transition map that we wrote as $\psi \circ \varphi^{-1}$ should actually be defined as $\psi|_{U\cap V}^{\psi(U\cap V)} \circ \left(\varphi|_{U\cap V}^{\varphi(U\cap V)}\right)^{-1}$, using the *restricted* charts

$$\varphi|_{U\cap V}^{\varphi(U\cap V)} : U \cap V \to \varphi(U \cap V), \quad \psi|_{U\cap V}^{\psi(U\cap V)} : U \cap V \to \psi(U \cap V).$$

In general we will not write the restrictions explicitly because it is cumbersome. When we compose functions, it should be understood that the resulting composite function is defined in principle at all points where it is possible. (Maybe no points at all!)

We may further restrict a function by specifying a reduced domain or codomain. On the other hand, we shall never specify a domain containing points where the function is not defined, nor a codomain that does not contain the image of the specified domain. Thus a function "$f : A \to B$" always has domain A and codomain B.

The next proposition shows that it suffices to give any $\mathcal{C}^k$-consistent atlas (not necessarily a maximal one) to determine a $\mathcal{C}^k$ structure.

Proposition 1.2.4 ($\mathcal{C}^k$ atlas defines $\mathcal{C}^k$ structure)**.** *For a fixed topological manifold M, each $\mathcal{C}^k$ atlas $\mathcal{A}$ is contained in a unique maximal $\mathcal{C}^k$ atlas $\overline{\mathcal{A}}$, which consists of all charts for M that are $\mathcal{C}^k$-compatible with those of $\mathcal{A}$. Any other $\mathcal{C}^k$ atlas $\mathcal{B}$ for M is equivalent to $\mathcal{A}$ if and only if it is contained in $\overline{\mathcal{A}}$.*

Proof. Let $\mathcal{A}$ be any $\mathcal{C}^k$ atlas for M. Define

$$\overline{\mathcal{A}} := \{\varphi \text{ chart for } M \text{ that is } \mathcal{C}^k \text{ compatible with all charts } \theta \in \mathcal{A}\}.$$

Clearly $\overline{\mathcal{A}}$ contains $\mathcal{A}$. We claim that $\overline{\mathcal{A}}$ is a $\mathcal{C}^k$ atlas. To prove this we have to show that if $\varphi, \psi \in \overline{\mathcal{A}}$ are charts with respective domains $U, V \subseteq M$, then the transition map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is $\mathcal{C}^k$. Take any point $\psi(p) \in \psi(U \cap V)$ and let $\theta \in \mathcal{A}$ be a chart whose domain W contains the point $p \in U \cap V$. Then $\psi(U \cap V \cap W)$ is an open neighborhood of $\psi(p)$ and we can write the restriction[6]

$$\varphi \circ \psi^{-1} : \psi(U \cap V \cap W) \to \varphi(U \cap V \cap W)$$

as the composition $(\varphi \circ \theta^{-1}) \circ (\theta \circ \psi^{-1})$, which is $\mathcal{C}^k$ because $\varphi \circ \theta^{-1}$ and $\theta \circ \psi^{-1}$ are $\mathcal{C}^k$ by assumption. This proves that $\varphi \circ \psi^{-1}$ in a neighborhood of $\psi(p)$, but the same reasoning is valid at any point of $\psi(U \cap V)$, therefore $\varphi \circ \psi^{-1}$ is $\mathcal{C}^k$.

Finally, from the definition of $\overline{\mathcal{A}}$ it is clear that it is maximal, and that any atlas $\mathcal{B}$ is equivalent to $\mathcal{A}$ if and only if it is contained in $\overline{\mathcal{A}}$. In particular, any atlas $\mathcal{B}$ containing $\mathcal{A}$ is equivalent to $\mathcal{A}$ (by definition of equivalent atlases), and is therefore contained in $\overline{\mathcal{A}}$. Therefore a maximal atlas containing $\mathcal{A}$ is contained in $\overline{\mathcal{A}}$, but in fact it must be equal to $\overline{\mathcal{A}}$ (by maximality). We conclude that $\overline{\mathcal{A}}$ is the unique maximal $\mathcal{C}^k$ atlas containing $\mathcal{A}$. □

In consequence, given a topological manifold M and some $\mathcal{C}^k$ atlas $\mathcal{A}$ on M we can speak without ambiguity of *the* $\mathcal{C}^k$ structure $\overline{\mathcal{A}}$ determined by $\mathcal{A}$.

Remark 1.2.5. For practical purposes the concept of a maximal $\mathcal{C}^k$ atlas is not really important. We usually work with a smaller $\mathcal{C}^k$ atlas and this is all we need e.g. for checking that a function is $\mathcal{C}^k$ (see next section). In general we won't give any name to the maximal atlas and we'll just speak about the $\mathcal{C}^k$ manifold M, with the maximal atlas $\mathcal{A}$ being implicit.

Also, when we say "a $\mathcal{C}^k$ chart" or simply "a chart" of M, we mean a chart $\varphi \in \mathcal{A}$. In the rare case that we may need a $\mathcal{C}^l$ chart φ with $l \leq k$ (which means φ is only $\mathcal{C}^l$ compatible with the charts of $\mathcal{A}$), we will say it explicitely. In particular, a "topological chart" is a $\mathcal{C}^0$ chart, i.e. a homeomorphism.

Remark 1.2.6. Every $\mathcal{C}^k$ manifold is automatically a $\mathcal{C}^l$ manifold for every $l \leq k$, because every $\mathcal{C}^k$ atlas is a $\mathcal{C}^l$ atlas. In the other direction, Whitney (*Differentiable Manifolds*, 1936) proved that every $\mathcal{C}^k$ structure contains a (non unique!) $\mathcal{C}^l$ structure for any $l > k$. The proof is reproduced in Munkres' *Elementary Differential Topology* and in Hirsch's *Differential Topology*.

[6] see Remark 1.2.3

Example 1.2.7 (Examples of smooth manifolds).

1. $\mathbb{R}^n$ (with the atlas consisting of the single chart $\mathrm{id}_{\mathbb{R}^n}$) is a smooth manifold. In general, any topological manifold endowed with a single-chart atlas is automatically a smooth manifold. For example, the graph Gra_f of any *continuous* (sic) function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ as described in Example 1.1.11, endowed with the projection chart, is a smooth manifold.
2. Any open subset U of a $\mathcal{C}^k$ manifold M has a natural $\mathcal{C}^k$ structure consisting of the $\mathcal{C}^k$ charts of M whose domain is contained in U. (Exercise.) We will also see in the exercices that finite products of $\mathcal{C}^k$ manifolds have a natural $\mathcal{C}^k$ structure.
3. The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a smooth manifold. Indeed, the atlas given by the graph coordinates (Example 1.1.11) is smooth. To see this, we compute the transition functions (wlog $i < j$):

$$\begin{aligned}\varphi_i^+ \circ (\varphi_j^\pm)^{-1} :&\varphi_j^\pm(U_j^\pm \cap U_i^+ \cap \mathbb{S}^n) \to \varphi_i^\pm(U_j^\pm \cap U_i^+ \cap \mathbb{S}^n)\\ &(y_0, \dots, y_{n-1}) \mapsto (y_0, \dots, \hat{y}_i, \dots, y_{j-1}, \pm\sqrt{1 - \textstyle\sum_i (y_i)^2}, y_j, \dots, y_{n-1})\end{aligned}$$

   and a similar formula works if we replace φ_i^+ by φ_i^-. Hence all the transition maps are smooth.
4. More generally, any subset M of $\mathbb{R}^k$ given as the *regular level set* of a smooth map $F : \mathbb{R}^k \to \mathbb{R}^\ell$ is a $k - \ell$ dimensional smooth manifold "in a natural way[7]". (Being a level set means $M = F^{-1}(\{c\})$ for some $c \in \mathbb{R}^\ell$ and being a regular level set means that, moreover, the Jacobian $D|_p F$ is surjective for all $p \in M$.) You can prove this quite easily using the implicit function theorem and writing M locally as a graph of smooth functions (analogous to the graph coordinates for the sphere). We will show a more general statement later on when discussing submanifolds (Chapter 3).
5. Projective space $\mathbb{P}^n$ is naturally a smooth manifold; see exercises.
6. The torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is naturally a smooth manifold. (Exercise.)
7. On the topological 1-manifold $M = \{(x, y) \in \mathbb{R}^2 \mid y = x^3\}$, each of the two projections $\pi_0, \pi_1 : M \to \mathbb{R}$ (given by $\pi_0(x, y) = x$ and $\pi_1(x, y) = y$) is a chart defined on the whole manifold M, but these two charts are not $\mathcal{C}^k$ compatible for any $k \geq 1$. (The transition functions are $\pi_1 \circ (\pi_0)^{-1} : x \mapsto x^3$ and its inverse $\pi_0 \circ (\pi_1)^{-1} : y \mapsto \sqrt[3]{y}$, which is not differentiable.) Thus these charts determine two different $\mathcal{C}^k$ structures on M.

1.2.1 One-step $\mathcal{C}^k$ manifolds*

Here it is convenient to work with local parametrizations rather than charts. Recall that an (n-dimensional) **local parametrization** of a topological space M is an homeomorphism from an open subset U of $\mathbb{R}^n$ to an open subset of M or, equivalently, an open injective map $U \to M$. A family $\mathcal{A}$ of such maps whose images cover M is called a **parametrization atlas** or **inverse atlas** for M, and such an atlas is $\mathcal{C}^k$ if the transition map $\psi^{-1} \circ \psi$ is $\mathcal{C}^k$ for each $\phi, \psi \in \mathcal{A}$. Equivalently, a local parametrization is the inverse of a chart, and $\mathcal{A}$ is a $\mathcal{C}^k$ parametrization atlas iff $\mathcal{A}^{-1} := \{\varphi^{-1} : \varphi \in \mathcal{A}\}$ is a $\mathcal{C}^k$ atlas.

Note that if M is a $\mathcal{C}^k$ manifold, then the topology of M is determined by the $\mathcal{C}^k$ structure. Indeed, if $\mathcal{A}$ is a $\mathcal{C}^k$ parametrization atlas, then

$$\text{a set } U \subseteq M \text{ is open iff } \phi^{-1}(U) \text{ is open in } \mathrm{Dom}(\phi) \subseteq \mathbb{R}^n \text{ for all } \phi \in \mathcal{A}. \tag{1.1}$$

[7] In particular, the inclusion map $M \hookrightarrow \mathbb{R}^k$ is smooth.

In other words, the topology of M is the *final topology* induced by the family $\mathcal{A}$.[8]

We can ask then, given a set M and a collection $\mathcal{A}$ of functions $\phi : U_\phi \subseteq \mathbb{R}^n \to M$, whether there exists a topology on M such that $\mathcal{A}$ is a $\mathcal{C}^k$ parametrization atlas of M. Such a topology is necessarily unique by (1.1). This provides a way to define a $\mathcal{C}^k$ manifold without constructing first a topological space.

Proposition 1.2.8 (One-step manifolds)**.** *Consider a set M endowed with a family $\mathcal{A}$ of injections $\phi : U_\phi \subseteq \mathbb{R}^n \to M$ satisfying the the following properties: (Denote $\widetilde{U}_\phi = \phi(U_\phi)$ the image of each $\phi \in \mathcal{A}$.)*

1. *For all $\phi, \psi \in \mathcal{A}$, the set $U_\phi^\psi := \phi^{-1}(\widetilde{U}_\psi)$ is open in $\mathbb{R}^n$,*
2. *and the transition map $\psi^{-1} \circ \phi : U_\phi^\psi \to U_\psi^\phi$ is $\mathcal{C}^k$.*
3. *Countably many of the sets $\widetilde{U}_\phi$ cover M.*
4. *For every two distinct points $p, q \in M$ there exist maps $\phi, \psi \in \mathcal{A}$ containing p, q in their respective images, and open neighborhoods $V \subseteq U_\phi$, $W \subseteq U_\psi$ of $\phi^{-1}(p)$, $\psi^{-1}(q)$ respectively, such that $\phi(V) \cap \psi(W) = \emptyset$.*

Then the set M endowed with the final topology w.r.t. $\mathcal{A}$ is a topological n-manifold admitting $\mathcal{A}$ as a $\mathcal{C}^k$ parametrization atlas.

Remarks:

- The hypothesis implies that each set $U_\phi \subseteq \mathbb{R}^n$ is open, because it is equal to U_ϕ^ϕ.
- Each transition map $\psi^{-1} \circ \phi : U_\phi^\psi \to U_\psi^\phi$ is a $\mathcal{C}^k$ isomorphism because it is $\mathcal{C}^k$ and bijective, with inverse $\phi^{-1} \circ \psi$, which is also $\mathcal{C}^k$.

Proof. We endow M with the final topology w.r.t. $\mathcal{A}$, so that a set $U \subseteq M$ is open if and only if $\psi_\alpha^{-1}(U)$ is open in U_ϕ for all α. This is clearly a topology on M.

We claim that each map $\phi \in \mathcal{A}$ is open. Indeed, if $W \subseteq U_\phi$ is an open set, then $\phi(W)$ is open in M because for any $\psi \in \mathcal{A}$, the set $\psi^{-1}(\phi(W)) = (\psi^{-1} \circ \phi)(W \cap U_\phi^\psi)$ is open in U_ψ^ϕ since $\psi^{-1} \circ \phi$ is a homeomorphism $U_\phi^\psi \to U_\psi^\phi$.

It follows that each $\phi \in \mathcal{A}$ is a homeomorphism onto its image, because it is open an injective. Therefore the maps ϕ are n-dimensional local parametrizations of M ensuring that M is locally Euclidean of dimension n. Moreover, M is a topological manifold because it is second countable by (3) and Hausdorff by (4). Finally, the parametrization atlas $\mathcal{A}$ is $\mathcal{C}^k$-consistent by (2). □

An example of manifold that can be constructed in this way is the Grassman manifold $G_k(V)$ of k-subspaces of a vector space $V \simeq \mathbb{R}^n$; see [Lee13, p. 1.36].

1.3 Differentiable maps

We are about to define $\mathcal{C}^k$ maps between $\mathcal{C}^k$ manifolds. The plan is to reduce the question of differentiability to the case of a map between Euclidean open sets. We will do so by using charts.

In general, when studying a map $f : M \to N$ between manifolds, charts allow us to *locally* express f as a map between subsets of Euclidean space.

[8] The **final topology** on a set M induced by a family of maps $\phi : U_\phi \to M$, where U_ϕ are topological spaces, is the topology defined by the formula (1.1). Exercise: check that it is indeed a topology.

Definition 1.3.1 (Local expression of a map)**.** Let M, N be $\mathcal{C}^k$ manifolds and let $f : M \to N$ be any function (not necessarily continuous). A **local expression** (or **coordinate representation**) of f at some point $p \in M$ is a composite map

$$f|_{\varphi}^{\psi} := \psi \circ f \circ \varphi^{-1},$$

where φ and ψ are charts of M and N whose domains U, V contain the points p and $f(p)$ respectively. The composite is defined at all possible points; see Remark 1.2.3.

Local expressions allow us, for instance, to determine whether a function is continuous or not.

Remark 1.3.2. Exercise: Show that f is continuous at p if and only if the local expression $f|_{\varphi}^{\psi}$ at p is defined in a neighborhood of $\varphi(p)$ and is continuous at $\varphi(p)$. This fact holds however we choose the charts φ, ψ (provided their domains contain the points p, $f(p)$ respectively, of course).

We use local expressions to define whether a map is $\mathcal{C}^k$ or not.

Definition 1.3.3 ($\mathcal{C}^k$ maps between manifolds)**.** We say a function $f : M \to N$ between $\mathcal{C}^k$ manifolds is $\mathcal{C}^k$ at a point $p \in M$ if there exists a local expression $f|_{\varphi}^{\psi}$ of f at p such that

$$f|_{\varphi}^{\psi} \text{ is defined in a neighborhood of } \varphi(p) \text{ and is } \mathcal{C}^k \text{ at the point } \varphi(p). \tag{1.2}$$

If this holds for all points $p \in M$, we say that f is a $\mathcal{C}^k$ map. The set of $\mathcal{C}^k$ maps $M \to N$ is denoted $\mathcal{C}^k(M, N)$. A $\mathcal{C}^k$ **isomorphism** is a $\mathcal{C}^k$ map that has a $\mathcal{C}^k$ inverse.

If $k \geq 1$, a $\mathcal{C}^k$ map is also called $\mathcal{C}^k$**-differentiable**, and a $\mathcal{C}^k$ isomorphism is also called a $\mathcal{C}^k$ **diffeomorphism** (or a $\mathcal{C}^k$ **diffeo**, for short).

In fact the condition (1.2), that determines whether f is $\mathcal{C}^k$ or not, does not depend on how we choose the charts φ, ψ.

Proposition 1.3.4. *If f is $\mathcal{C}^k$ at the point p, then every local expression $f|_{\varphi}^{\psi}$ of f at p satisfies* (1.2)*.*

Proof. Assume that the local expression $f|_{\varphi}^{\psi}$ satisfies (1.2). This implies that f is continuous at p, by Remark 1.3.2. Consider a second local expression $f|_{\widetilde{\varphi}}^{\widetilde{\psi}}$ of f at p. This new local expression is defined on some neighborhood of $\widetilde{\varphi}(p)$ (again by Remark 1.3.2), and is related to the old one by the **chart-change formula**

$$f|_{\widetilde{\varphi}}^{\widetilde{\psi}} \equiv (\psi \circ \widetilde{\psi}^{-1}) \circ f|_{\varphi}^{\psi} \circ (\varphi \circ \widetilde{\varphi}^{-1}), \tag{1.3}$$

which holds at all points where the two expressions are defined (in particular, in some neighborhood of $\widetilde{\varphi}(p)$). Since the transition maps $(\psi \circ \widetilde{\psi}^{-1})$ and $(\varphi^{-1} \circ \widetilde{\varphi})$ are $\mathcal{C}^k$, we conclude that $f|_{\widetilde{\varphi}}^{\widetilde{\psi}}$ is $\mathcal{C}^k$ at the point $\widetilde{\varphi}(p)$. $\square$

Example 1.3.5.
1. The identity map of any $\mathcal{C}^k$ manifold is a $\mathcal{C}^k$ map. (Exercise.)
2. A composite map $g \circ f$ is $\mathcal{C}^k$ at a point p if f is $\mathcal{C}^k$ at p and g is $\mathcal{C}^k$ at $f(p)$. (Exercise.)
3. If M is a $\mathcal{C}^k$ manifold, then every $\mathcal{C}^k$ chart of M, as well as its inverse, are $\mathcal{C}^k$ maps. (Exercise.)

A $\mathcal{C}^k$ structure on a topological manifold M allows us to determine which maps that go to M are $\mathcal{C}^k$. But the reciprocal property also holds: if we know which maps to M are $\mathcal{C}^k$, this information determines the $\mathcal{C}^k$ structure of M.

Proposition 1.3.6. *Let $\mathcal{A}_0$, $\mathcal{A}_1$ be two $\mathcal{C}^k$ atlases on a topological manifold M, defining two $\mathcal{C}^k$ manifolds $M_i = (M, \overline{\mathcal{A}_i})$. Then the two atlases $\mathcal{A}_i$ are equivalent if and only if the following property holds:*
For every function $f : N \to M$ (where N is a $\mathcal{C}^k$ manifold), the function f is $\mathcal{C}^k$ as a map $N \to M_0$ if and only if it is $\mathcal{C}^k$ as a map $N \to M_1$.

Proof. Exercise. □

1.4 Partitions of unity

In this section we develop *partitions of unity*, a tool often used to turn a local construction, obtained by working in coordinates, into a global one. (Don't worry if this sounds vague; we will see examples later on.) The existence of partitions of unity on a manifold is easier to prove if the manifold is compact, and we will deal with this case first. The general proof relies on a topological property of manifolds called *paracompactness*, which is in turn a consequence of being Hausdorff, second countable and locally compact.

Recall that the **support** of a continuous function $\eta : M \to \mathbb{R}$ is the closed set

$$\operatorname{supp}(\eta) := \overline{\{p \in M \mid \eta(p) \neq 0\}}.$$

Definition 1.4.1. A $\mathcal{C}^k$ **partition of unity** (or **POU**, for short) on a $\mathcal{C}^k$ manifold M is a family $(\eta_i)_i$ of $\mathcal{C}^k$ functions $\eta_i : M \to [0, +\infty)$ satisfying

- Sum condition: $\sum_i \eta_i(x) = 1$ for every $x \in M$,
- Local finiteness: Every point of M has a neighborhood where all except finitely many of the functions η_i vanish.

A partition of unity $(\eta_i)_i$ is **subordinate** to an open cover $\mathcal{U}$ of M if every η_i has its support contained in some open set $U \in \mathcal{U}$.

Theorem 1.4.2 (Existence of Partitions of Unity.)**.** *For any open cover $\mathcal{U}$ of a $\mathcal{C}^k$ manifold M there exists a partition of unity $(\eta_i)_i$ subordinate to $\mathcal{U}$ such that the functions η_i have compact support (in fact, their supports are closed coordinate balls; see definition below).*

Another version of the theorem is the following.

Corollary 1.4.3 (Existence of Partitions of Unity, alternate form)**.** *For any open cover $\mathcal{U} = \{U_j\}_{j \in J}$ of a $\mathcal{C}^k$ manifold M there exists a partition of unity $(\xi_j)_{j \in J}$ such that* $\operatorname{supp}(\xi_j) \subseteq U_j$ *for all j.*

This version of the theorem can be deduced from the first one (exercise). Note that the functions g_j may not have compact support in this case.

For the proof of Theorem 1.4.2 we will use *bump functions.*

Lemma 1.4.4 (Bump functions on Euclidean space)**.** *For any numbers $0 < a < b$ there exists a smooth **bump function** $h : \mathbb{R}^n \to [0, 1]$ satisfying $h(x) = 1$ iff $\|x\| \leq a$ and $h(x) = 0$ iff $\|x\| > b$.*

Proof. It suffices to let $h(x) = g(\|x\|)$, where $g : \mathbb{R} \to [0,1]$ is a smooth **cutoff function** satisfying $g(t) = 1$ iff $t < a$ and $g(t) = 0$ iff $t > b$. This cutoff function, in turn, may be defined as

$$g(t) = \frac{f(b-t)}{f(b-t) + f(t-a)},$$

where $f : \mathbb{R} \to [0, +\infty)$ is a smooth function such that $f(t) > 0$ iff $t > 0$. Such a function f may be given e.g. by the formula

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \le 0 \end{cases}.$$

□

We may use charts to transport these bump functions from Euclidean space to any manifold. The resulting bump function will be supported on a closed *coordinate ball.*

Definition 1.4.5. A **closed coordinate ball** in a $\mathcal{C}^k$ manifold M is the preimage $\varphi^{-1}(\overline{B})$ of a closed Euclidean ball $\overline{B}$ by a $\mathcal{C}^k$ chart φ containing $\overline{B}$ in its codomain.

Proof of Thm. 1.4.2 for compact manifolds. Let M be a compact $\mathcal{C}^k$ manifold and let $\mathcal{U}$ be an open cover of M. In this case we'll obtain a *finite* $\mathcal{C}^k$ partition of unity $(\eta_i)_i$ subordinate to $\mathcal{U}$. In fact, it is sufficient to find finitely many $\mathcal{C}^k$ functions $\widetilde{\eta_i} : M \to [0, +\infty)$, each of which has its support $\mathrm{supp}(\widetilde{\eta_i})$ contained in some $U \in \mathcal{U}$, and such that $\sum_i \widetilde{\eta_i}(x) > 0$ for all x. The functions η_i can then be obtained by dividing each $\widetilde{\eta_i}$ by the strictly positive $\mathcal{C}^k$ function $\widetilde{\eta} = \sum_i \widetilde{\eta_i}$.

To construct the functions η_i we proceed as follows. Each point of a set $U \in \mathcal{U}$ is contained in the interior of some closed coordinate ball $D \subseteq U$. Thus there is a family of closed coordinates balls, each of them contained in some $U \in \mathcal{U}$, whose interiors cover M. By compactness, we may take a finite subfamily of balls D_i whose interiors still cover M. Write each ball D_i as $\varphi_i^{-1}(\overline{B_i})$, where B_i is an open Euclidean ball and φ_i is a $\mathcal{C}^k$ chart containing $\overline{B_i}$ in its codomain. Then let $h_i : \mathbb{R}^n \to [0, +\infty)$ be a $\mathcal{C}^k$ bump function that is supported on the closed ball $\overline{B_i}$ and strictly positive on the interior B_i. Finally, define a $\mathcal{C}^k$ function $\widetilde{\eta_i} : M \to [0,1]$ by the formula

$$\widetilde{\eta_i} = \begin{cases} h_i \circ \varphi_i & \text{on } \mathrm{Dom}(\varphi) \\ 0 & \text{on } M \setminus D_i \end{cases}$$

This function is supported on D_i, which is contained in some $U \in \mathcal{U}$, thus the function family $(\widetilde{\eta_i})$ is subordinate to $\mathcal{U}$. In addition, $\widetilde{\eta_i}$ is strictly positive on B_i, and since the balls B_i cover M, we conclude that $\sum_i \widetilde{\eta_i}(x) > 0$ for all $x \in M$, as required. □

1.4.1 Paracompactness*

Definition 1.4.6 (Paracompact space)**.** Let X be a topological space.

- An **open cover** of X is a family $\mathcal{U}$ of open sets whose union is X.
- Another open cover $\mathcal{V}$ **refines** $\mathcal{U}$ if every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$.
- The space X is **paracompact** if every open cover $\mathcal{U}$ admits as refinement some open cover $\mathcal{V}$ that is locally finite.
- (A family of subsets of X is **locally finite** if every point $x \in X$ has a neighborhood that intersects only finitely many sets of the family.)

Proposition 1.4.7. *If a topological space X is Hausdorff, second countable and locally compact, then it is paracompact. (In particular, a manifold is paracompact.)*

In fact, given any topological base $\mathcal{B}$ of X, every open cover $\mathcal{U}$ has a locally finite refinement $\mathcal{V}$ consisting of open sets $B \in \mathcal{B}$ with their closures $\overline{B}$ contained in some $U \in \mathcal{U}$.

Proof.

Lemma 1.4.8. *X admits an exhaustion by compact sets, i.e. a sequence $(K_i)_{i\in\mathbb{N}}$ of compact sets that cover X and satisfy $K_i \subseteq \operatorname{Int}(K_{i+1})$.*

Proof of lemma. Let $(V_j)_{j\in\mathbb{N}}$ be a countable open cover of M where each V_j has compact closure. Let $K_0 = \emptyset$, and define inductively for each $i \in \mathbb{N}$ an integer $j_i \geq i$ such that the sets $(V_j)_{j<j_i}$ cover K_i, and a compact set $K_{i+1} = \overline{V_i} \cup \bigcup_{j<j_i} \overline{V_j}$. These compact sets K_i cover M and satisfy $K_i \subseteq \operatorname{Int}(K_{i+1})$. □

Consider the compact sets $L_i = K_i \setminus \operatorname{Int}(K_{i-1})$ and their respective open neighborhoods $W_i = \operatorname{Int} K_{i+1} \setminus K_{i-2}$.

Each point $x \in L_i$ is contained in some open $U_x \in \mathcal{U}$ and has a basic neighborhood $B^i_x \in \mathcal{B}$ such that $\overline{B^i_x} \subseteq W_i \cap U$.

Take a finite subfamily $(B^i_{x_j})_j$ that covers L_i. Doing this for each i we obtain a family of basics $(B^i_{x_j})_{i,j}$ that cover M. Their closures satisfy $\overline{B^i_{x_j}} \subseteq U_{x_j}$, which gives the subordination condition, and $\overline{B^i_{x_j}} \subseteq W_i$, which ensures local finiteness since every point $x \in L_i$ is contained in at most three sets W_ℓ, namely, those with $|\ell - i| \leq 1$. □

Using paracompactness, we can prove the existence of partitions of unity on a noncompact manifold.

Proof of Thm. 1.4.2. Let M be a manifold and $\mathcal{U}$ an open cover.

Let $\mathcal{B}$ be the topological base of M consisting of the interiors of closed coordinate balls. By Proposition 1.4.7, there exists a family of closed coordinate balls D_i, each contained in some open set $U \in \mathcal{U}$, whose interiors cover M.

The proof finishes as in the compact case. We take for each i a $\mathcal{C}^k$ function $\widetilde{\eta}_i : M \to [0,1]$ that is strictly positive on $\operatorname{Int}(D_i)$ and supported on D_i. Then the functions $\eta_i = \frac{\widetilde{\eta}_i}{\sum_j \widetilde{\eta}_j}$ form a $\mathcal{C}^k$ partition of unity $(\eta_i)_i$ that is subordinate to the open cover $\mathcal{U}$. □

1.4.2 Applications

Corollary 1.4.9 (Bump functions). *If M is a $\mathcal{C}^k$ manifold, $A \subset M$ a closed set and $U \subseteq M$ an open neighborhood of A, then there exists a $\mathcal{C}^k$ function $\eta : M \to [0,1]$ such that $\eta \equiv 1$ on A and $\operatorname{supp}(\eta) \subset U$.*

We call η a *bump function* for A supported in U.

Proof. Just take the open cover $\{V_0 = U, V_1 = M \setminus A\}$ of M and a partition of unity $(\eta_i)_{i=0,1}$ satisfying $\operatorname{supp} \eta_i \subseteq V_i$, then set $\eta = \eta_0$. □

We have already defined differentiability for a function $f : M \to N$ between $\mathcal{C}^k$ manifolds. If instead f is defined on a closed subset of M, we define differentiability as follows:

Definition 1.4.10. Let $f : A \to N$ be a function where $A \subseteq M$ is a closed set and M, N are $\mathcal{C}^k$ manifolds. We say that f is $\mathcal{C}^k$ if it can be extended to a $\mathcal{C}^k$ function defined on an open neighborhood of A.

As a corollary of the existence of bump functions we can extend a smooth function defined on a closed set to a smooth function on the whole manifold:

Corollary 1.4.11 (Extension lemma)**.** *Let $f : A \to \mathbb{R}$ be a $\mathcal{C}^k$ function, where $A \subseteq M$ is a closed subset of a $\mathcal{C}^k$ manifold M, and let $U \subseteq M$ be an open set containing A. Then there exists a $\mathcal{C}^k$ function $\widetilde{f} : M \to \mathbb{R}$ such that $\widetilde{f}|_A = f$ and* $\operatorname{supp} \widetilde{f} \subseteq U$.

Proof. By definition f can be extended to a $\mathcal{C}^k$ function (say, also called f) on some open set $W \supseteq A$; wlog $W \subseteq U$. We take a $\mathcal{C}^k$ bump function η for A supported in W. Then ηf has support in W and therefore extending the function by 0 outside W we obtain a $\mathcal{C}^k$ function $\widetilde{f}$ on M with the desired properties. $\square$

2 Tangent vectors

Recall that if a map $f : U \to V$ between Euclidean open sets $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ is $\mathcal{C}^1$ at a point $p \in U$, then there exists a unique linear transformation $\mathrm{D}_p f : \mathbb{R}^m \to \mathbb{R}^n$, called the **differential transformation** of f at p, which gives a first-order approximation

$$f(p+v) = f(p) + \mathrm{D}_p f(v) + r_p(v)$$

where $\frac{r_p(v)}{\|v\|} \to 0$ as $v \to 0$.

To define the differential of a map between $\mathcal{C}^k$ manifolds, we need a notion of tangent space.

Definition 2.0.1. Let M be an n-dimensional $\mathcal{C}^k$ manifold with $k \geq 1$. A **coordinatized tangent vector** on M is a triple (p, φ, v) where $p \in M$ is a point, φ is a $\mathcal{C}^k$ chart of M defined at p, and $v \in \mathbb{R}^n$ is a vector in Euclidean space. A **tangent vector** on M is the equivalence class $[p, \varphi, v]$ of a coordinatized tangent vector (p, φ, v) under the equivalence relation

$$(p, \varphi, v) \sim (\widetilde{p}, \widetilde{\varphi}, \widetilde{v}) \quad \Longleftrightarrow \quad \widetilde{p} = p \quad \text{and} \quad \widetilde{v} = \mathrm{D}_{\varphi(p)}(\widetilde{\varphi}\,\varphi^{-1})(v)$$

The set TM of tangent vectors is the **tangent bundle** of M, and there is a canonic projection map $\pi_{TM} : TM \to M$ sending $[p, \varphi, v] \mapsto p$.

The **tangent space** at a point $p \in M$ is the set $T_pM := \pi^{-1}(p)$. It is a vector space with vector addition

$$[p, \varphi, v] + [p, \varphi, w] := [p, \varphi, v + w]$$

and vector scaling

$$\lambda\,[p, \varphi, v] := [p, \varphi, \lambda v] \quad \text{for } \lambda \in \mathbb{R}.$$

Remark 2.0.2.

1. $\sim$ is indeed an equivalence relation. (Exercise.)
2. Fixed a point $p \in M$ and a $\mathcal{C}^k$ chart φ defined on p, the function $\iota : \mathbb{R}^n \to T_pM$ sending $v \mapsto [p, \varphi, v]$ is a bijection. (Exercise.)
3. A tangent vector $X = [p, \varphi, v] \in T_pM$ can be considered as a function

$$\begin{array}{ccc} \{\text{charts of } M \text{ def. at } p\} & \to & \mathbb{R}^n \\ \psi & \mapsto & D_{\varphi(p)}(\psi\,\varphi^{-1})(v). \end{array}$$

   The vector $D_{\varphi(p)}(\psi\,\varphi^{-1})(v)$ is the only $w \in \mathbb{R}^n$ such that $[p, \psi, w] = X$.
4. Vector addition and scaling are well defined and make T_pM a vector space isomorphic to $\mathbb{R}^n$. (Exercise.)

Remark 2.0.3. If $U \subseteq \mathbb{R}^n$ is an open set (considered as a smooth manifold), we identify $TU \equiv U \times \mathbb{R}^n$ by the bijection

$$(p, v) \in U \times \mathbb{R}^n \mapsto [p, \mathrm{id}_U, v] \in TU.$$

Thus for each $p \in U$ we have $T_pU \equiv \{p\} \times \mathbb{R}^n \equiv \mathbb{R}^n$, and for $p \in U$ and $v \in \mathbb{R}^n$ we write

$$v|_p := [p, \mathrm{id}_U, v] \in T_pU. \tag{2.1}$$

Coordinate base for the tangent space We can construct a base of the tangent space T_pM as follows. Consider the canonic base $(e_i)_i$ of $\mathbb{R}^n$, and take a chart φ defined at p. Then the vectors

$$\left.\frac{\partial}{\partial \varphi^i}\right|_p := [p, \varphi, e_i],$$

called the **coordinate vectors** at p associated to the chart φ, form a base of T_pM.

If ψ is another chart defined at p, then this chart determines a second base consising of vectors $\left.\frac{\partial}{\partial \psi^j}\right|_p$. This second base is related to the first one by the formula

$$\left.\frac{\partial}{\partial \varphi^i}\right|_p = \sum_j \left.\frac{\partial \psi^j}{\partial \varphi^i}\right|_{\varphi(p)} \left.\frac{\partial}{\partial \psi^j}\right|_p.$$

where $\left.\frac{\partial \psi^j}{\partial \varphi^i}\right|_{\varphi(p)}$ is the partial derivative of $\psi \circ \varphi^{-1}$ that appears as the coefficient (j, i) of the matrix expression of the linear map $\mathrm{D}_{\varphi(p)}(\psi \circ \varphi^{-1}) : \mathbb{R}^n \to \mathbb{R}^n$. To see this we compute as follows:

$$\begin{aligned}
\left.\frac{\partial}{\partial \varphi^i}\right|_p &= [p, \varphi, e_i] = \big[p, \psi,\, \mathrm{D}_{\varphi(p)}(\psi \circ \varphi^{-1})e_i\big] \\
&= \left[p, \psi, \sum_j \left.\frac{\partial \psi^j}{\partial \varphi_i}\right|_{\varphi(p)} e_j\right] \\
&= \sum_j \left.\frac{\partial \psi^j}{\partial \varphi_i}\right|_{\varphi(p)} [p, \psi, e_j] \\
&= \sum_j \left.\frac{\partial \psi^j}{\partial \varphi_i}\right|_{\varphi(p)} \left.\frac{\partial}{\partial \psi^j}\right|_p.
\end{aligned}$$

Remark 2.0.4. In concrete examples it is common to use more intuitive symbols for φ^i, e.g. the polar coordinates (r, φ) or the spherical coordinates (r, ϕ, θ). The standard coordinates $(x^0, \ldots, x^{n-1})$ on $\mathbb{R}^n$ are usually written (x, y) for $n = 2$ and (x, y, z) for $n = 3$. Bear in mind that an expression like (r, φ) can mean either the map (chart) or the coordinates of a particular point; see example below.

Example 2.0.5 (Polar coordinates)**.** Let $W := \mathbb{R}^+ \times (0, 2\pi)$. The map

$$\Psi : W \to \mathbb{R}^2 : (r, \varphi) \mapsto (r\cos\varphi, r\sin\varphi)$$

is a diffeomorphism onto its image $U := \Psi(W) = \mathbb{R}^2 \setminus (\mathbb{R}_{\geq 0} \times \{0\})$. Its inverse $\Psi^{-1} : U \to W$ is therefore a smooth chart for $\mathbb{R}^2$. The components of Ψ^{-1} are usually written (r, φ) and called *polar coordinates*. On the other hand, we have the standard coordinates (x, y) (i.e. the identity map) on $\mathbb{R}^2$. Take a point $p = (x, y) \in U$ and let $(r, \varphi) = \Psi^{-1}(x, y)$ be its polar coordinates. The polar coordinate vectors $\left.\frac{\partial}{\partial r}\right|_p$, $\left.\frac{\partial}{\partial \varphi}\right|_p$ can be expressed as a linear combination of the standard coordinate vectors $\left.\frac{\partial}{\partial x}\right|_p$, $\left.\frac{\partial}{\partial y}\right|_p$, using the change of coordinates formula:

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos(\varphi)\frac{\partial}{\partial x} + \sin(\varphi)\frac{\partial}{\partial y} \\
\frac{\partial}{\partial \varphi} &= \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} = -r\sin(\varphi)\frac{\partial}{\partial x} + r\cos(\varphi)\frac{\partial}{\partial y}.
\end{aligned}$$

Here we do certain standard abuses of notation. First, the letters r, φ, x, y represent functions when preceeded by a ∂, otherwise they are numbers (obtained by evaluating these functions at the point p). Second, we have omitted the "$|_p$" on the vectors and the evaluation at (r, φ) for the partial derivatives.

2.0.1 Differential of a $\mathcal{C}^k$ map between manifolds

Now that we have tangent spaces, we can define the differential of a $\mathcal{C}^k$ map.

Definition 2.0.6. The **differential transformation** (or **differential**, for short) of a map $f : M \to N$ that is $\mathcal{C}^k$ at a point $p \in M$ is the linear operator

$$\begin{array}{rccc} D_p f: & T_pM & \to & T_{f(p)}N \\ & [p,\varphi,v] & \mapsto & [f(p),\psi,D_{\varphi(p)}f|_\varphi^\psi(v)]. \end{array}$$

where φ, ψ are charts of M, N defined at the points p, $f(p)$ respectively, and $f|_\varphi^\psi = \psi \circ f \circ \varphi^{-1}$ is the local expression of f with respect to the charts φ, ψ.

If f is $\mathcal{C}^k$ everywhere, the union of the maps D_pf for all $p \in M$ is a map $Df : TM \to TN$, also denoted f_* and called the **pushforward** by f.

Note that D_pf is well defined (independent of φ, ψ) and linear. To see that it is well defined, we compute twice

$$\begin{array}{rrcl} D_pf: & X=[p,\varphi,v] & \mapsto & Y=[f(p),\psi,w=D_{\varphi(p)}f|_\varphi^\psi(v)]. \\ D_pf: & \widetilde{X}=[p,\widetilde{\varphi},\widetilde{v}] & \mapsto & \widetilde{Y}=[f(p),\widetilde{\psi},\widetilde{w}=D_{\widetilde{\varphi}(p)}f|_{\widetilde{\varphi}}^{\widetilde{\psi}}(v)]. \end{array}$$

and verify that $X = \widetilde{X}$ implies $Y = \widetilde{Y}$. That is, we must check that $\widetilde{v} = D_{\varphi(p)}(\widetilde{\varphi}\varphi^{-1})(v)$ (or, equivalently, $v = D_{\widetilde{\varphi}(p)}(\varphi\widetilde{\varphi}^{-1})(\widetilde{v})$) implies $\widetilde{w} = D_{\psi(p)}(\widetilde{\psi}\psi^{-1})(w)$. This follows from the equation $f|_{\widetilde{\varphi}}^{\widetilde{\psi}} = (\widetilde{\psi}\psi^{-1})f|_\varphi^\psi(\varphi\widetilde{\varphi}^{-1})$ by the chain rule (in its Euclidean version).

The linearity of D_pf follows from the linearity of $D_{\varphi(p)}f|_\varphi^\psi$.

The chain rule has the following version for maps between manifolds.

Proposition 2.0.7 (Chain rule)**.** *If $f : M \to N$ is $\mathcal{C}^k$ at some point p and $g : N \to L$ is $\mathcal{C}^k$ at $f(p)$, then $g \circ f$ is $\mathcal{C}^k$ at p and has differential*

$$D_p(g \circ f) = D_{f(p)}g \circ D_pf.$$

In particular, for a diffeo f, the differential D_pf is a linear isomorphism whose inverse is $D_{f(p)}(f^{-1})$. (Exercise.)

Example 2.0.8 (Derivative of a chart)**.** Let M be an n-dimensional $\mathcal{C}^k$-manifold, let $p \in M$, and let $\varphi : U \to V$ be a chart of M defined at p. Then we have

$$\mathrm{D}\varphi(\left.\frac{\partial}{\partial\varphi^i}\right|_p) = e_i|_{\phi(p)} \in \mathrm{T}_{\phi(p)}\mathbb{R}^n.$$

To see this, using the local expression $\varphi|_\varphi^{\mathrm{id}_V} = \mathrm{id}_V \circ\varphi\circ\varphi^{-1} = \mathrm{id}_V$ we compute

$$\begin{aligned} \mathrm{D}\varphi(\left.\frac{\partial}{\partial\varphi^i}\right|_p) &= \mathrm{D}\varphi[p,\varphi,e_i] \\ &= [\varphi(p),\mathrm{id}_V,D_{\varphi(p)}(\varphi|_\varphi^{\mathrm{id}_V})(e_i))] = [\varphi(p),\mathrm{id}_V,e_i] = e_i|_p \in \mathrm{T}_p\mathbb{R}^n. \end{aligned}$$

Example 2.0.9 (Velocity of a curve)**.** For a differentiable curve $\gamma : I \subseteq \mathbb{R} \to M$ on a manifold M, we define its **velocity vector** at an instant $t \in I$ as the vector $\gamma'(t) := D_t\gamma(1|_t) \in T_{\gamma(t)}M$ where $1|_t$ represents the element $[t,\mathrm{id}_I,1]$ of T_tI according to the identification $TI \cong I \times \mathbb{R}$ given in Remark 2.0.3.
Exercise: Show that for any vector $X \in TM$ there is a curve $\gamma : (-\varepsilon,\varepsilon) \to M$ such that $\gamma'(0) = X$.

Tangent vectors as derivations

Definition 2.0.10. A **derivation** on a $\mathcal{C}^k$ differentiable manifold M at a point $p \in M$ is a linear function $D : \mathcal{C}^k(M, \mathbb{R}) \to \mathbb{R}$ satisfying the Leibniz identity

$$D(f\,g) = D(f)\,g(p) + f(p)\,D(g).$$

The set Der_pM of derivations on M at p is a vector space with the operations

$$(D+E)(f) := D(f) + E(f),$$
$$(\lambda D)(f) := \lambda(D(f))$$

defined for $D, E \in Der_pM$ and $\lambda \in \mathbb{R}$.

Each vector $X \in \mathrm{T}_pM$ induces a derivation $D_X \in Der_pM$ defined by the formula

$$D_X(f) := \mathrm{D}_pf(X) \in \mathrm{T}_{f(p)}\mathbb{R} \equiv \mathbb{R}.$$

(Here we use the identification of Remark 2.0.3).

Proposition 2.0.11. *The map $\nu_p : X \in \mathrm{T}_pM \mapsto D_X \in Der_pM$ is a linear injection. (Exercise.)*

Therefore we may identify a tangent vector X with the derivation D_X and write $X(f) := D_X(f)$. In fact, the map ν is bijective if M is smooth (see e.g. [Lee13, Prop. 3.2]). Therefore in some books (e.g. [Lee13]), the tangent space T_pM of a smooth manifold M is *defined* as the vector space of derivations at p. Here we do not use that definition because it does not work for non-smooth manifolds (i.e. C^k manifolds with $k < \infty$).

Tangent vectors as derivations

In this chapter we give an alternative definition of the "tangent space" using derivations. This second definition is equivalent to the one given in the previous chapter, but is more similar to the one used in most books (e.g. Lee's book [Lee13]).

In more detail, for a point p of a differentiable manifold M, we will consider first a vector space $\operatorname{Der}_p M$ of *derivation operators* (or *derivations*, for short) and then we will define the tangent space T_pM as a certain subspace of $\operatorname{Der}_p M$. Tangent vectors are the derivations that correspond (via a chart) to a derivation defined by a vector of $\mathbb{R}^n$. Equivalently, tangent vectors are those derivations that can be expressed as the velocity vector of a curve.

The objects that we will define are analogous to those defined in the previous chapter (except that the differential of a function will be called "tangent map"). The plan is the following:

(1) For each $\mathcal{C}^k$-differentiable n-manifold M and each point $p \in M$, define a vector space $\mathrm{T}_pM \simeq \mathbb{R}^n$, called the *tangent space* of M at p.

(2) For each $\mathcal{C}^k$ map $f : M \to N$ and each point $p \in M$, define a linear map $\mathrm{T}_pf : \mathrm{T}_pM \to \mathrm{T}_pN$ called the *tangent map* of f at p.

If the manifolds M, N are open subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ respectively, then the map T_pf will be essentially equivalent to the **differential transformation** studied in calculus courses, that is, the unique linear map $\mathrm{D}_pf : \mathbb{R}^m \to \mathbb{R}^n$ which gives the first-order approximation

$$f(p+v) = f(p) + \mathrm{D}_pf(v) + r(v)$$

where $\frac{r(v)}{\|v\|} \to 0$ as $v \to 0$.

Motivation for definition of tangent vectors Let M be a $\mathcal{C}^k$ manifold M with $k \geq 1$ and let $p \in M$. We want to have a vector space T_pM of vectors tangent to M at p. How can we define the notion of a "tangent vector"?

For example, let $\gamma : I \subseteq \mathbb{R} \to M$ be a differentiable curve such that $\gamma(0) = p$. This curve should have a "velocity vector" X at the instant $t = 0$, which should be vector tangent to M at p. This object X should tells us the *direction and speed* of movement of $\gamma(t)$ at time $t = 0$. How can we measure this "direction and speed" ?

Idea: take a function $h \in \mathcal{C}^k(M, \mathbb{R})$. The composite $h \circ \gamma$ is a function $I \to \mathbb{R}$ that is differentiable at $t = 0$, and the number number $(h \circ \gamma)'(0)$ tells us something about how γ is moving at $t = 0$. If we know this number for all functions h, let us agree that we know the velocity vector X completely. Therefore we propose to *define* the velocity vector X simply as the function $h \mapsto (h \circ \gamma)'(0)$. This function has certain algebraic properties that make it a *derivation*.

Definition 2.0.12. A **derivation operator** (or **derivation**, for short) on a $\mathcal{C}^k$-differentiable manifold M at a point $p \in M$ is a function $X : \mathcal{C}^k(M, \mathbb{R}) \to \mathbb{R}$ that is $\mathbb{R}$-linear, that is, such that

$$X(a \cdot g + b \cdot h) = a \cdot X(g) + b \cdot X(h) \quad \text{for } g, h \in \mathcal{C}^k(M, \mathbb{R}) \text{ and } a, b \in \mathbb{R},$$

and satisfies the **Leibniz identity**

$$X(g \cdot h) = X(g) \cdot h(p) + g(p) \cdot X(h). \tag{2.2}$$

The set $\mathrm{Der}_p(M)$ of derivations on M at p is a real vector space: for $X, Y \in \mathrm{Der}_p M$ and $a, b \in \mathbb{R}$ we define the derivation[1] $a\,X + b\,Y \in \mathrm{Der}_p M$ by the formula

$$(a \cdot X + b \cdot Y)(h) := a \cdot X(h) + b \cdot Y(h).$$

We also define the set $\mathrm{Der}\, M := \coprod_{p \in M} \mathrm{Der}_p M$.

Before getting to the derivations that we are interested in (namely, those we'll call "tangent vectors"), let us prove some properties of general derivations.

Proposition 2.0.13. *Consider any derivation $X \in \mathrm{Der}_p M$ at a point p of a $\mathcal{C}^k$-diferentiable manifold M. Then for all functions $g, h \in \mathcal{C}^k(M, \mathbb{R})$ we have:*

(a) If h is constant, then $X(h) = 0$.

(b) If both g and h vanish at p (i.e. $g(p) = h(p) = 0$), then $X(g \cdot h) = 0$.

(c) Locality: If $g \equiv h$ on a neighborhood of p, then $X(g) = X(h)$.

Proof. To prove (a), since X is linear, it suffices to show that $X(h) = 0$ if $h \equiv 1$. And indeed, in this case we have

$$X(h) = X(h^2) = X(h) \cdot h(p) + h(p) \cdot X(h) = 2X(h)$$

which implies $X(h) = 0$.

Fact (b) follows from the Leibniz identity.

To prove (c), since X is linear, it suffices to show that $X(f) = 0$ if $f \equiv 0$ in some open neighborhood U of p. Let $\eta : \mathcal{C}^k(M, \mathbb{R})$ be a bump function that is constantly 1 on the closed set $M \setminus U$ and whose support is contained in the open set $M \setminus \{p\}$. Note that $f \cdot \eta = f$, therefore $X(f) = X(f \cdot \eta) = 0$ since $f(p) = \eta(p) = 0$. $\square$

One first example of derivation is the one we already mentioned: velocity vectors.

Definition 2.0.14. The **velocity vector** of a curve $\gamma : I \subseteq \mathbb{R} \to M$ at an instant $t \in I$ (where γ is differentiable) is the derivation $\mathrm{Vel}_\gamma(t) \in \mathrm{Der}_{\gamma(t)} M$ defined by

$$\begin{array}{rccc} \mathrm{Vel}_\gamma(t): & \mathcal{C}^k(M, \mathbb{R}) & \to & \mathbb{R} \\ & h & \mapsto & (h \circ \gamma)'(t). \end{array}$$

Let us check that the function $X = \mathrm{Vel}_\gamma(t_0)$ is indeed a derivation on M at the point $p = \gamma(t_0)$. For a function $h \in \mathcal{C}^k(M, \mathbb{R})$ we denote $h_\gamma := h \circ \gamma \in \mathcal{C}^k(I, \mathbb{R})$. The fact that X satisfies the Leibniz identity for functions $g, h \in \mathcal{C}^k(M, \mathbb{R})$ follows from the Leibniz identity for the single-variable functions $g_\gamma, h_\gamma : I \to \mathbb{R}$ (that you already know from calculus):

$$\begin{aligned} X(g \cdot h) = (g \cdot h)'_\gamma(t_0) = (g_\gamma \cdot h_\gamma)'(t_0) &= g'_\gamma(t_0) \cdot h_\gamma(t_0) + g_\gamma(t_0) \cdot h'_\gamma(t_0) \\ &= X(g) \cdot h(p) + g(p) \cdot X(h) \end{aligned}$$

The fact that X is $\mathbb{R}$-linear can be proven in a similar way (exercise).

Remark 2.0.15. The velocity vector $\mathrm{Vel}_\gamma(t)$ is sometimes denoted $\gamma'(t)$, but it is different from the usual $\gamma'(t)$ defined for a curve in $\mathbb{R}^n$, because it also contains the information of the point $\gamma(t)$ where the derivation is located.
Exercise: If γ, β are differentiable curves in $\mathbb{R}^n$, show that $\mathrm{Vel}_\gamma(t) = \mathrm{Vel}_\beta(t)$ if and only if $\gamma(t) = \beta(t)$ and $\gamma'(t) = \beta'(t)$.

[1] Exercise: check that $a \cdot X + b \cdot Y$ is indeed a derivation and convince yourself that these operations make $\mathrm{Der}_p M$ a vector space.

Another kind of derivations are *vectorial derivations* (also known as *directional derivations*[2]). Before considering general manifolds, let us recall how these derivation operators are defined on an open subset of $\mathbb{R}^n$.

Definition 2.0.16. A **vectorial derivation** at a point p of an open set $U \subseteq \mathbb{R}^n$ is a derivation $\delta_{p,v} \in \operatorname{Der}_p U$ determined by a vector $v \in \mathbb{R}^n$ by the formula

$$\begin{array}{rccc} \delta_{p,v}: & \mathcal{C}^k(U,\mathbb{R}) & \to & \mathbb{R} \\ & h & \mapsto & \mathrm{D}_p h(v), \end{array} \tag{2.3}$$

where $\mathrm{D}_p h : \mathbb{R}^n \to \mathbb{R}$ is the differential operator of h at the point p.

Note that $\delta_{p,v}$ is a derivation; in fact, it is the velocity vector at time $t = 0$ of any curve $\gamma : (-\varepsilon, \varepsilon) \to U$ satisfying $\gamma'(0) = p$ and $\gamma'(0) = v$. (For example, the curve $\gamma(t) = p + tv$.) Indeed, if γ is such a curve, then for any function $h \in \mathcal{C}^k(U,\mathbb{R})$ we have

$$\delta_{p,v}(h) = \mathrm{D}_p h(v) = \mathrm{D}_p h(\gamma'(0)) = (h \circ \gamma)'(0) = \mathrm{Vel}_\gamma(0)(h).$$

The definition of vectorial derivation can be adapted to a general manifold by using a chart.

Definition 2.0.17. A **vectorial derivation** on a $\mathcal{C}^k$ differentiable n-manifold M at a point $p \in M$ is a derivation of the form

$$\begin{array}{rccc} \delta_{p,\phi,v}: & \mathcal{C}^k(M,\mathbb{R}) & \to & \mathbb{R} \\ & h & \mapsto & \mathrm{D}_{\phi(p)}(h \circ \phi^{-1})(v) \end{array} \tag{2.4}$$

where ϕ is a chart defined at p and $v \in \mathbb{R}^n$. The triple (p, ϕ, v) is called a **coordinate expression** of $\delta_{p,\phi,v}$.

A **tangent vector** to M at p is the same thing as a vectorial derivation on M at p.

The **tangent space** of M at p is the set $\mathrm{T}_p M \subseteq \operatorname{Der}_p M$ of vectors tangent to M at p. (We will show that $\mathrm{T}_p M$ is in fact an n-dimensional subspace of $\operatorname{Der}_p M$.)

The **tangent bundle** of M is the set $TM := \coprod_{p \in M} \mathrm{T}_p M$.

Exercise 2.0.18. Prove that tangent vectors are derivations indeed. In fact, tangent vectors are the same thing as velocity vectors. That is, every tangent vector is the velocity vector of some curve, and every velocity vector can also be expressed as a vectorial derivation.

Remark 2.0.19 (Domain of tangent vectors)**.** Although the "official" domain of a tangent vector $X \in \mathrm{T}_p M$ is the set $\mathcal{C}^k(M,\mathbb{R})$, in fact we can apply X to any real-valued h that is defined on a neighborhood U of p and is differentiable at p. The number $X(h)$ is defined using the same formula (2.4) that we would use for a function $h \in \mathcal{C}^k(M,\mathbb{R})$.

Remark 2.0.20 (Locality of tangent vectors)**.** It is also clear from the definition that the number $X(h)$ depends only on the values of h near p. Thus if two functions g, h coincide on a neighborhood of p, then $X(g) = X(h)$. (We have already shown in Proposition 2.0.13 that this is a property of general derivations, but in the case of vectorial derivations it is especially evident.)

[2] I don't like saying "directional" because it suggests that the derivation depends only on the direction of the vector, and not on its size.

Let us show that the tangent space T_pM of an n-dimensional differentiable manifold M at any point $p \in M$ is an n-dimensional vector space, as promised. We defined T_pM as a subset of $\operatorname{Der}_p M$, which is itself a vector space (with the obvious operations that we already defined). Thus we need just show that T_pM is an n-dimensional subspace of $\operatorname{Der}_p M$, and we will do so by showing that T_pM is the image of an injective linear transformation $\mathbb{R}^n \to \operatorname{Der}_p M$.

Proposition 2.0.21. *Let M be a $\mathcal{C}^k$-differentiable manifold and let $p \in M$. Then for any chart ϕ of M defined at p, the map*

$$\begin{array}{rccc} \iota_{p,\phi} : & \mathbb{R}^n & \to & \operatorname{Der}_p M \\ & v & \mapsto & \delta_{p,\phi,v} \end{array}$$

is a linear isomorphism onto its image, which is the tangent space T_pM. In consequence, T_pM is an n-dimensional subspace of $\operatorname{Der}_p M$.

In other words, *any single* chart ϕ defined at p allows us to express *all* tangent vectors $X \in \mathrm{T}_pM$ in the form $X = \delta_{p,\phi,v}$, with the vector $v \in \mathbb{R}^n$ being unique for each X.

Proof. To show that $\iota_{p,\phi}$ is an isomorphism we develop some formulas that will be useful elsewhere.

Lemma 2.0.22. *Any tangent vector $X = \delta_{p,\phi,v} \in \mathrm{T}_pM$ applied to a component function ϕ^i of the chart ϕ gives*

$$X(\phi^i) = v^i. \tag{2.5}$$

In consequence, $\iota_{p,\phi}$ is injective.

Proof of lemma. Note first that $\phi^i = \pi^i \circ \phi$, where $\pi^i : \mathbb{R}^n \to \mathbb{R}$ is the projection on the i-th coordinate, defined by $\pi^i(x) = x^i$. Using this, we can compute

$$X(\phi^i) = \mathrm{D}_p(\phi^i \circ \phi^{-1})(v) = \mathrm{D}_p(\pi^i)(v) = \pi^i(v) = v^i.$$

Here we used the fact that $\mathrm{D}_p(\pi^i) = \pi^i$ since π^i is linear.

Note that the functions ϕ^i are not in the "official" domain $\mathcal{C}^k(M,\mathbb{R})$ of the operator $\delta_{p,\phi,v}$, because they are defined on $U \subseteq M$ (the domain of ϕ) rather than on the whole manifold M. However, we can construct functions $\widetilde{\phi^i} \in \mathcal{C}^k(M,\mathbb{R})$ that coincide with ϕ^i on a neighborhood of p. (For example, using bump functions; recall Lemma 1.4.11.) Using the functions $\widetilde{\phi^i}$ instead of ϕ^i we obtain an analogous formula $X(\widetilde{\phi^i}) = v^i$. This formula allows us to recover the components of the vector $v \in \mathbb{R}^n$ from the derivation $\delta_{p,\phi,v}$. It follows that the map $\iota_{p,\phi} : v \mapsto \delta_{p,\phi,v}$ is injective. □

Lemma 2.0.23. *For any two charts ϕ, ψ defined at p, and for any two vectors $v, w \in \mathbb{R}^n$, we have*

$$\begin{aligned} \delta_{p,\phi,v} = \delta_{p,\psi,w} \quad &\iff \quad w = \mathrm{D}_{\phi(p)}(\psi \circ \phi^{-1})(v) \\ &\iff \quad v = \mathrm{D}_{\psi(p)}(\phi \circ \psi^{-1})(w) \end{aligned} \tag{2.6}$$

In consequence, the map $\iota_{p,\phi}$ is surjective onto T_pM.

Proof of lemma. Note first that the two equations on the right are equivalent since the linear transformations $\mathrm{D}_{\phi(p)}(\psi \circ \phi^{-1})$ and $\mathrm{D}_{\psi(p)}(\phi \circ \psi^{-1})$ are inverse of each other.

Suppose first that these equations hold. Then for any function $h \in \mathcal{C}^k(M, \mathbb{R})$ we have

$$\begin{aligned}
\delta_{p,\psi,w}(h) &= \mathrm{D}_{\psi(p)}(h \circ \psi^{-1})(w) \\
&= \mathrm{D}_{\psi(p)}(h \circ \psi^{-1})(\mathrm{D}_{\phi(p)}(\psi \circ \phi^{-1})(v)) \\
&= \mathrm{D}_{\phi(p)}(h \circ \phi^{-1})(v) \\
&= \delta_{p,\phi,v}(h).
\end{aligned}$$

It follows that $\iota_{p,\phi}$ is surjective onto $\mathrm{T}_p M$, since any vector $X \in \mathrm{T}_p M$ is of the form $X = \delta_{p,\psi,w}$ for some chart ψ and some vector $w \in \mathbb{R}^n$, and we can obtain $\iota_{p,\phi}(v) = X$ by putting $v = \mathrm{D}_{\psi(p)}(\phi \circ \psi^{-1})(w)$.

Finally, suppose that $\delta_{p,\phi,v} = \delta_{p,\psi,w}$. We have already shown that this equation holds when $v = \mathrm{D}_{\psi(p)}(\phi \circ \psi^{-1})(w)$, and it cannot hold for any other value of v because the map $\iota_{p,\phi} : v \mapsto \delta_{p,\phi,v}$ is injective by Lemma 2.0.22. We conclude that $v = \mathrm{D}_{\psi(p)}(\phi \circ \psi^{-1})(w)$. □

This finishes the proof that $\iota_{p,\phi}$ is an isomorphism, and hence $\mathrm{T}_p M \simeq \mathbb{R}^n$. □

Remark 2.0.24. Lemma 2.0.23 shows that a tangent vector $X \in \mathrm{T}_p M$ can be thought of as an equivalence class of triples (p, ϕ, v) (where ϕ is a chart of M and $v \in \mathbb{R}^n$) under the equivalence relation

$$\begin{aligned}
(p, \phi, v) \sim (p, \psi, w) \quad &\iff \quad w = \mathrm{D}_{\phi(p)}(\psi \circ \phi^{-1})(v) \\
&\iff \quad v = \mathrm{D}_{\psi(p)}(\phi \circ \psi^{-1})(w)
\end{aligned}$$

This is another way to define tangent vectors, equivalent to ours.

Example 2.0.25 (Tangent bundle of an open subset of $\mathbb{R}^n$)**.** Recall that any open set $U \subseteq \mathbb{R}^n$ is naturally a $\mathcal{C}^k$ differentiable manifold with an atlas consisting of the single chart id_U. This single chart is sufficient to express all tangent vectors, by Proposition 2.0.21. That is, all tangent vectors $X \in \mathrm{T}U$ are of the form $X = \delta_{p,\mathrm{id}_U,v}$, with $p \in U$ and $v \in \mathbb{R}^n$. Thus there is a natural identification[3]

$$\begin{aligned}
\mathrm{T}U \quad &\equiv \quad U \times \mathbb{R}^n \\
\delta_{p,\mathrm{id}_U,v} \quad &\leftrightarrow \quad (p, v).
\end{aligned}$$

Note also that the operator $\delta_{p,\mathrm{id}_U,v}$ coincides with the operator $\delta_{p,v}$ of Definition 2.0.17, therefore the two definitions of "vectorial derivation" (2.0.17 and 2.0.16) are equivalent for an open set $U \subseteq \mathbb{R}^n$.

2.0.2 Pushforward and tangent operators

Any differentiable map between manifolds induces a map on derivations called the **pushforward operator**.

Definition 2.0.26 (Pushforward of derivations)**.** The **pushforward operator** of a $\mathcal{C}^k$-differentiable map $f : M \to N$ at some point $p \in M$ is the linear map

$$\begin{aligned}
f_*|_p : \ \mathrm{Der}_p M \quad &\to \quad \mathrm{Der}_{f(p)} N \\
X \quad &\mapsto \quad (h \in \mathcal{C}^k(N, \mathbb{R}) \mapsto X(h \circ f) \in \mathbb{R}).
\end{aligned}$$

The union of the operators $f_*|_p$ over all points $p \in M$ is the **pushforward map**

$$f_* := \coprod_{p \in M} f_*|_p : \mathrm{Der}\, M \to \mathrm{Der}\, N.$$

[3] We write $A \equiv B$ when there is a natural isomorphism between two objects A, B.

Example 2.0.27. Each derivation $\delta_{p,\phi,v}$ of Definition 2.0.17 is the image of the derivation $\delta_{\phi(p),v}$ of Definition 2.0.16 by the pushforward operator of the map ϕ^{-1} at the point $\phi(p)$. (Check it!)

The **tangent operator** of a differentiable map is obtained by restricting the pushforward operator to tangent vectors.

Definition 2.0.28. The **tangent operator** of a $\mathcal{C}^k$ map $f : M \to N$ at a point $p \in M$ is the map

$$\mathrm{T}_p f : \mathrm{T}_p M \to \mathrm{T}_{f(p)} N$$

obtained by restricting the pushforward map $f_*|_p : \mathrm{Der}_p M \to \mathrm{Der}_{f(p)} N$.

The **tangent map** of f is the union of the tangent operators $\mathrm{T}_p f$ over all points $p \in M$,

$$\mathrm{T}f := \coprod_{p \in M} \mathrm{T}_p f : \mathrm{T}M \to \mathrm{T}N.$$

To show that the tangent operator $\mathrm{T}_p f$ is well defined, we have to verify that the pushforward operator $f_*|_p$ indeed maps tangent vectors to tangent vectors (or, equivalently, it maps velocity vectors to velocity vectors).

Proposition 2.0.29 (Local expression of the tangent operator)**.** *If $f : M \to N$ is differentiable at a point $p \in M$, then for any tangent vector $\delta_{p,\phi,v} \in \mathrm{T}_p M$ and any chart ψ of N defined at $f(p)$ we have*

$$f_*|_p(\delta_{p,\phi,v}) = \delta_{\phi(p),\psi,w} \quad \textit{where} \quad w = \mathrm{D}_{\phi(p)}(f|_\phi^\psi)(v). \tag{2.7}$$

Proof. For any $h \in \mathcal{C}^k(N, \mathbb{R})$ we have

$$\begin{aligned}
(f_*|_p\, \delta_{p,\phi,v})(h) &= \delta_{p,\phi,v}(h\, f) \\
&= \mathrm{D}_{\phi(p)}(h\, f\, \phi^{-1})(v) \\
&= \mathrm{D}_{\psi(f(p))}(h\, \psi^{-1})\left(\mathrm{D}_{\phi(p)}(\psi\, f\, \phi^{-1})(v)\right) \\
&= \delta_{f(p),\psi,w}(h).
\end{aligned}$$

□

Proposition 2.0.30 (Pushforward of velocity vectors; exercise)**.** *Let $f : M \to N$ be a $\mathcal{C}^k$-differentiable map, and let $\gamma : I \subseteq \mathbb{R} \to M$ be a curve that is differentiable at some instant $t_0 \in I$. Then $f_*(\mathrm{Vel}_\gamma(t_0)) = \mathrm{Vel}_{f \circ \gamma}(t_0)$.*

Remark 2.0.31. The tangent transformation $\mathrm{T}_p f : \mathrm{T}_p M \to \mathrm{T}_{f(p)} N$ can in fact be defined for any function $f : U \subseteq M \to N$ that is defined on a neighborhood U of p and is differentiable at p. This follows from Remarks 2.0.19 and 2.0.20 (check it!).

Example 2.0.32. For a $\mathcal{C}^k$ differentiable map $f : U \to V$, where $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ are open sets, the tangent operator $\mathrm{T}_p f : \mathrm{T}_p U \to \mathrm{T}_p V$ is essentially equivalent to the differential $\mathrm{D}_p f : \mathbb{R}^m \to \mathbb{R}^n$ because for all vectors $X \in \mathrm{T}_p U$ (which can be written as $X = \delta_{p,v}$ with $v \in \mathbb{R}^m$) we have

$$\mathrm{T}_p f(X) = \mathrm{T}_p f(\delta_{p,v}) = \delta_{p,w} \quad \text{where} \quad w = \mathrm{D}_p f(v).$$

The chain rule that you know from calculus also holds for maps between manifolds.

Proposition 2.0.33 (Chain rule; exercise)**.** *Let $f : M \to N$ and $g : N \to L$ be maps between differentiable manifolds.*

1. *If f and g are differentiable at points $p \in M$, $f(p) \in N$ respectively, then the composite map $g \circ f$ is differentiable at p, and*

$$\mathrm{T}_p(g \circ f) = \mathrm{T}_{f(p)}g \circ \mathrm{T}_p f. \tag{2.8}$$

2. *For any point $p \in M$ we have $T_p \operatorname{id}_M = \operatorname{id}_{\mathrm{T}_pM}$.*
3. *If f is a diffeomorphism, then $\mathrm{T}_p f$ is a linear isomorphism with inverse $\mathrm{T}_{f(p)}(f^{-1})$.*

2.0.3 Classical notation for tangent vectors

Here we could do the same as in the last chapter.

Given a chart ϕ of M at p, define a basis of T_pM consisting of vectors $\frac{\partial}{\partial \phi^i}\Big|_p := \delta_{p,\phi,e_i}$.

Note that for any function $f \in \mathcal{C}^k(M, \mathbb{R})$ $\frac{\partial}{\partial \phi^i}\Big|_p f = \frac{\partial}{\partial x^i}\big|_{\phi(p)} f|\phi(x)$.

The vector $\frac{\partial}{\partial \phi^i}\Big|_p$ is the velocity vector at $t = 0$ of a curve $\gamma_i(t)$ that satisfies $\phi^j(\gamma_i(t)) = \phi^j(p) + \delta_i^j t$. That is, along this curve $\gamma_i(t)$ the i-th coordinate increases at unit speed while the other coordinates remain constant.

One can prove that $\frac{\partial}{\partial \phi^i}\Big|_p (\phi^j) = \delta_i^j$ and also give the coordinate change formula.

2.0.4 General derivations*

If M is a smooth manifold, then the space $\operatorname{Der}_p M$ contains no other derivations than the tangent vectors (see exercies). Thus one can define the tangent space T_pM of a smooth manifold M simply as the space $\operatorname{Der}_p M$ of derivations. However, if M is a $\mathcal{C}^k$ manifold with $k < \infty$, then the space $\operatorname{Der}_p M$ is infinite dimensional (see exercises).

3 Submanifolds

Let us start with a short motivation. The "bent line" in $\mathbb{R}^2$, defined as $C = \{(x, y) \in \mathbb{R}^2 \mid xy = 0, x \geq 0, y \geq 0\}$, is intuitively not "smoothly contained" in $\mathbb{R}^2$. It is topological submanifold of $\mathbb{R}^2$ because, when endowed with the subspace topology, it is homeomorphic to $\mathbb{R}$. Moreover, it follows that C can be given the structure of a smooth manifold (using a single-chart atlas), but the smooth structure does not "agree" with the ambient one; the bent line is not a *smooth submanifold* of $\mathbb{R}^2$. To make this precise we need a few definitions:

Definition 3.0.1. Let $f : M \to N$ be a $\mathcal{C}^k$-differentiable map between manifolds of respective dimensions m, n.

We say that f is **submersive** at a point $p \in M$ if $\mathrm{T}_p f$ is surjective. If this holds for all points $p \in M$, then f is a **submersion**.

We say that f is **immersive** at a point $p \in M$ if $\mathrm{T}_p f$ is injective. If this holds for all points $p \in M$, then f is an **immersion**.

We say that f is an **embedding** if it is an immersion and also a topological embedding (i.e. a homeomorphism onto its image, the latter being endowed with the subspace topology). Its image is called a $\mathcal{C}^k$ **embedded submanifold** of N (or a $\mathcal{C}^k$ **embedded m-submanifold**, to highlight its dimension).

The **rank** of f at a point $p \in M$, denoted $\operatorname{rank}_p f$, is the rank of the tangent operator $\mathrm{T}_p f$. If f has the same rank k at all points ("constant rank") then we write $\operatorname{rank} f = k$.

So to sum up, a $\mathcal{C}^k$ map $f : M \to N$ is a submersion iff $\operatorname{rank} f = n = \dim N$ and is an immersion iff $\operatorname{rank} f = m = \dim M$. In both cases the tangent operator $\mathrm{T}_p f$ has maximal rank for all points $p \in M$.[1] Note that the set of points where f has maximal rank is open because the rank is an upper semicontinuous function: if $k = \operatorname{rank}_p f$, then there is a neighborhood of p where f has rank $\geq k$.

Example 3.0.2. The **standard immersion** $\iota|_k^n : \mathbb{R}^k \to \mathbb{R}^n$, defined for $k \leq n$, is the map

$$\iota_k^n : (x^0 \ldots, x^{k-1}) \mapsto (x^0, \ldots, x^{k-1}, 0, \ldots, 0).$$

This is also a smooth embedding.
The **standard submersion** $\pi_m^k = \mathbb{R}^m \to \mathbb{R}^k$, defined for $m \geq k$, is the map

$$\pi_m^k : (x^0, \ldots, x^{m-1}) \mapsto (x^0, \ldots, x^{k-1}).$$

Composing these we obtain the **standard map of rank** k

$$\iota_k^n \circ \pi_m^k d : \mathbb{R}^m \to \mathbb{R}^n : (x^0, \ldots, x^{m-1}) \mapsto (x^0, \ldots, x^{k-1}, 0, \ldots, 0).$$

Example 3.0.3.

- A smooth curve $\gamma : J \to M$, where $J \subset \mathbb{R}$ is an open interval and M is a smooth manifold, is an immersion iff $\gamma'(t) \neq 0$ for all $t \in J$. For example, the curve on $\mathbb{R}^2$ given by $t \mapsto (\cos t, \sin t)$, $t \in \mathbb{R}$, is an immersion, but the curve on $\mathbb{R}^2$ given by $t \mapsto (t^2, t^3)$, $t \in \mathbb{R}$, is not.

[1] A linear map $V \simeq \mathbb{R}^m \to W \simeq \mathbb{R}^n$ has **maximal rank** if its rank is equal to $\min\{m, n\}$, which is the maximum rank that a linear map $V \to W$ can have.

- The *bent line* $C = \{(x,y) \in \mathbb{R}^2 | xy = 0,\ x,y \geq 0\}$ is not an embedded submanifold. In fact, it is not the image of an immersion. (Exercise.)
- The inclusion $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding (where we endow $\mathbb{S}^n$ and $\mathbb{R}^{n+1}$ with the standard smooth structures).
- Let $M = M_0 \times M_1$ be a product of smooth manifolds. Then the canonical projections $\pi^i : M \to M_i$ are submersions.
- *"The figure 8"*: Let $J = (-\frac{\pi}{2}, \frac{3\pi}{2})$ and consider the curve

$$\gamma : J \to \mathbb{R}^2, \quad \gamma(t) = (\sin(2t), \cos t).$$

  It is easy to verify that γ is an injective immersion. However, it is not a homeomorphism onto its image $N := \gamma(J) \subset \mathbb{R}^2$ (endowed with the subspace topology): removing e.g. the point $(0,1) = \gamma(0)$ the set $N \backslash \{(0,1)\}$ is connected, but its preimage under γ is $(-\frac{\pi}{2}, 0) \cup (0, \frac{3\pi}{2})$ hence disconnected. Therefore γ is not an embedding.
- *Irrational line on the torus*: Consider the curve in the torus $f : \mathbb{R} \to \mathbb{T}^2$ defined by $f(t) = [t, \alpha\, t]$, where $\alpha \in \mathbb{R}$ is an irrational number. This map is an injective immersion, but is not an embedding since one can find a divergent sequence of numbers $t_i \in \mathbb{R}$ such that $f(t_i)$ converges to some point in the image of f. (See [Lee13, Example 4.20] for more details.)

Example 3.0.4 (Graphs as submanifolds)**.** The **graph** of a $\mathcal{C}^k$ map $f : M \to N$, defined as the set $\mathrm{Gra}_f = \{(x, f(x)) : x \in M\}$, is a submanifold of $M \times N$. Indeed, it is the image of the **graphing map** of f, i.e. the map $g : M \to M \times N$ defined by $g(x) = (x, f(x))$, which is an embedding.

To see that g is an immersion we note that for any point $p \in M$, the tangent operator $\mathrm{T}_p g : T_p M \to T_{p,f(p)}(M \times N) \equiv T_p M \times T_{f(p)} N$ maps every nonzero vector $v \in \mathrm{T}_p M$ to the vector $\mathrm{T}_p g(v) = (v, \mathrm{T}_p f(v))$, which is nonzero because its first component is nonzero.

To see that the map g is a topological embedding we note that it admits a continuous retraction, namely, the projection map $\pi : M \times N \to M : (x,y) \mapsto x$. (A **retraction** of g is a left inverse, i.e. a map ρ satisfying $\rho \circ g = \mathrm{id}$.) Equivalently, the map $g|^{\mathrm{Gra}_f} : M \to \mathrm{Gra}_f$ is a homeomorphism because it admits as inverse the continuous map $\pi|_{\mathrm{Gra}_f} : \mathrm{Gra}_f \to M$.

3.1 How to recognize an embedding

Suppose we are given a differentiable map $f : M \to N$ between manifolds, and we have already checked that it is an injective immersion. How can we prove that it is an embedding ? This is essentially a topological question: how can we show that a given injective map[2] is actually a topological embedding ? Here are some ways to do so:

Proposition 3.1.1. *An injective $\mathcal{C}^r$ immersion $f : M \to N$ is an embedding in any of the following cases:*

(a) *f is an open map.*

(b) $\dim M = \dim N$.

(c) *f is a closed map.*

(d) *The domain M is compact.*

(e) *f is a proper map.*

[2] A **map** between topological spaces is a continuous function.

(f) *The image $f(M)$ has an open neighborhood W such that the map $f|^W : M \to W$ is closed or proper.*

Proof. (a) In this case f sends open subsets of M to open subsets of N, which are clearly also open in the image $f(M)$. Thus f is an open (and bijective) map onto its image, hence an homeomeomorphism onto its image.

(b) In this case f is a local diffeomorphism by the IFT, hence an open map.

(c) Similar to (a), replacing "open" by "closed".

(d) A map from a compact space to a Hausdorff space is closed.

(e) A proper map is closed if its codomain is Hausdorff and locally compact; see e.g. [Lee13, A.57] or [Lee10, p. 4.95].

(f) In this case f is the composite of two embeddings: the closed embedding $f|^W$, followed by the inclusion map $\iota_W : W \to M$, which is an open embedding. □

Remark 3.1.2. The last case (f) is interesting because it gives a general criterion: for every embedding $f : M \to N$ there exists an open neighborhood W of $\mathrm{Img}(f)$ such that $f|^W : M \to W$ is closed. (Exercise.)

A different, arguably more direct way to check that a $\mathcal{C}^k$ map $f : M \to N$ is an embedding is by constructing a $\mathcal{C}^k$ **retraction** (a.k.a left inverse) of f, that is, a $\mathcal{C}^k$ map $g : N \to M$ satisfying $g \circ f = \mathrm{id}_M$. In fact, the retraction need not be defined everywhere; an open neighborhood of the image suffices.

Proposition 3.1.3. *Let $f : M \to N$ be a $\mathcal{C}^k$ map. Then f is an embedding if it admits a $\mathcal{C}^k$ **neighborhood retraction**, that is, a $\mathcal{C}^k$ map $g : U \to M$, where $U \subseteq N$ is an open neighborhood of the image $f(M)$, such that $g \circ f = \mathrm{id}_M$.*

Proof. Suppose $f : M \to N$ has a neighborhood retraction $g : U \to M$. Then f is clearly injective. Its tangent operator $\mathrm{T}_p f$ at any point $p \in M$ is injective as well because

$$\mathrm{T}_p \,\mathrm{id}\, M = \mathrm{T}_p(g \circ f) = \mathrm{T}_{f(p)} g \circ \mathrm{T}_p f$$

is an isomorphism. Therefore f is an immersion. Finally, f is a homeomorphism onto its image $f(M)$ because it admits as inverse the continuous map $g|_{f(M)}$. □

Note that this is the argument we used in Example 3.0.4 to prove that the graph of a $\mathcal{C}^r$ map $f : M \to N$ is a $\mathcal{C}^r$ submanifold of $M \times N$. The graphing map $g : M \to M \times N : x \mapsto (x, f(x))$ admits as a retraction the projection $\pi : M \times N \to M : (x, y) \mapsto x$.

Remark 3.1.4. This method for showing that a map is a submanifold is also general in the sense that every embedding $f : M \to N$ admits a neighborhood retraction (defined on a so-called *tubular neighborhood* of the submanifold $f(M)$). We won't prove that, but we'll show how to produce local neighborhood retractions, which are in fact sufficient to prove that a given subset of a manifold is an embedded submanifold (see Propositions 3.3.1 and 3.4.1 below).

3.2 Constant rank theorem

Many theorems about submanifolds are based on the inverse function theorem.

Theorem 3.2.1 (Inverse Function Theorem, or IFT)**.** *Let $f : M \to N$ be a $\mathcal{C}^k$ map, where M, N are n-manifolds, and let $p \in M$ be a point where the tangent operator $\mathrm{T}_p f$ is invertible. Then f is a **local diffeomorphism** at p, i.e. there exists respective open neighborhoods U, V of p, $f(p)$ such that $f|_U^V$ is a $\mathcal{C}^k$ diffeomorphism.*

In case that M, N are open subsets of $\mathbb{R}^n$, this is just the inverse function theorem that you know from calculus. The general case can be deduced by using charts. (Exercise.)

A useful generalization (and corollary) of the IFT is the *constant rank theorem.*

Theorem 3.2.2 (Constant Rank Theorem, or CRT)**.** *Let $f : M \to N$ be $\mathcal{C}^r$ map that has rank k at some point $p \in M$, and let $m = \dim M$ and $n = \dim N$. Suppose, further that f has constant rank k in a neighborhood of p. Then f admits at the point p a local expression $\widetilde{f} = f|_\phi^\psi$ of the form*

$$\begin{array}{rccc} \widetilde{f}: & \widetilde{U} \subseteq \mathbb{R}^m & \to & \widetilde{V} \subseteq \mathbb{R}^n \\ & x & \mapsto & (x^0, \dots, x^{k-1}, 0, \dots, 0). \end{array}$$

where $\phi : U \to \widetilde{U}$, $\psi : V \to \widetilde{V}$ are charts of M, N defined and centered at p, $f(p)$ respectively. (This means that $\phi(p) = 0_{\mathbb{R}^n}$ and $\psi(f(p)) = 0_{\mathbb{R}^n}$.) Moreover, one can assume that the chart images are product sets $\widetilde{U} = \widetilde{W} \times \widetilde{U}'$, $\widetilde{V} = \widetilde{W} \times \widetilde{V}'$, for some open sets $\widetilde{W} \subseteq \mathbb{R}^k$, $\widetilde{U}' \subseteq \mathbb{R}^{m-k}$ and $\widetilde{V}' \subseteq \mathbb{R}^{n-k}$.

The hypothesis of constant rank near p is satisfied automatically in the following cases:

- *If f is submersive at p (i.e. if $n = k$): In this case the chart ψ may be obtained by restricting an arbitrary chart. Equivalently, if N is an open subset of $\mathbb{R}^k$, one can take $\psi = \mathrm{id}_V$, where $V \subseteq N$ is an open neighborhood of $f(p)$.*
- *If f is immersive at p (i.e. if $m = k$): In this case the chart ϕ may be obtained by restricting an arbitrary chart. Equivalently, if M is an open subset of $\mathbb{R}^k$ one can take $\phi = \mathrm{id}_U$, where $U \subseteq M$ is an open neighborhood of p.*

This theorem says that any map of constant rank (in particular, any immersion or submersion) is locally equivalent to the standard map given in Example 3.0.2 above. If $\mathrm{T}_p f$ does not have maximum rank (i.e. if $\mathrm{rank}_p f =: k < m, n$), then the hypothesis of having constant rank k near p is hard to satisfy because the rank is lower semicontinuous but not upper semicontinuous in general. We will prove the constant rank theorem only in the cases $k = m$ and $k = n$, which are the most useful.

Example 3.2.3. If $f : V \to W$ is a *linear* map of rank k between topological spaces $V \simeq \mathbb{R}^m$ and $W \simeq \mathbb{R}^n$, then there are *linear* isomorphisms $\phi : V \to \mathbb{R}^m$, $\psi : W \to \mathbb{R}^n$ such that $\psi \circ f \circ \phi^{-1}$ is a standard map of rank k. (Exercise.)

Before proving the constant rank theorem, let us discuss the situation. We use the following terminology and notation for a point $x = (x^i)_{0 \le i < m} \in \mathbb{R}^m$. Its **horizontal part** is the tuple of its first k-coordinates,

$$x^{hor} := (x^0, \dots, x^{k-1}) = (x^i)_{0 \le i < k} \in \mathbb{R}^{[0,k)} = \mathbb{R}^k,$$

and its **vertical part** is the tuple of the remaining coordinates

$$x^{ver} := (x^k, \dots, x^{m-1}) = (x^i)_{k \le i < m} \in \mathbb{R}^{[k,m)} \equiv \mathbb{R}^{m-k}.$$

Similarly, for a point $y \in \mathbb{R}^n$, its horizontal and vertical parts are $y^{[0,k)}$ and $y^{[k,n)}$.

From the constant rank theorem we conclude that the map $f|_U^V$ has the following features: its image is a k-submanifold of V, and each fiber (i.e. the preimage of a point of the image) is an $(m-k)$-submanifold of U. These features are evident in the local expression $f|_\phi^\psi$, but they cannot be created by the charts, therefore they must be already present in the function $f|_U^V$.

The main job of the chart ϕ is to straighten the fibers of $f|_U^V$ so that they appear in the chart image $\widetilde{U} \subseteq \mathbb{R}^m$ as the vertical slices $\{x^{[0,k)}\} \times \widetilde{U}'$. For this task, the first k coordinates must be chosen carefully, so that they are constant on the fibers of f. This straightening is not necessary if $k = m$ (because the fibers are points), hence we may choose ϕ arbitrarily in this case.

Similarly, the main job of the chart ψ is to straighten the image of $f|_U^V$ so that it appears in the chart image $\widetilde{V} \subseteq \mathbb{R}^m$ as the horizontal null plane $\widetilde{W} \times \{0_{\mathbb{R}^{[k,n)}}\}$. For this task, the last $n-k$ coordinates are the most important; they must vanish on the image of $f|_U^V$. This straightening is not necessary if $k = n$ (because the image is an open set, not a submanifold of smaller dimension), hence we may choose ψ arbitrarily in this case.

Finally, we must ensure that the local expression $f|_\phi^\psi$ maps each vertical fiber $\{x^{hor}\} \times \widetilde{U}'$ to the right point $(x^{hor}, 0, \ldots, 0)$ of the image. (To see that something is needed consider a map $f : \mathbb{R}^m \to \mathbb{R}^n$ of the form $f(x) = (\theta(x^{hor}), 0_{\mathbb{R}^{n \setminus k}}) \in \mathbb{R}^n$, where $\theta : \mathbb{R}^k \to \mathbb{R}^k$ is a diffeomorphism. This map f has constant rank k but it is not in the standard form, even though its fibers are the vertical slices and its image is the horizontal null plane.) This third job can be done either by ϕ or by ψ, but it must be done by one of the two charts. This is why in the case $k = m = n$ we cannot choose both ϕ and ψ arbitrarily.

Proof of the constant rank theorem (for submersions and immersions). Since the theorem is local, by taking coordinates we may assume that M, N are open subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ respectively and that $p = 0_{\mathbb{R}^m}$ and $f(p) = 0_{\mathbb{R}^n}$. Each point $x \in M$ is a tuple of coordinates $x = (x^i)_{i \in [0,m)}$, and, similarly a point $y \in N$ is of the form $y = (y^j)_{j \in [0,n)}$. Since $\mathrm{D}_p f$ has rank k, we may assume wlog (by permutting the indices, if necessary) that the $k \times k$ matrix

$$\left(\frac{\partial y^j}{\partial x^i}\bigg|_p \right)_{i \in [0,k), j \in [0,k)}, \text{ where } y = f(x) \text{, is invertible.} \tag{3.1}$$

Note that if $k = m$, then no permutation of the i indices is needed, and if $k = n$, then no permutation of the j indices is needed.

Case f submersive at p (i.e. $k = n$): We define on M near p a coordinate system ϕ by the formula

$$\phi^j(x) = u^j = \begin{cases} y^j & \text{if } j \in [0,k) \\ x^j & \text{if } j \in [k,m) \end{cases}$$

where $y = f(x)$. The first k coordinates are defined in the only possible way that ensures that the map $f|_\phi^{\mathrm{id}_V} = f \circ \phi^{-1}$ has the right form, sending $u \mapsto (u^j)_{j \in [0,k)}$. The remaining $m-k$ coordinates are just meant to complete the coordinate system. To verify that this correspondence $x \mapsto u$ really defines a chart ϕ on a neighborhood of p, we check that its differential $\mathrm{D}_p\phi$ is invertible. And indeed, the matrix

$$\left(\frac{\partial u^j}{\partial x^i}\bigg|_{x=p} \right)_{\substack{j \in m \\ i \in m}} = \left(\begin{array}{c|c} \left(\frac{\partial y^j}{\partial x^i}\Big|_{x=p} \right)_{\substack{j \in k \\ i \in k}} & * \\ \hline 0 & I_{m-k} \end{array} \right)$$

is invertible because the block $\left(\frac{\partial y^j}{\partial x^i}\Big|_{x=p} \right)_{\substack{j \in k \\ i \in k}}$ by the hypothesis (3.1). Thus by the IFT, there are respective open neighborhoods $U \subseteq M$, $\widetilde{U} \subseteq \mathbb{R}^m$ of p, $\phi(p)$, such that the restriction $\phi|_U^{\widetilde{U}}$ is a diffeomorphism. Moreover, by shrinking the sets U, $\widetilde{U}$

we can ensure that $\widetilde{U} = \widetilde{W} \times \widetilde{U}'$, where $\widetilde{W} \subseteq \mathbb{R}^k$ and $\widetilde{U}' \subseteq \mathbb{R}^{m-k}$ are open sets. On the manifold N we choose the chart $\psi = \mathrm{id}_{\widetilde{W}}$.

Note: The fact that we were able to put $\psi = \mathrm{id}_{\widetilde{W}}$ means that when M is a general manifold (instead of an open subset of $\mathbb{R}^m$) we can obtain ψ as the restriction of an arbitrary chart.

Case f immersive at p (i.e. $k = m$): Note that the horizontal part of f, i.e. the map $\theta : x \mapsto f(x)^{ver}$, is a diffeomorphism when restricted to a suitable domain and codomain (respective open neighborhoods of p and $f(p)^{hor}$ in $\mathbb{R}^k$). This follows from condition (3.1) by the IFT.

We define on N near $f(p)$ a coordinate system ψ by the formula

$$\begin{cases} \phi^{hor}(y) = \underline{x} \\ \phi^{ver}(y) = y^{ver} - f^{ver}(\underline{x}) \end{cases}$$

where $\underline{x} = \theta^{-1}(y^{hor})$. For a point $y = f(x)$ we have

$$\underline{x} = \theta^{-1}(y^{hor}) = \theta^{-1}(f^{hor}(x)) = \theta^{-1}(\theta(x)) = x,$$

therefore the coordinates $\phi(y)$ have the expected behaviour: the horizontal part $\phi(y)^{hor}$ equals x, while the vertical part vanishes:

$$\phi(y)^{ver} = f(x)^{ver} - f^{ver}(\underline{x}) = 0.$$

Therefore the map $f|_{\mathrm{id}}^{\psi} = \psi \circ f$ sends $x \mapsto (x, 0)$, as required.

To ensure that ψ is a chart on a neighborhood of $f(p)$, we verify that its differential $\mathrm{D}_{f(p)}\phi$ is invertible. And indeed, its matrix of partial derivatives

$$\left(\frac{\partial v^i}{\partial y^j}\bigg|_{y=f(p)} \right)_{\substack{i\in n\\ j\in n}} = \left(\begin{array}{c|c} \left(\frac{\partial \underline{x}^i}{\partial y^j}\Big|_{y=f(p)} \right)_{\substack{i\in k\\ j\in k}} & 0 \\ \hline * & I_{n-k} \end{array} \right),$$

is invertible. This follows because the matrix $\left(\frac{\partial \underline{x}^i}{\partial y^j}\Big|_{y=f(p)} \right)_{i\in k, j\in k}$, which represents the linear transformation $\mathrm{D}_{f(p)}\theta^{-1} = (\mathrm{D}_p\theta)^{-1}$, is invertible. Thus by the IFT, there are respective open neighborhoods $V \subseteq N$, $\widetilde{V} \subseteq \mathbb{R}^n$ of $f(p)$, $\psi(f(p))$, such that the restriction $\phi|_V^{\widetilde{V}}$ is a diffeomorphism. Moreover, by shrinking the sets $\widetilde{V}$, V we can ensure that $\widetilde{V} = \widetilde{W} \times \widetilde{V}'$, where $\widetilde{W} \subseteq \mathbb{R}^k$ and $\widetilde{V}' \subseteq \mathbb{R}^{n-k}$ are open sets. On the manifold M we choose the chart $\psi = \mathrm{id}_{\widetilde{W}}$.

General case* (i.e. $k \leq m, n$) Here one can combine the two constructions. If interested, you can take it as an exercise or read the proof e.g. on p. 42 of Spivak's "Comprehensive introduction to Differential Geometry", volume 1. $\square$

3.3 Some properties of immersions and embeddings

3.3.1 Local properties: slices and retractions

The following corollary of the constant rank theorem says that for any immersion $f : M \to N$, if we restrict f to a sufficiently small neighborhood U of any point $p \in M$, its image $f(U)$ looks like a "slice" of some open set $V \subseteq N$. This implies that f has a local retraction and is locally an embedding.

Lemma 3.3.1 (Slice property (and local retractions) for immersions)**.** *Let N be an n-manifold and let $f : M \to N$ be an immersion with $\dim M = k$. Then for each point p of M there exists a chart $\psi : V \to \widetilde{V}$ of N and an open neighborhood U of p such that*

$$f(U) = \{q \in V : \psi^k(q) = \cdots = \psi^{n-1}(q) = 0\},$$

and such that the map $f|_U^V$ is an embedding and has a $\mathcal{C}^k$ retraction $\rho : V \to U$.

Proof. By the constant rank theorem, the map f admits at the point p a local expression $\widetilde{f} = f|_\phi^\psi$ of the form $\widetilde{f}(x) = (x^0, \ldots, x^{k-1}, 0, \ldots 0)$, where $\phi : U \to \widetilde{U}$, $\psi : V \to \widetilde{V}$ are charts of M, N defined at the points p, $f(p)$ respectively. Moreover, we can assume that $\widetilde{V} = \widetilde{U} \times \widetilde{V}'$, where $\widetilde{V}' \subseteq \mathbb{R}^{n-k}$ is an open neighborhood of $0_{\mathbb{R}^{n-k}}$. It follows that $\widetilde{f}(\widetilde{U}) = \widetilde{U} \times \{0_{\mathbb{R}^{n-k}}\}$ and hence

$$f(U) = \psi^{-1}(\widetilde{U} \times \{0_{\mathbb{R}^{n-k}}\}) = \{q \in V : \psi^k(q) = \cdots = \psi^{n-1}(q) = 0\}.$$

To construct the retraction ρ, consider the projection map

$$\begin{array}{rrcl} \pi : & \widetilde{V} = \widetilde{U} \times \widetilde{V}' & \to & \widetilde{U} \\ & (x, y) & \mapsto & x. \end{array}$$

There is a unique map $\rho : V \to U$ that has local expression $\rho|_\psi^\phi = \pi$, namely, the map $\rho = \phi^{-1} \circ \pi \circ \psi : V \to U$. This map ρ is a $\mathcal{C}^r$ retraction of $f|_U^V$. The existence of this retraction implies that $f|_U^V$ is an embedding by Proposition 3.1.3. □

We have shown that $f(U)$ is a slice of V, but note that the set $f(M \setminus U)$ may also intersect the set V. (Exercise: Find an example (i.e. an immersion f and a point p) such that this happens for any V.) However, this situation can be avoided if f is an embedding.

Proposition 3.3.2 (Slice property for embedded submanifolds)**.** *Let S be a the image of a $\mathcal{C}^r$ embedding $f : M \to N$, and let $k = \dim M$ and $n = \dim N$. Then each point of S is contained in the domain V of a chart ψ that is k-**sliced** by S, meaning that*

$$S \cap V = \{q \in V : \phi^k(q) = \cdots = \phi^{n-1}(q) = 0\}. \tag{3.2}$$

Proof. Take a point $q_0 = f(p_0) \in S$. By Lemma 3.3.1, there exists a chart ψ of M with domain V' and a neighborhood U of p_0 such that $f(U) = \{q \in V' : \psi^k(q) = \cdots = \psi^{n-1}(q) = 0\}$. Since $f(U)$ is open in S, we can write $f(U) = S \cap V''$ using some open set $V'' \subseteq M$. But since $f(U) \subseteq V'$, we have $f(U) = S \cap V$, and therefore

$$S \cap V = f(U) = \{q \in V : \psi^k(q) = \cdots = \psi^{n-1}(q) = 0\}.$$

□

3.3.2 The initial property

Recall that topological embeddings have the following *initial property*.

Proposition 3.3.3 (Initial property of topological embeddings)**.** *If a continuous map $f : X \to Y$ between topological spaces is an embedding, then any function $h : Z \to X$ (where Z is a topological space) is continuous if the composite $f \circ h$ is continuous.*

A $\mathcal{C}^r$ immersion has a similar property that allows us to show that a continuous function is $\mathcal{C}^r$.

Proposition 3.3.4 (Initial property of immersions and embeddings)**.** *Let $f : M \to N$ be a $\mathcal{C}^r$-differentiable map, and let L be another $\mathcal{C}^r$ manifold.*

(a) If f is an immersion, then any continuous map $h : L \to M$ is $\mathcal{C}^r$ if the composite $f \circ h$ is $\mathcal{C}^r$.

(b) If f is an embedding, then any function $h : L \to M$ is $\mathcal{C}^r$ if the composite $f \circ h$ is $\mathcal{C}^r$.

The proof is based on the fact that an immersion admits local retractions.

Proof. (a) Suppose that f is an immersion and that $h : L \to M$ is continuous at some point $z \in L$, and the composite $f \circ h$ is $\mathcal{C}^r$ at some point $z \in L$. Let us show that h is $\mathcal{C}^r$ at z.

Let $U \subseteq M$ and $V \subseteq N$ be open neighborhoods of the points $h(z)$ and $f(h(z))$ such that $f(U) \subseteq V$ and the map $f|_U^V$ has a $\mathcal{C}^r$ retraction $\rho : V \to U$. Thus we have $\rho \circ f|_U^V = \mathrm{id}_U$. Note that the set $W = h^{-1}(U)$ is a neighborhood of x since h is continuous at x. Thus we can write

$$h|_W^U = \mathrm{id}_U \circ h|_W^U = (\rho \circ f|_U^V) \circ h|_W^U = \rho \circ (f \circ h)|_W^V.$$

This factorization shows that h is $\mathcal{C}^r$ at the point z, since $f \circ h$ is $\mathcal{C}^r$ at z and ρ is $\mathcal{C}^r$ everywhere.

(b) Now suppose that f is a $\mathcal{C}^r$ embedding and that $h : L \to M$ is any function such that the composite $f \circ h$ is $\mathcal{C}^r$. Since f is a topological embedding, the function h is continuous by Proposition 3.3.3. Since f is an immersion, the map h is $\mathcal{C}^r$ by part (a). □

3.3.3 Consequences of the initial property

- If two embeddings $f_0 : M_0 \to N$, $f_1 : M_1 \to M$ have the same image, then they are equivalent in the sense that there is a diffeomorphism $g : M_0 \to M_1$ such that $f_1 \circ g = f_0$. (Exercise.)
- If S is an embedded submanifold of M, then there is a unique topology and smooth structure on S such that the inclusion map $S \to M$ is an embedding. (Exercise.)

This allows us to define the tangent space of an embedded submanifold.

Definition 3.3.5. The **tangent space** of an embedded submanifold S of a differentiable manifold M is the subspace $T_pS \subseteq T_pM$ defined as the image $T_pS = \mathrm{Img}(T_{f^{-1}(p)}f)$ of the differential of any embedding $f : N \to M$ whose image is S.

This subspace is well defined (i.e. independent of f). This follows from the fact that all the embeddings with image S are equivalent. (Exercise.)

3.4 How to recognize an embedded submanifold

Sometimes we are just given a subset S of a manifold M, and we have to determine whether it is an embedded submanifold or not. There are several ways to do so.

Proposition 3.4.1. *Let S be a subset of a $\mathcal{C}^r$ manifold M, and let $k \leq n = \dim M$. The following are equivalent:*

(a) S is a $\mathcal{C}^r$-embedded k-submanifold of M.

(b) Local embedded submanifold: Each point of S has an open neighborhood W in M such that the set $S \cap W$ is a $\mathcal{C}^r$-embedded k-submanifold of W.

(c) *Local slice: Each point of S is contained in the domain W of a chart ψ of M that is k-sliced by S.*

(d) *Local retract: Each point of S has an open neighborhood W in M such that the set $S \cap W$ is the image of a $\mathcal{C}^r$ map $\phi : U \to W$ (where $U \subseteq \mathbb{R}^k$ is an open set), which in turn admits a $\mathcal{C}^r$ retraction $\rho : W \to U$.*

Proof. (a) $\Rightarrow$ (b) Trivial: take $W = M$.
(a) $\Rightarrow$ (c) This is Proposition 3.3.2.
(b) $\Rightarrow$ (c) Same proof as (a) $\Rightarrow$ (c).
(c) $\Rightarrow$ (d) Let p be a point of S, and let $\psi : W \to \widetilde{W}$ be a chart of M, defined at p, that is k-sliced by S. By shrinking the sets W and $\widetilde{W}$, we may assume that $\widetilde{W} = \widetilde{U} \times \widetilde{V}$, for some open sets $\widetilde{U} \subseteq \mathbb{R}^k$ and $\widetilde{V} \subseteq \mathbb{R}^{n-k}$. The slice condition implies that

$$S \cap W = \psi^{-1}(\widetilde{U} \times \{0\}).$$

Consider the projection map $\pi : \widetilde{U} \times \widetilde{V} \to \widetilde{U} : (x, y) \mapsto x$ and its section $\sigma : \widetilde{U} \to \widetilde{U} \times \widetilde{V} : x \mapsto (x, 0)$. Note that $\pi \circ \sigma = \mathrm{id}_{\widetilde{U}}$.

Now we use the diffeomorphism $\psi : W \to \widetilde{W}$ to transport the maps π and ρ from the Euclidean set $\widetilde{W} \subseteq \mathbb{R}^n$ to the manifold subset $W \subseteq M$. That is, we define the $\mathcal{C}^r$ maps $\phi = \psi^{-1} \circ \sigma : \widetilde{U} \to W$ and $\rho = \pi \circ \psi : W \to \widetilde{U}$. Clearly ρ is a retraction of ϕ, that is, we have $\rho \circ \phi = \mathrm{id}_U$. The image of ϕ is the set

$$\mathrm{Img}(\phi) = \phi^{-1}(\mathrm{Img}(\sigma)) = \phi^{-1}(\widetilde{U} \times \{0\}) = S \cap W.$$

(d) $\Rightarrow$ (a) Suppose S is covered by open sets W_i such that each set $V_i := S \cap W_i$ is the image of a $\mathcal{C}^r$ maps $\phi_i : U_i \to W_i$ admitting a retraction $\rho_i : W_i \to U_i$.

Let us show that S admits a structure of k-dimensional $\mathcal{C}^r$ manifold such that the inclusion map $\iota_S : S \to M$ is a $\mathcal{C}^r$ embedding. The only topology we may put on S (so that ι_S a topological embedding) is the subspace topology. This topology is Hausdorff and second countable.

Let us construct a $\mathcal{C}^r$ atlas on S. The maps ϕ_i and ρ_i of course satisfy $\rho_i \circ \phi_i = \mathrm{id}_{U_i}$. Composing in the other order we get a map $\phi_i \circ \rho_i$ that fixes the image of ϕ_i, which is the set V_i. Therefore the continuous maps

$$\overline{\phi}_i := \phi_i|^{V_i} : U_i \to V_i, \quad \overline{\rho}_i := \rho_i|_{V_i} : V_i \to U_i$$

are inverse of each other. The maps $\overline{\rho}_i$ form a $\mathcal{C}^r$ atlas of S since the transition maps

$$\overline{\rho}_i \circ \overline{\rho}_j^{-1} = \overline{\rho}_i \circ \overline{\phi}_j = \rho_i \circ \phi_j$$

are $\mathcal{C}^r$.

The inclusion $\iota_S : S \to M$ is an immersion because composing with each local parametrization $\overline{\phi}_i$ we obtain a map $\iota_S \circ \overline{\phi}_i = \phi_i$ that is a $\mathcal{C}^k$ immersion, since it admits a $\mathcal{C}^k$ retraction. $\square$

Example 3.4.2. A subset $S \subseteq \mathbb{R}^n$ that can be locally expressed as a graph of a $\mathcal{C}^r$ function (with some $n - k$ coordinates expressed as a function of the other k coordinates) is a k-submanifold of $\mathbb{R}^n$ because it satisfies condition (d), since graphing maps admit projections as retractions. This is essentially what we used to construct the manifold structure on the sphere $\mathbb{S}^{n-1}$.

3.5 How to produce a submanifold as a level set

Definition 3.5.1. Let $f : M \to N$ be a $\mathcal{C}^k$-differentiable map. A point $q \in N$ is called a **regular value** of f if $\mathrm{T}_p f$ is surjective for all points $p \in f^{-1}(q)$.

Theorem 3.5.2 (Regular preimage theorem). *Let c be a regular value of a $\mathcal{C}^r$-differentiable map $f : M \to N$, and let $m = \dim M$, $n = \dim N$. Then the set $S = f^{-1}(c)$ is a $\mathcal{C}^r$-embedded submanifold of M of dimension $k = m - n$. Its tangent space at any point $p \in S$ is $\mathrm{T}_p S = \mathrm{Ker}(\mathrm{T}_p f)$.*

Proof. We will use the constant rank theorem to show that S fulfills the "local slice" condition of Proposition 3.4.1, therefore S is a submanifold.

At a point $p_0 \in S$ the map f is submersive, therefore by the CRT it admits at the point p_0 a local expression $\widetilde{f} = f|_\phi^\psi$ of the form $\widetilde{f}(x) = (x^k, \dots, x^{m-1})$, where $\phi : U \to \widetilde{U}$, $\psi : V \to \widetilde{V}$ are charts of M, N defined and centered at the points p_0, $f(p_0) = c$ respectively. In addition, we may assume that $\widetilde{U} = \widetilde{V} \times \widetilde{U}'$ where $U' \subseteq \mathbb{R}^{m-n}$ is an open set.

Note that $\psi(c) = 0$. Thus for a point $p \in U$, denoting $x = \phi(p) \in \mathbb{R}^m$, we have $p \in S$ iff $f(p) = c$ iff $\widetilde{f}(x) = 0$ iff the last n coordinates x^k, ..., x^{m-1} vanish. In other words, the chart ϕ is k-sliced by $S = f^{-1}(c)$. Since we can find such a chart ϕ for any point $p_0 \in S$, we conclude that S is a $\mathcal{C}^r$-embedded k-submanifold of M.

Tangent space: exercise. □

3.6 Whitney's theorem*

Theorem 3.6.1 (Whitney). *Every compact $\mathcal{C}^k$ manifold M can be $\mathcal{C}^k$ embedded in $\mathbb{R}^m$ for some m.*

Proof. Let $n = \dim M$, and denote $B(r) = \{x \in \mathbb{R}^n : \|x\| < r\}$ for $r > 0$. Since M is compact, there is a finite family $(\phi_i)_{i \in N}$ of charts $\phi_i : U_i \to B(2)$ such that the open sets $B_i := \phi_i^{-1} B(1)$ cover M.

For each i, take a $\mathcal{C}^k$ bump function $\eta_i : M \to [0, 1]$ with support $\operatorname{supp} \eta_i \subseteq U_i$ such that $\eta_i \equiv 1$ on B_i, and define a function $f_i : M \to \mathbb{R}^n$ by

$$f_i(x) = \begin{cases} \eta_i(x)\, \phi_i(x) & \text{if } x \in U_i \\ 0 & \text{if } x \notin \operatorname{supp}(\eta_i) \end{cases}$$

Finally, define a $\mathcal{C}^k$ map $f : M \to \mathbb{R}^{Nn+N}$ by

$$f(x) = ((f_i(x))_{i \in N}, (\eta_i(x))_{i \in N}).$$

We claim that f is an embedding.

f is an immersion because each f_i is immersive on B_i, since it coincides with the chart ϕ_i.

f is injective: suppose that $f(x) = f(y)$. Take some i such that $x \in B_i$. Then $\eta_i(x) = 1$, and therefore $\eta_i(y) = 1$. But then $x, y \in U_i$ and we have $\phi_i(x) = f_i(x) = f_i(y) = \phi_i(y)$, which implies that $x = y$.

Since f is an injective map with compact domain (and Hausdorff codomain), it is a topological embedding. □

4 Vector bundles and vector fields

So far we have looked at each of the vector spaces T_pM, where $p \in M$, separately. Their union $\mathrm{T}M := \bigcup_{p\in M} \mathrm{T}_pM$ is called the **tangent bundle** of M. It has a special structure that we will encounter several times and so it is worthwhile introducing the following definition:

Definition 4.0.1. Let M be a $\mathcal{C}^r$ manifold. A $\mathcal{C}^r$ **vector bundle** of rank k over M is a $\mathcal{C}^r$ manifold E together with a $\mathcal{C}^r$ map $\pi = \pi_E : E \to M$ (the **projection map**) satisfying

(a) For each $p \in M$, the set $\pi^{-1}(p) =: E_p$ (called the **fibre** of E over p) is a k-dimensional vector space.

(b) Each point of M is contained in an open set $U \subseteq M$ such that there is a $\mathcal{C}^r$ diffeomorphism
$$\Phi : U \times \mathbb{R}^k \to \pi^{-1}(U)$$
(called a **local trivialization** of E over U) such that for each $p \in U$, Φ maps $\{p\} \times \mathbb{R}^k$ to E_p , and the restriction $\Phi_p : \{p\} \times \mathbb{R}^k \equiv \mathbb{R}^k \to E_p$ is a vector space isomorphism.

We call E the **total space** of the bundle and M the **base space**.

Note: When talking about a vector bundle (E, π), sometimes we omit mentioning the projection map π and we just say that E is a vector bundle.

If (E, π) and (E', π') are smooth vector bundles over the same smooth manifold M, a **vector bundle map** (or **vector bundle morphism**) $(E, \pi) \to (E', \pi')$ is a smooth map $f : E \to E'$ such that $\pi' \circ f = \pi$ and with the property that for each $p \in M$ the restricted map $f|_{E_p} : E_p \to E'_p$ is linear. A **vector bundle isomorphism** is a bundle map that has an inverse.

The product $M \times \mathbb{R}^k$ with the standard projection $\pi : M \times \mathbb{R}^k \to M$ is called **the trivial vector bundle** of rank k over M. (This is indeed a vector bundle since one can take a single local trivialization: the identity map.) A vector bundle of rank k over M is called **trivial** if it is isomorphic to *the* trivial bundle $M \times \mathbb{R}^k$.

4.1 The tangent bundle

By this point you are certainly suspecting that the tangent bundle $\mathrm{T}M$ of a differentiable manifold is a vector bundle. First, let us show that it is a differentiable manifold.

Lemma 4.1.1. *Let M be a $\mathcal{C}^{k+1}$ manifold. Then the tangent bundle $\mathrm{T}M$ has a natural topology and $\mathcal{C}^k$ structure making it a $2n$-dimensional manifold. The natural projection $\pi : \mathrm{T}M \to M$, which sends $X \in \mathrm{T}_pM$ to p, is a $\mathcal{C}^k$ map.*

Proof. We define the maps that will become our charts as follows: Given a local parametrization $\varphi : \widetilde{U} \subseteq \mathbb{R}^n \to U \subseteq M$, let

$$\begin{array}{rccl} \Phi : & \widetilde{U} \times \mathbb{R}^n & \to & \pi^{-1}(U) \\ & (x, v) & \mapsto & [\varphi(x), \varphi^{-1}, v], \end{array}$$

or equivalently, in terms of the coordinate vectors of the chart $\psi = \varphi^{-1}$, $\Phi(x,v) = \sum_i \frac{\partial}{\partial\psi^i}\Big|_{\varphi(x)}$. This function Φ is a bijection onto its image $\pi^{-1}(U)$. We define on $\mathrm{T}M$ a topology as follows:

$$\text{A set } A \subseteq \mathrm{T}M \text{ is open } \iff \text{ for each } \varphi, \text{ the set } \Phi^{-1}(A) \subseteq \widetilde{U} \times \mathbb{R}^n \text{ is open}$$

You can check that this is indeed a topology that is Hausdorff and second countable. (Use Proposition 1.2.8.) Each map Φ is a homeomorphism from an open subset of $\mathbb{R}^n$ onto its image, which is an open subset of M. Hence $\mathrm{T}M$ is a topological manifold of dimension $2n$.

We show that the collection of maps Φ obtained as above from local parametrizations φ of M is a parametrization atlas for $\mathrm{T}M$. Let $\varphi : \widetilde{U} \to U$, $\psi : \widetilde{V} \to V$ be smooth local parametrizations of M and let Φ, Ψ be the corresponding parametrizations of $\mathrm{T}M$. Then the transition map is

$$\begin{array}{rccc} \Psi^{-1} \circ \Phi : & \varphi^{-1}(U \cap V) \times \mathbb{R}^n & \to & \psi^{-1}(U \cap V) \times \mathbb{R}^n \\ & (x, v) & \mapsto & (\psi^{-1} \circ \varphi(x),\, \mathrm{D}_x(\psi^{-1} \circ \varphi)(v)) \end{array}$$

which is smooth. This formula comes from the fact that if two tangent vectors $\Phi(x,v) = [\varphi(x), \varphi^{-1}, v]$ and $\Psi(y,w) = [\psi(y), \psi^{-1}, w]$ are equal, then $y = \psi^{-1}\varphi(x)$ and $w = \mathrm{D}_x(\psi^{-1} \circ \varphi)(v)$.

To check that the projection map π is smooth, just note that its coordinate representation with respect to the local parametrizations φ and Φ is $\pi|^{\varphi}_{\Phi}(x,v) = x$. □

The smooth structure defined in the previous proof is called the **standard smooth structure** for $\mathrm{T}M$. (Similarly, if M is a $\mathcal{C}^{r+1}$ manifold, then $\mathrm{T}M$ is naturally a $\mathcal{C}^r$ manifold.)

Proposition 4.1.2. *The tangent bundle* $\mathrm{T}M$ *with the standard smooth structure and the standard projection* $\pi : \mathrm{T}M \to M$ *is a smooth vector bundle of rank* n *over* M.

Proof. Each fibre $\mathrm{T}_pM = \pi^{-1}(\{p\})$, $p \in M$, is a vector space. Given a smooth local parametrization $\varphi : \widetilde{U} \subseteq \mathbb{R}^n \to U \subseteq M$, the map

$$\begin{array}{ccc} U \times \mathbb{R}^n & \to & \pi^{-1}(U) \\ (p = \varphi(x), v) & \mapsto & [p, \varphi^{-1}, v] \end{array}$$

is a local trivialization of $\mathrm{T}M$ over U. □

4.2 Sections, frames, and trivial bundles

Definition 4.2.1. Let (E, π) be a $\mathcal{C}^k$ vector bundle over a manifold M. A **section** of a E is a continuous map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$. It is $\mathcal{C}^k$ if it is $\mathcal{C}^k$ as a map between manifolds.

If $U \subset M$ is open it is easy to verify that $E|_U := \pi^{-1}(U)$ is again a vector bundle with the restriction of π as its projection map. A $\mathcal{C}^k$ section of $E|_U$ is called a $\mathcal{C}^k$ **section of** E **over** U.

For example, if (U, ϕ) is a chart of a manifold M, then the **coordinate vector fields** $p \in U \mapsto \frac{\partial}{\partial\phi^i}\Big|_p \in \mathrm{T}_pM$ are sections of the tangent bundle TM over U. (The sections of TM are called *vector fields*; we will talk about them in next section.)

Definition 4.2.2. If $U \subset M$ is open, a **local frame** for E over U is an ordered k-tuple $(\sigma_0, \ldots, \sigma_{k-1})$ where each σ_i is a smooth section of E over U such that $(\sigma_0(p), \ldots, \sigma_{k-1}(p))$ is a base for the fibre E_p for each $p \in U$. It is called a **global frame** if $U = M$.

Remark 4.2.3. Note that if E is a rank 1 vector bundle over M, a frame consists of a single **nonvanishing** section, i.e. a section $\sigma : M \to E$ such that $\sigma_p \neq 0$ for all $p \in M$. This is because a basis of a 1-dimensional vector space is the same sing as a set containing a single nonzero vector.

For the vector bundles we encounter in this lecture we usually write the value of a section σ at a point p as $\sigma_p := \sigma(p) \in E_p$. Addition of two sections σ, τ of E over an open subset $U \subset M$ is defined pointwise: $(\sigma + \tau)_p := \sigma_p + \tau_p$. We can also multiply a section over U by a function $f \in C^\infty(U)$ by defining the operation pointwise: $(f\sigma)_p := f(p)\sigma_p$. For those who know what it means: this makes the set of smooth sections of E over U a module over the ring $C^\infty(U)$.

Proposition 4.2.4. *A smooth vector bundle is trivial iff it has a global frame.*

Proof. Exercise. □

E.g. the tangent bundle $\mathrm{T}\mathbb{R}^n$ is trivial, since it has a global frame consisting of the coordinate vector fields $p \mapsto \frac{\partial}{\partial x^i}\big|_p$.

Let us give an example of a non-trivial vector bundle.

Example 4.2.5. (Möbius band) The Möbius band E is the quotient space of $\mathbb{R}^2$ under the equivalence relation

$$(x,t) \sim (x',t') \iff x' - x = n \in \mathbb{Z} \text{ and } t' = (-1)^n t.$$

We endow E with the quotient topology and one can check that E is Hausdorff and second countable. We write $[x,t]$ for the equivalence class of $(x,t) \in \mathbb{R}^2$ in E.

We want to make E a vector bundle over the 1-torus $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. The projection map will be $\pi : E \to \mathbb{T}^1 : [x,t] \mapsto [x]$. Note that each fibre $E_t = \pi^{-1}[x] = \{[x,t] : t \in \mathbb{R}\}$ is a vector space with the operations $a[x,t] + a'[x,t'] = [x, at + a't']$.

Recall the smooth structure of the torus $\mathbb{T}^1$. The quotient map $\kappa : \mathbb{R} \to \mathbb{T}^1 : x \mapsto [x]$ is open, and it is injective when restricted e.g. to any open interval $W \subseteq \mathbb{R}$ of length ≤ 1. Consider a family of such intervals W_i whose images $U_i = \kappa(W_i)$ cover $\mathbb{T}^1$ (for example, $W_1 = (0,1)$ and $W_0 = (-\frac{1}{2}, \frac{1}{2})$). Then the maps $\kappa|_{W_i} : W_i \to U_i$ form a smooth inverse atlas of $\mathbb{T}^1$ (i.e. its inverse functions $W_i \to U_i$ form a smooth atlas).

Let us give E a smooth structure. We cover E with open sets

$$V_i = \pi^{-1}(U_i) = \{[x,t] : x \in W_i, t \in \mathbb{R}\}$$

and we define homeomorphisms

$$\psi_i : W_i \times \mathbb{R} \to V_i : (x,t) \mapsto [x,t]$$

These maps form a smooth inverse atlas of E since the transition maps are smooth (check it!). Therefore E is a smooth manifold. In addition, (E, π) is a smooth vector bundle of rank 1 over $\mathbb{T}^1$ since it admits the local trivializations

$$\widetilde{\psi}_i : U_i \times \mathbb{R} \to V_i : ([x], t) \mapsto [x,t].$$

Proposition 4.2.6. *The Möbius bundle (E, π) is not trivial.*

Proof. Indeed, suppose that E is trivial. Then there is a global frame which consists of a single nowhere vanishing section $\sigma : \mathbb{T}^1 \to E$.

Note that a smooth section σ is of the form $\sigma([x]) = [x, f(x)]$ for some function $f : \mathbb{R} \to \mathbb{R}$, and we claim that f is smooth. To see that f is smooth on each interval W_i, we express it as a composite of smooth functions

$$\begin{array}{ccccccccccc} f|_{W_i} : & W_i & \xrightarrow{\kappa} & U_i & \xrightarrow{\sigma} & V_i & \xrightarrow{\psi_i^{-1}} & W_i \times \mathbb{R} & \xrightarrow{\pi_1} & \mathbb{R} \\ & x & \mapsto & [x] & \mapsto & [x,t] & \mapsto & (x,t) & \mapsto & t. \end{array}$$

Also note that f is "antiperiodic" with period 1, i.e., it satisfies $f(x+1) = -f(x)$ because

$$[x, f(x)] = \sigma[x] = \sigma[x+1] = [x+1, f(x+1)] = [x, -f(x+1)].$$

Since f is continuous and antiperiodic, by the intermediate value theorem f vanishes at some point between x and $x+1$. On the other hand, f does not vanish if σ is a nonvanishing section. Therefore there is no nonvanishing section σ.

□

4.3 Vector fields

Definition 4.3.1. A **tangent vector field** (or a **vector field**, for short) on a $\mathcal{C}^{k+1}$ manifold M is a function $X : M \to \mathrm{T}M$ such that

$$X_p \in \mathrm{T}_p M \quad \text{for each } p \in M$$

where $X_p := X(p)$.

Its **component functions** with respect to a chart (U, ϕ) are the functions $X^i : U \to \mathbb{R}$ such that $X \equiv \sum_i X^i \frac{\partial}{\partial \phi^i}$ on U.

We say that X is $\mathcal{C}^k$ at a point $p \in M$ if its component functions (w.r.t. any chart ϕ defined at p) are $\mathcal{C}^k$ at the point p. (Exercise: it suffices to check just one chart.) We say that X is $\mathcal{C}^k$ if it is $\mathcal{C}^k$ at all points. The set of $\mathcal{C}^k$ tangent vector fields on M is denoted $\mathfrak{X}(M)$.

Equivalently, in terms of vector bundles, a tangent vector field is a section of the tangent bundle $\pi : \mathrm{T}M \to M$, i.e. a function $X : M \to \mathrm{T}M$ such that $\pi \circ X = \mathrm{id}_M$. It is $\mathcal{C}^k$ if it is $\mathcal{C}^k$ as a function $M \to \mathrm{T}M$. (Exercise.)

Addition of two vector fields and multiplication by functions on M is defined in the natural way, i.e. pointwise: $(X+Y)_p = X_p + Y_p$, $(fX)_p = f(p)X_p$ where $f \in C^\infty(M)$.

Remark 4.3.2. A vector field over an open subset $U \subset M$ is a section of $\mathrm{T}M$ over U. In particular, given (U, φ) a $\mathcal{C}^{k+1}$ chart, the so-called **coordinate vector fields**

$$\frac{\partial}{\partial \varphi^i} : U \to \mathrm{T}_p M : p \mapsto \left. \frac{\partial}{\partial \varphi^i} \right|_p$$

are $\mathcal{C}^k$ vector fields over U.

Example 4.3.3. Recall the polar coordinates (r, φ) defined on the open subset $U \subset \mathbb{R}^2$ (Example 2.0.5). The coordinate vector field $\frac{\partial}{\partial \varphi}$ is a smooth vector field on U. Writing the vector field in terms of the Euclidean coordinate vectors

$$\frac{\partial}{\partial \varphi} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

we see that $\frac{\partial}{\partial\varphi}$ extends to a smooth vector field on $\mathbb{R}^2$, which we denote by the same symbol $\frac{\partial}{\partial\varphi}$. So while the coordinate φ cannot be extended to the whole space $\mathbb{R}^2$ as a smooth function, the corresponding coordinate vector field can be extended.

Example 4.3.4. On $\mathbb{S}^1$ we can define a nowhere vanishing vector field (exercise). Therefore, the vector bundle $\mathrm{T}\mathbb{S}^1$ is trivial. The tangent bundle $\mathrm{T}\mathbb{S}^2$ is *not* trivial: the *hairy ball theorem* says that there is no nowhere vanishing vector field on $\mathbb{S}^2$.

4.4 Integral curves and flow of a vector field

Definition 4.4.1. An **integral curve** of a tangent vector field X on a manifold M is a $\mathcal{C}^1$ curve $\gamma : I \to M$ (where $I \subseteq \mathbb{R}$ is an open interval) such that

$$\gamma'(t) = X|_{\gamma(t)} \quad \text{for all } t \in I. \tag{4.1}$$

(Here we use the notation $\gamma'(t)$ for the velocity vector $\mathrm{Vel}_t\,\gamma$.)

Equation (4.1) is an ordinary differential equation (ODE) on the manifold M. In a local chart, it boils down to solving the system of ODEs:

$$(\gamma^i)'(t) = X^i(\gamma(t)), \quad i = 0, \dots, n-1$$

in $\mathbb{R}^n$. Similarly to ODE's in $\mathbb{R}^n$, for manifolds there is a theorem of existence and uniqueness of solutions.

Proposition 4.4.2 (Existence and uniqueness of solutions for an ODE on a manifold)**.** *If X is a $\mathcal{C}^1$ vector field, then for each point $p \in M$ there exists a unique maximal integral curve $\gamma_{X,p} : I_{X,p} \to M$ of X that satisfies the initial condition $\gamma_{X,p}(0) = p$. Any other integral curve of X satisfying the same initial condition is equal to $\gamma_{X,p}$, restricted to a subinterval of $I_{X,p}$.*

This proposition can be deduced from the corresponding proposition in $\mathbb{R}^n$. We will not study the proof.

Definition 4.4.3. The **flow** of a $\mathcal{C}^1$ vector field on a manifold M is the function

$$\begin{array}{rccc} \Phi_X = & \coprod_{p\in M} I_{X,p} & \to & M \\ & (p,t) & \mapsto & \gamma_{X,p}(t), \end{array}$$

where $\gamma_{X,p} : I_{X,p} \to M$ is the maximal integral curve of Proposition 4.4.2. We also write $\Phi_X^t(p) := \Phi_X(p,t)$.

Proposition 4.4.4 (Differentiability of flow for an ODE on a manifold)**.** *If X is a $\mathcal{C}^k$ vector field with $k \geq 1$, the flow Φ_X is a $\mathcal{C}^k$ map defined on an open subset of $M \times \mathbb{R}$.*

This theorem also follows from a corresponding theorem in $\mathbb{R}^n$, found e.g. in Hartman's ODE book, Thm. 4.1. Again, we skip the proof.

Definition 4.4.5. We say that a vector field X is **complete** if the flow Φ_X is defined over $M \times \mathbb{R}$, or equivalently, if every maximal integral curve of X is defined on $\mathbb{R}$.

Note that a vector field on a compact manifold (or, more generally, a vector field with compact support) is complete. (Exercise.)

Example 4.4.6. On Euclidean space $\mathbb{R}^n$, a constant vector field $X \equiv a$ has maximal integral curves $\gamma_{X,p}(t) = p + ta$, defined for all $t \in \mathbb{R}$. Hence X is complete.

Example 4.4.7. On the line $\mathbb{R}$ consider the vector field X given by $X_p = p^2 \left.\frac{\partial}{\partial\phi}\right|_p$, where $\phi = \mathrm{id}_{\mathbb{R}}$ is the identity map considered as a chart with just one component function $\phi^0 = \phi$. Let us find an integral curve γ with starting point $\gamma(0) = 1$. This means that we have to solve

$$\gamma'(t) = (\gamma(t))^2, \quad \gamma(0) = 1.$$

As is shown easily by separation of variables, the maximal solution is $\gamma(t) = \frac{1}{1-t}$, $t \in (-\infty, 1)$, i.e. it goes to infinity in a finite time and cannot be defined for all $t \in \mathbb{R}$. Therefore X is not complete.

The notion of *related vector fields* is useful for transporting solutions of ODE's from one manifold to another.

Definition 4.4.8. Let $f : M \to N$ be a smooth map. A vector field $X \in \mathfrak{X}(M)$ is f**-related** to a vector field $Y \in \mathfrak{X}(N)$ if $\mathrm{T}_p f(X_p) = Y_{f(p)}$ for all $p \in M$.

The following facts follow from the definition (exercise):

(a) X is f-related to Y if and only if $X_p(f \circ h) = Y_{f(p)}(h)$ for all functions $h \in \mathcal{C}^\infty(N, \mathbb{R})$ and all points $p \in M$.

(b) If X is f-related to Y and γ is an integral curve of X, then $f \circ \gamma$ is an integral curve of Y.

(c) If f is a local diffeo, for every vector field $Y \in \mathfrak{X}(N)$ there exists a unique $X \in \mathfrak{X}(M)$ that is f-related to Y. This field X determined by Y is denoted f^*Y. Thus if f is a diffeo, then f-relatedness is a bijection from $\mathfrak{X}(M)$ to $\mathfrak{X}(N)$. In this case, if X is f-related to Y, we write $X = f^*Y$ and $Y = f_*X$.

Finally, a nice property of any vector field X is that its aspect near any point p where $X_p \neq 0$ is quite standard:

Exercise 4.4.9. If X is a smooth vector field on a manifold M and $p \in M$ is a point where $X_p \neq 0$, then there exists a chart (U, ϕ) of M defined at p such that $X|_U = \frac{\partial}{\partial\phi^0}$.

5 Covector fields, or 1-forms

5.1 Vectors and covectors; superindices and subindices

Let us recall some facts from linear algebra and explain how we use subindices and superindices in differential geometry. Consider a vector space $V \simeq \mathbb{R}^n$ and its dual space V^*. (Typically, V is the tangent space T_pM of a manifold M.)

Dual bases A sequence of vectors $\mathcal{B} = (E_i)_i \subseteq V$ and a sequence of covectors $\mathcal{B}' = (\varepsilon^j)_j \subseteq V^*$ are called **dual** to each other if

$$\varepsilon^j(E_i) = \delta_i^j \quad \text{for all } i, j. \tag{5.1}$$

Proposition 5.1.1 (Dual bases)**.** *If a sequence of vectors $\mathcal{B} = (E_i)_i \subseteq V$ and a sequence of covectors $\mathcal{B}' = (\varepsilon^j)_j \subseteq V^*$ are dual to each other, then they are bases of V and V^* respectively. Every base $\mathcal{B}$ of V has a unique dual base, denoted $\mathcal{B}^*$, and every base $\mathcal{B}^*$ of V^* has a unique predual base $\mathcal{B}$.*

Coordinates of a vector or covector Let $\mathcal{B} = (E_i)_i$ and $\mathcal{B}^* = (\varepsilon^j)_j$ be dual bases of V and V^* as above. For any vector $X \in V$ there exists a unique sequence of numbers $(X^i)_i$ (called the **coordinates** of X w.r.t. the base $\mathcal{B}$) such that $X = \sum_i X^i E_i$. Similarly, for any covector $\xi \in V^*$ there exists a unique sequence of numbers $(\xi_j)_j$ (called the **coordinates** of ξ w.r.t. the base $\mathcal{B}^*$) such that $\xi = \sum_j \xi_j \varepsilon^j$. Note that

$$\xi(X) = \sum_k \xi_k X^k = \begin{pmatrix} \xi_0 \cdots \xi_{n-1} \end{pmatrix} \begin{pmatrix} X^0 \\ \vdots \\ X^{n-1} \end{pmatrix}$$

because

$$\xi(X) = \left(\sum_j \xi_j \, \varepsilon^j \right) \left(\sum_i X^i \, E_i \right) = \sum_{i,j} \xi_j \, X^i \, \varepsilon^j(E_i) = \sum_{i,j} \xi_j \, X^i \, \delta_i^j.$$

Note that we always sum over an index that appears exactly twice: once as a subindex, once as a superindex.

Proposition 5.1.2 (Coefficients of vectors and covectors)**.** *The coordinates of a vector $X = \sum_i X^i E_i \in V$ can be obtained by applying to X the covectors of the dual base $\mathcal{B}^* = (\xi^j)_j$, that is,*

$$X^i = \varepsilon^i(X).$$

Similarly, the coordinates of a covector $\xi = \sum_j \xi_j \, \varepsilon^j \in V^*$ are obtained by evaluating ξ on the base vectors, that is,

$$\xi_j = \xi(E_j).$$

Exercise 5.1.3. Draw a picture of $V = \mathbb{R}^2$ with the vector $X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and represent also the covector $\xi = \begin{pmatrix} 1 & 2 \end{pmatrix}$ by drawing the lines $\xi^{-1}(c)$ for $c = -1, 0, 1, 2, 3, 4$.

In general, a good way to represent graphically a nonzero covector $\xi \in V^*$ is by drawing the hyperplane $\xi^{-1}(1)$.

5.2 Cotangent vectors

Definition 5.2.1. The **cotangent space** T_p^*M of a differentiable manifold M at a point $p \in M$ is the dual of the tangent space, that is,

$$\mathrm{T}_p^*M = (\mathrm{T}_pM)^*$$

i.e. its elements (called **cotangent vectors**, or **covectors** of M at p) are the linear functions $\mathrm{T}_pM \to \mathbb{R}$.

We can obtain a covector by differentiating a function at a point.

Definition 5.2.2. The **differential** of a function $h : M \to \mathbb{R}$ at a point $p \in M$ where f is $\mathcal{C}^1$ is the covector $\mathrm{d}h|_p \in \mathrm{T}_p^*M$ given by

$$\mathrm{d}h|_p(X) = X(h) \qquad \text{for all vectors } X \in \mathrm{T}_pM.$$

Example 5.2.3 (Differential of chart components)**.** If φ is a chart of M defined at $p \in M$, the differential of a chart component φ^i applied to a coordinate vector $\frac{\partial}{\partial \varphi^j}$ gives

$$\mathrm{d}\varphi^i|_p \left(\left. \frac{\partial}{\partial \varphi^j} \right|_p \right) = \left. \frac{\partial}{\partial \varphi^j} \right|_p \varphi^i = \delta^i_j. \tag{5.2}$$

This shows that the covectors $\mathrm{d}\varphi^i|_p$ form a basis of T_pM^* that is dual to the basis of T_pM consisting of the coordinate vectors $\left.\frac{\partial}{\partial \varphi^j}\right|_p$. Therefore we can apply the formulas of Section 5.1.

Definition 5.2.4. The **coefficients** or **components** of a covector $\xi \in \mathrm{T}_p^*M$ with respect to the chart φ are the real numbers ξ_i such that

$$\xi = \sum_i \xi_i \, \mathrm{d}\varphi^i|_p.$$

These coefficients can be obtained by evaluating ξ on the coordinate vectors:

$$\xi_j = \xi \left(\left. \frac{\partial}{\partial \varphi^j} \right|_p \right); \tag{5.3}$$

this formula follows from the duality relation (5.2) by Proposition 5.1.2.

Proposition 5.2.5 (Transformation law for covector coefficients)**.** *Let* $(\xi^i)_i, (\widetilde{\xi}^j)_j \in \mathbb{R}^n$ *be the tuples coefficients of a covector* $\xi \in \mathrm{T}_p^*M$ *with respect to two charts* φ, $\widetilde{\varphi}$. *These tuples are related by the formula*

$$\widetilde{\xi}_j = \sum_i \left. \frac{\partial \varphi^i}{\partial \widetilde{\varphi}^j} \right|_p \xi_i. \tag{5.4}$$

Proof. By Proposition 5.1.2, we have

$$\begin{aligned}
\widetilde{\xi}_j = \xi \left(\left. \frac{\partial}{\partial \widetilde{\varphi}^j} \right|_p \right) &= \xi \left(\sum_i \left. \frac{\partial \varphi^i}{\partial \widetilde{\varphi}^j} \right|_p \left. \frac{\partial}{\partial \varphi^i} \right|_p \right) \\
&= \sum_i \left. \frac{\partial \varphi^i}{\partial \widetilde{\varphi}^j} \right|_p \xi \left(\left. \frac{\partial}{\partial \varphi^i} \right|_p \right) \\
&= \sum_i \left. \frac{\partial \varphi^i}{\partial \widetilde{\varphi}^j} \right|_p \xi_i.
\end{aligned}$$

□

5.3 Covector fields

Definition 5.3.1. The **cotangent bundle** T^*M of a differentiable manifold M is the disjoint union of the cotangent spaces: $T^*M = \coprod_{p\in M} T^*_pM$.

This set has a natural structure of vector bundle over M...

Definition 5.3.2. A **covector field** or 1-**form** on a $\mathcal{C}^{k+1}$ manifold M is a function $\xi : M \to T^*M$ such that $\xi_p \in T^*_pM$ for all $p \in M$. Its **component functions** with respect to a chart (U,φ) are the functions $\xi_j : U \to \mathbb{R}$ such that

$$\xi_p = \sum_j \xi_j(p)\, \mathrm{d}\varphi^j|_p \qquad \text{for all } p \in U.$$

Note that $\xi_j(p) = \xi|_p \left(\frac{\partial}{\partial \varphi^j}\Big|_p \right)$ for all $p \in U$.

A covector field ξ is $\mathcal{C}^k$ at a point $p \in M$ if its component functions (w.r.t. any chart that is defined at p)[1] are $\mathcal{C}^k$ at the point p, and ξ is $\mathcal{C}^k$ on M if it is $\mathcal{C}^k$ at all points of M.

We denote $\Omega^1(M)$ the set of $\mathcal{C}^k$ covector fields on a $\mathcal{C}^{k+1}$ manifold M. Note that $\Omega^1(M)$ is a module over the ring $\mathcal{C}^k(M)$, i.e. we can obtain a new covector field by adding two covector fields, or by multiplying a covector field $\xi \in \Omega^1(M)$ by a function $h \in \mathcal{C}^k(M)$.

5.4 Contraction of a 1-form with a vector field

The **contraction** of a 1-form $\xi \in \Omega^1(M)$ with a vector field $X \in \mathfrak{X}(M)$ is the function $\langle \xi, X\rangle : M \to \mathbb{R}$, also denoted $\iota_X \xi$, defined by

$$\langle \xi, X\rangle(p) = \xi|_p(X|_p) \qquad \text{for all } p \in M.$$

Thus the components of ξ w.r.t. a chart ϕ are $\xi_i = \langle \xi, \frac{\partial}{\partial \phi^i}\rangle$.

5.5 Differential of a function

An example of covector field is the differential of a function.

Definition 5.5.1. The **differential** of a function $h \in \mathcal{C}^{k+1}(M)$ is the 1-form $\mathrm{d}h \in \Omega^1(M)$ defined by

$$\mathrm{d}h|_p(X) = X(h) \qquad \text{for all points } p \in M \text{ and all vectors } X \in T_pM.$$

A 1-form is **exact** if it is the differential of some function.

On a chart (U,ϕ) we can express $\mathrm{d}h$ as

$$\mathrm{d}h|_U = \sum_i \frac{\partial h}{\partial \phi^i}\, \mathrm{d}\phi^i$$

since the components of $\mathrm{d}h$ are $\mathrm{d}h\left(\frac{\partial}{\partial \phi^i}\right) = \frac{\partial}{\partial \phi^i} h$ by (5.3). This shows that $\mathrm{d}h$ is $\mathcal{C}^k$ if h is $\mathcal{C}^{k+1}$.

Example 5.5.2 (Differential of coordinate functions)**.** If (U,φ) is a chart on M, each coordinate differential $\mathrm{d}\varphi^i$ is a $\mathcal{C}^k$ covector field since its component functions $\langle \mathrm{d}\varphi^i, \frac{\partial}{\partial \varphi^j}\rangle \equiv \delta^i_j$ are constant.

[1] It doesn't matter which chart we use; this follows from the transformation law.

In the exercises we prove some basic properties of the differential:

Proposition 5.5.3. *(i) For $f, g \in C^\infty(M)$, $a, b \in \mathbb{R}$ we have:* $\mathrm{d}(a \cdot f + b \cdot g) = a \cdot \mathrm{d}f + b \cdot \mathrm{d}g$, $\mathrm{d}(f \cdot g) = f \cdot \mathrm{d}g + g \cdot \mathrm{d}f$, $\mathrm{d}(\frac{f}{g}) = \frac{g \cdot \mathrm{d}f - f \cdot \mathrm{d}g}{g^2}$ *(on the set where $g \neq 0$)*

(ii) Chain rule: If $h : \mathbb{R} \to \mathbb{R}$ is a smooth function then $\mathrm{d}(h \circ f) = (h' \circ f) \cdot \mathrm{d}f$.

(iii) If $\mathrm{d}f = 0$ then f is constant on each connected component of M.

5.6 Pullback of a 1-form

Definition 5.6.1. The **pullback** of a covector field $\xi \in \Omega^1(N)$ by a $\mathcal{C}^1$ map $f : M \to N$ is the covector field $f^*\xi \in \Omega^1(M)$ defined by

$$(f^*\xi)|_p = (\mathrm{T}_p f)^* \xi|_{f(p)} \qquad \text{for all } p \in M,$$

that is,

$$(f^*\xi)|_p(X) = \xi|_{f(p)}(\mathrm{T}_p f X) \qquad \text{for all } X \in \mathrm{T}_p M.$$

Proposition 5.6.2. *Pullback commutes with differentiation. That is, for any $\mathcal{C}^1$ map $f : M \to N$ and any function $h \in \mathcal{C}^1(N, \mathbb{R})$, we have $f^*(\mathrm{d}h) = \mathrm{d}(f^*h)$ where $f^*h = h \circ f$.*

Also, pullback commutes with the module operations on Ω^1, i.e. we have

$$f^*(\theta + \xi) = f^*\theta + f^*\xi \quad \textit{and} \quad f^*(h \cdot \xi) = (f^*h) \cdot f^*\xi.$$

for covector fields $\xi, \theta \in \Omega^1(N)$ and a function $h \in \mathcal{C}^1(N, \mathbb{R})$.

Proof. To show that the 1-forms $f^*(\mathrm{d}h)$ and $\mathrm{d}(h \circ f)$ are equal, we apply them to a tangent vector $X \in \mathrm{T}_p M$:

$$f^*(\mathrm{d}h)(X) = \mathrm{d}h(\mathrm{T}_p f X) = (\mathrm{T}_p f X)h = X(h \circ f) = \mathrm{d}(h \circ f)(X).$$

The rest: exercise. □

5.7 Integration of a 1-form along a curve

Definition 5.7.1. The **integral** (or **line integral**) of a continuous 1-form $\xi \in \Omega^1(M)$ along a $\mathcal{C}^1$ curve $\gamma : I = [a, b] \to M$ in a differentiable manifold M is defined as

$$\int_\gamma \xi := \int_a^b \xi_{\gamma(t)}(\gamma'(t))\, \mathrm{d}t.$$

Example 5.7.2. Let us compute the integral of the 1-form $\xi = -y\,\mathrm{d}x + x\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$ along the radius r circle $\gamma : t \in [0, 2\pi] \mapsto (r\cos t, r\sin t)$. Thus

$$\begin{aligned}
\int_\gamma \xi &= \int_0^{2\pi} \xi|_{\gamma(t)}\gamma'(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} (-r\sin t\,\mathrm{d}x + r\cos t\,\mathrm{d}y)\,(-r\sin t\frac{\partial}{\partial x} + r\cos t\frac{\partial}{\partial y})\,\mathrm{d}t \\
&= \int_0^{2\pi} r^2\big(\sin^2 t\,\underbrace{\mathrm{d}x(\frac{\partial}{\partial x})}_{=1} - \sin t\cos t\,\underbrace{\mathrm{d}x(\frac{\partial}{\partial y})}_{=0} \\
&\qquad\qquad - \cos t\sin t\,\underbrace{\mathrm{d}y(\frac{\partial}{\partial x})}_{=0} + \cos^2 t\,\underbrace{\mathrm{d}y(\frac{\partial}{\partial y})}_{=1}\big)\,\mathrm{d}t \\
&= \int_0^{2\pi} r^2\,\mathrm{d}t = 2\pi r^2.
\end{aligned}$$

Theorem 5.7.3 (1-dimensional Stokes)**.** *The integral along a $\mathcal{C}^1$ curve $\gamma : [a,b] \to M$ of the differential of a function $h \in \mathcal{C}^1(M, \mathbb{R})$ is*

$$\int_\gamma \mathrm{d}h = h(\gamma(b)) - h(\gamma(a)).$$

Proof.

$$\begin{aligned}\int_\gamma \mathrm{d}h &= \int_a^b \mathrm{d}h(\gamma'(t))\ \mathrm{d}t \\ &= \int_a^b (h \circ \gamma)'(t)\,\mathrm{d}t = h(\gamma(b)) - h(\gamma(a)).\end{aligned}$$

□

Corollary 5.7.4. *If a 1-form $\xi \in \Omega^1(M)$ is exact, i.e. $\xi = \mathrm{d}f$ for some $f \in C^{k+1}(M)$, then its line integral over any closed differentiable curve is zero.*

In the exercises we prove the following property of integrals of 1-forms.

Proposition 5.7.5 (Reparametrization invariance of curve integrals)**.** *If two $\mathcal{C}^1$ curves $\gamma : J \to M$, $\beta : I \to M$ are equivalent as oriented curves, in the sense that $\beta = \gamma \circ \tau$, for some positive (= increasing) diffeo $\tau : I \to J$, then*

$$\int_\beta \xi = \int_\gamma \xi \quad \text{for any 1-form } \xi \in \Omega^1(M).$$

Proof. The integrals we are comparing are

$$\int_\gamma \xi = \int_J g(t)\,\mathrm{d}t \quad \text{and} \quad \int_\beta \xi = \int_I f(s)\,\mathrm{d}s$$

where $g(t) = \xi_{\gamma(t)}(\gamma'(t))$ and $f(s) = \xi_{\beta(s)}(\beta'(s))$. Since $\beta = \gamma \circ \tau$, applying the chain rule we get

$$\begin{aligned}f(s) = \xi_{\beta(s)}(\beta'(s)) &= \xi_{\gamma(\tau(s))}((\gamma \circ \tau)'(s)) \\ &= \xi_{\gamma(\tau(s))}(\tau'(s) \cdot \gamma'(\tau(s))) \\ &= \tau'(s) \cdot \xi_{\gamma(\tau(s))}(\gamma'(\tau(s))) \\ &= \tau'(s) \cdot g(\tau(s)).\end{aligned}$$

Thus the two integrals are equal by the change of variables formula. □

5.8 Integration along curves via pullback

There is an equivalent definition of the integral $\int_\gamma \xi$ using the pullback $\gamma^*\xi$.

Note that any 1-form $\theta \in \Omega^1(J)$ defined on an open interval $J \subseteq \mathbb{R}$ can be written as $\theta = \theta_0\,\mathrm{d}\phi$ where $\theta_0 : J \to \mathbb{R}$ is a function. This function θ_0 is the single component function of θ w.r.t. the identity chart φ and can be computed as

$$\theta_0(t) = \theta|_t \left(\frac{\partial}{\partial \varphi}|_t \right).$$

Definition 5.8.1. The **integral** of a 1-form $\theta \in \Omega^1(I)$ along a compact interval $I = [a,b] \subseteq \mathbb{R}$ is defined as

$$\int_a^b \theta := \int_a^b \theta_0(t)\,\mathrm{d}t,$$

where $\theta_0 \in \mathcal{C}(I, \mathbb{R})$ is the component function of θ with respect to the standard coordinate $\varphi = \mathrm{id}_I$.

Proposition 5.8.2. *The integral of a continuous 1-form $\xi \in \Omega(M)$ over a $\mathcal{C}^1$ curve $\gamma : I = [a, b] \to M$ in a differentiable manifold M is*

$$\int_\gamma \xi := \int_I \gamma^* \xi.$$

Remark 5.8.3. The concept of 1-form on I is not yet defined because the interval I is not a manifold. (It is a *manifold-with-boundary*, but we have not yet defined this kind of objects.) For the moment we can just consider I as a subset of the manifold $\mathbb{R}$ (or of any open interval I^+ that contains I). A 1-form on I is the restriction of a 1-form defined on $\mathbb{R}$ (or on I^+). We can assume that γ can be extended to some open interval $I^+ \supseteq I$, then the 1-form $\gamma^*\xi$ can be extended to I^+, so it is a 1-form on I.

In the exercises we prove that this second definition of $\int_\gamma \xi$ is equivalent to the previous one, and use this to give a second proof of the reparametrization invariance of line integrals.

6 Differential k-forms

Roughly speaking, a differential k-form on a manifold M is a k-dimensional integrand, i.e. something that can be integrated on the k-dimensional submanifolds of M. We will give a precise definition later.

6.1 Motivation

Let us think first about integrating over the whole n-manifold M (rather than a submanifold).

First, note that it is not a good idea to try to integrate a function $h : M \to \mathbb{R}$ over M. Why ? Let's try to do it and see what is the problem.

Since we know how to integrate functions on $\mathbb{R}^n$, we could attempt to define $\int f$ using a chart as follows. Suppose, for simplicity, that the support of h is contained on the domain of some chart (U, φ). (More generally, we could break h as a sum $h = \sum_i h_i$ where each h_i is supported in the domain of a chart (U_i, φ_i). We can do this using a partition of unity.) Then we could propose the definition

$$\int_M h := \int_{\varphi(U)} h|_\varphi$$

where $h|_\varphi := h \circ \varphi^{-1}$ is the local expression of h in the chart φ. The problem with this is that the integral on the right hand side depends on the coordinate system. For example, for the function $h \equiv 1$, the result of the integral is the volume of $\varphi(U)$, which clearly depends on the coordinate system. E.g. if we pass to a new coordinate system $\widetilde{\varphi} = 2\varphi$, then we have $vol(\widetilde{\varphi}(U)) = 2^n vol(\varphi(U))$, therefore the value of the integral grows by a factor 2^n.

Line integrals When we study calculus in an open set $U \subseteq \mathbb{R}^n$, we define the integral of a vector field $X : U \to \mathbb{R}^n$ along a curve $\gamma : [a, b] \to \mathbb{R}^n$ by the formula

$$\begin{aligned}\int_\gamma X &:= \int_a^b \langle X, \gamma' \rangle \,\mathrm{d}t \qquad (6.1)\\ &= \int_a^b \sum_i X^i \cdot (\gamma^i)' \,\mathrm{d}t.\end{aligned}$$

where $\langle Y, Z \rangle := \sum_i Y^i \cdot Z^i$ is the standard inner product in $\mathbb{R}^n$. (In this formula we are omitting the evaluation points to simplify the expression, i.e. we write γ' instead of $\gamma'(t)$ and X instead of $X|_{\gamma(t)}$.)

The integral (6.1) is well defined only because in the set $U \subseteq \mathbb{R}^n$ we always use the standard coordinate system—let's call it φ. Suppose that we switch to a new chart $\widetilde{\varphi} = 2\varphi$, so that the new coordinate covectors and vectors are

$$\mathrm{d}\widetilde{\varphi}^i = 2\,\mathrm{d}\varphi^i \qquad \text{and} \qquad \frac{\partial}{\partial \widetilde{\varphi}^i} = \frac{1}{2}\frac{\partial}{\partial \varphi^i}.$$

The first equation comes immediately from the fact that $\widetilde{\varphi}^i = 2\varphi^i$, and the second one follows because the coordinate vectors are dual to the coordinate covectors. The coefficients of the vector field X w.r.t. the new chart $\widetilde{\varphi}$ are

$$\widetilde{X}^i = \mathrm{d}\widetilde{\varphi}^i(X) = 2\,\mathrm{d}\varphi^i(X) = 2X^i.$$

and, similarly, the new coefficients of $\gamma'(t)$ are $(\widetilde{\gamma}^i)' = 2(\gamma^i)'(t)$. (Here we are using (5.3).) Therefore the result of the integral (6.1) increases by a factor 4. This is why we cannot define line integrals in this way on a manifold which does not have a specified coordinate system.

The simplest way to fix the problem is to integrate a covector field θ instead of a vector field X. The integral of θ is defined by the formula

$$\begin{aligned}\int_\gamma \theta &:= \int_a^b \theta(\gamma')\,\mathrm{d}t \\ &= \int_a^b \sum_i \theta_i \cdot (\gamma^i)'\,\mathrm{d}t.\end{aligned} \tag{6.2}$$

(Again, we write γ' instead of $\gamma'(t)$, and θ instead of $\theta|_{\gamma(t)}$.) Given a vector field X, we can replace it by the covector field θ defined by

$$\theta|_p(Y) = \langle X|_p, Y\rangle \qquad \text{for all points } p \in U \text{ and vectors } Y \in \mathrm{T}_pU \equiv \mathbb{R}^n.$$

Then the integral (6.2) gives the same value as (6.1) for any curve γ, which means that θ is equivalent to X as a curvilinear integrand. However, the new integral (6.2) behaves better than (6.1) when we switch to the coordinate system $\widetilde{\varphi} = 2\varphi$, because the new coefficients of the covector field θ are, by (5.3),

$$\widetilde{\theta}_i = \theta\left(\frac{\partial}{\partial\widetilde{\varphi}^i}\right) = \theta\left(\frac{1}{2}\frac{\partial}{\partial\varphi^i}\right) = \frac{1}{2}\theta\left(\frac{\partial}{\partial\varphi^i}\right) = \frac{1}{2}\theta_i$$

which compensates the change in the coefficients of the velocity vector γ', which are $(\widetilde{\gamma}')^i = 2(\gamma')^i$, so that the value of the integral (6.2) is unchanged.

Heuristically, we can think the curve γ as a union of infinitesimal "line elements" $\gamma'(t)\cdot\delta t$, which is a vector at the point $p = \gamma(t)$. To each of these vectors we apply the covector $\theta|_p$ to obtain a number, and then we sum these numbers over the curve to compute the value of the integral (6.2).

Surface integrals Now suppose we have in an open set $U \subseteq \mathbb{R}^n$ a surface parametrized by a smooth function $f : R \to \mathbb{R}^n$, where $R = [a,b]\times[c,d] \subseteq \mathbb{R}^2$ is a rectangle. We want to define an integral $\int_f \omega$. What kind of field ω can we integrate over a surface ?

Heuristically, we subdivide the rectangle R into infinitesimal rectangles $\delta R = [s, s+\delta s]\times[t, t+\delta t]$. The image of a rectangle δR is a "surface element" $f(\delta R)$, which is an infinitesimal parallelogram located at the point $p = f(s,t) \in U$ and spanned by the infinitesimal vectors $X = \frac{\partial f}{\partial s}\cdot\delta s$ and $Y = \frac{\partial f}{\partial t}\cdot\delta t$. We have to use the object $\omega|_p$ to traduce this pair of vectors X, Y into a scalar value, so that we can integrate this scalars over all the surface elements. We propose that ω should be a *field of real-valued bilinear forms* i.e., ω should assign to each point $p \in U$ a bilinear form $\omega|_p$ which is a function $\omega|_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that is linear on each of its two variables. Then we can compute

$$\omega(\delta R) = \omega|_p(X,Y) = \omega\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\delta s\,\delta t$$

to obtain an infinitesimal number, which we can integrate over all infinitesimal rectangles to obtain the value of $\int_f \omega$. We also require that each bilinear form $\omega|_p$ be *alternating*, which means that

$$\omega(X,Y) = 0 \quad \text{if } X = Y.$$

This ensures that $\omega(\delta R) = 0$ if the parallelogram δR is degenerate. Note that if ω is alternating, then it satisfies the identity

$$\omega(Y, X) = -\omega(X, Y), \text{ for any vectors } X, Y.$$

because

$$\begin{aligned} 0 &= \omega(X+Y, X+Y) \\ &= \omega(X,X) + \omega(X,Y) + \omega(Y,X) + \omega(Y,Y) = \omega(X,Y) + \omega(Y,X). \end{aligned}$$

Thus we can define the notion of a *2-form* on $\mathbb{R}^n$ as follows.
(This is just for having a quick glance; we will later repeat the definition in more generality when we define k-forms on any manifold.)

Definition 6.1.1. A **differential 2-form** on an open set $U \subseteq \mathbb{R}^n$ is a field ω of alternating bilinear forms, i.e., ω assigns to each point $p \in U$ to a bilinear form $\omega|_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

The **component functions** of ω (w.r.t. the standard coordinates $\varphi = \mathrm{id}_U$) are the functions $\omega_{i,j} : U \to \mathbb{R}$ given by $\omega_{i,j}(p) := \omega|_p(E_i, E_j)$, where $(E_i)_i$ is the standard basis of $\mathbb{R}^n$.

We say that ω is $\mathcal{C}^r$ if its component functions are $\mathcal{C}^r$.

Note that since $\omega|_p$ is alternating for each point p, we have $\omega_{j,i} = -\omega_{i,j}$; in particular $\omega_{i,i} = 0$.

Definition 6.1.2. The **integral** of a continuous 2-form ω over a $\mathcal{C}^1$ parametric surface $f : R = [a,b] \times [c,d] \to \mathbb{R}^n$ is defined as

$$\int_f \omega := \iint_{(s,t)\in[a,b]\times[c,d]} \omega\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) \mathrm{d}s\, \mathrm{d}t \tag{6.3}$$

(Here we are omitting the evaluation points, i.e. we write f instead of $f(s,t)$ and ω instead of $\omega|_{f(s,t)}$.)

To compute the integral we write it in a more explicit way:

$$\begin{aligned} \int_f \omega &= \iint_{(s,t)\in R} \omega\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) \mathrm{d}s\, \mathrm{d}t. \\ &= \iint_{(s,t)\in R} \omega\left(\sum_i \frac{\partial f^i}{\partial s}, \sum_j \frac{\partial f^j}{\partial t}\right) \mathrm{d}s\, \mathrm{d}t. \\ &= \iint_{(s,t)\in R} \sum_{i,j} \omega_{i,j} \frac{\partial f^i}{\partial s} \frac{\partial f^j}{\partial t}\, \mathrm{d}s\, \mathrm{d}t. \end{aligned}$$

This shows that the integral (6.3) is well defined and finite valued, since the function

$$(s,t) \mapsto \omega|_{f(s,t)}\left(\frac{\partial f(s,t)}{\partial s}, \frac{\partial f(s,t)}{\partial t}\right) = \sum_{i,j} \omega_{i,j}|_{f(s,t)} \frac{\partial f^i(s,t)}{\partial s} \frac{\partial f^j(s,t)}{\partial t} \in \mathbb{R}$$

is continuous because the functions $\omega_{i,j}(f(s,t))$, $\frac{\partial f^i(s,t)}{\partial s}$ and $\frac{\partial f^i(s,t)}{\partial t}$ are continuous.

Exercise 6.1.3. (a) What happens if we switch from the standard coordinates φ to a chart $\widetilde{\varphi} = 2\varphi$? Compute the new components $\widetilde{\omega}_{i,j}$, $\widetilde{f}^i$, $\frac{\partial \widetilde{f}^i}{\partial s}$ and verify that the value of the integral does not change.

(b) What happens if we reparametrize the surface ? For example: consider the function $g(t,s) := f(s,t)$, defined on the rectangle $[c,d] \times [a,b]$. Show that $\int_g \omega = -\int_f \omega$.

6.2 Tensors

As suggested above, a differential form on a manifold will be defined as a field of alternating multilinear functions on the tangent space. But before getting there, we need to understand multilinear functions on a single vector space $V \simeq \mathbb{R}^n$. We'll study in this section the general multilinear functions, and in the next section we restrict to those that are alternating.

Definition 6.2.1. A **multilinear form** or **covariant k-tensor** on $V \simeq \mathbb{R}^n$ is a function

$$T : V^k = \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}$$

that is linear on each of its k variables. The number k is called the **degree** of the tensor T. The set of covariant k-tensors on V is denoted by $\operatorname{Ten}^k V$ or $\bigotimes^k V^*$. (The explanation for the second notation will come later.) This set is a vector space with the operations of pointwise addition and multiplication by a scalar:

$$(S + \lambda T)(X_0, \dots, X_{k-1}) := S(X_0, \dots, X_{k-1}) + \lambda T(X_0, \dots, X_{k-1}).$$

Example 6.2.2.

- A 0-tensor is a function of zero variables, i.e. a constant, thus $\operatorname{Ten}^0 V = \mathbb{R}$.
- Covariant 1-tensors on V are just the linear maps $V \to \mathbb{R}$, thus $\operatorname{Ten}^1 V = V^*$.
- Covariant 2-tensors are precisely the bilinear forms on V. E.g. any inner product is a bilinear form i.e. a covariant 2-tensor. Note that covariant 2-tensors on $\mathbb{R}^n$ can be identified with $n \times n$ matrices, namely a matrix A corresponds to the bilinear form $(v, w) \mapsto v^T A w$.
- The determinant $A \in \mathbb{R}^{n \times n} \mapsto \det(A)$, thought of as a function of the n vectors (the columns of the matrix A), is a covariant n-tensor on $\mathbb{R}^n$.

Remark 6.2.3. There is also the notion of *contravariant tensors*, which are multilinear functions from $(V^*)^k$ to $\mathbb{R}$. A tensor on V, in general, is a multilinear function on $V^k \times (V^*)^\ell$, where $k, \ell \in \mathbb{N}$. For the moment we are only interested in covariant tensors.

One way to obtain a tensor is by multiplying two tensors of smaller degree.

Definition 6.2.4. The **tensor product** of two covariant tensors $S \in \operatorname{Ten}^k V$, $T \in \operatorname{Ten}^\ell V$ is the covariant tensor $S \otimes T \in \operatorname{Ten}^{k+\ell} V$ defined by

$$S \otimes T(X_0, \dots, X_{k-1}, Y_0, \dots Y_{\ell-1}) = S(X_0, \dots, X_{k-1}) \cdot T(Y_0, \dots, Y_{\ell-1}).$$

We see immediately that this is indeed a tensor on V and that the tensor product operation is bilinear and associative. We can therefore write the tensor product of three or more tensors unambiguously without parentheses. I.e. on a product $S_0 \otimes \cdots \otimes S_{r-1}$ of tensors $S_i \in \operatorname{Ten}^{\ell_i} V$, the tensor S_0 acts on the first ℓ_0 vectors, S_1 on the next ℓ_1 vectors etc.

Note that for $c \in \operatorname{Ten}^0 V = \mathbb{R}$ and $T \in \operatorname{Ten}^k V$ we have $c \otimes T = cT$.

A base of tensors

Consider a vector space $V \simeq \mathbb{R}^n$ with a fixed base $(E_i)_{i \in \underline{n}}$, and let $(\varepsilon^j)_{j \in \underline{n}}$ be the dual base of V^*. (Here we use the notation $\underline{n} := \{0, \dots, n-1\}$.) We can construct a base of $\operatorname{Ten}^k V$ by taking tensor products of the base covectors ε^i.

Definition 6.2.5. The **elementary covariant k-tensors** are the k-tensors of the form
$$\varepsilon^I_\otimes = \varepsilon^{i_0} \otimes \cdots \otimes \varepsilon^{i_{k-1}} \in \operatorname{Ten}^k V,$$
and the **elementary vector k-tuples** are the vector k-tuples
$$E_I = (E_{i_0}, \ldots, E_{i_{k-1}})$$
defined for any sequence of indices $I = (i_0, \ldots, i_{k-1}) \in \underline{n}^k$ (called a k-component **multi-index**, or **k-index**).

Note that
$$\varepsilon^I E_J = \delta^I_J \qquad \text{for all } I, J \in \underline{n}^k.$$

Proposition 6.2.6. *The elementary k-tensors $\varepsilon^I_\otimes$, with $I \in \underline{n}^k$, form a base of $\operatorname{Ten}^k V$, and for any tensor $T \in \operatorname{Ten}^k V$ we have*
$$T = \sum_{(i_0,\ldots,i_{k-1})\in\underline{n}^k} T_{i_0,\ldots,i_{k-1}} \varepsilon^{i_0} \otimes \cdots \otimes \varepsilon^{i_{k-1}} = \sum_{I\in\underline{n}^k} T_I\, \varepsilon^I_\otimes \tag{6.4}$$
where $T_I = T(E_I) = T(E_{i_0}, \ldots, E_{i_{k-1}})$. In particular, $\dim \operatorname{Ten}^k V = n^k$.

Proof. For any vectors $X_0, \ldots, X_{k-1} \in V$, decomposing $X_s = \sum_i X^i_s E_i$ we obtain
$$\begin{aligned} T(X_0, \ldots, X_{k-1}) &= T\left(\sum_{i_0\in\underline{n}} X^{i_0}_0 E_{i_0}, \ldots, \sum_{i_{k-1}\in\underline{n}} X^{i_{k-1}}_{k-1} E_{i_{k-1}}\right) \\ &= \sum_{I=(i_0,\ldots,i_{k-1})\in\underline{n}^k} X^{i_0}_0 \cdots X^{i_{k-1}}_{k-1}\, T(E_{i_0}, \cdots, E_{i_{k-1}}) \end{aligned}$$
since T is multilinear. Therefore T is determined by the n^k numbers
$$T_I = T_{i_0,\ldots,i_{k-1}} := T(E_{i_0}, \ldots, E_{i_{k-1}}) = T(E_I) \in \mathbb{R}.$$
Furthermore, we have
$$T = \sum_{(i_0,\ldots,i_{k-1})\in\underline{n}^k} T_{i_0,\ldots,i_{k-1}} \varepsilon^{i_0} \otimes \cdots \otimes \varepsilon^{i_{k-1}}$$
since both sides are multilinear and the equation is true when we evaluate on $E_I = (E_{i_0}, \ldots, E_{i_{k-1}})$. This shows that the elementary k-tensors span $\operatorname{Ten}^k V$. To show that they are linearly independent, suppose that for some coefficients $T_{i_0,\ldots,i_{k-1}}$ we have
$$\sum_{i_0,\ldots,i_{k-1}} T_{i_0,\ldots,i_{k-1}} \varepsilon^{i_0} \otimes \cdots \otimes \varepsilon^{i_{k-1}} = 0.$$
Evaluating on $E_J = (E_{j_0}, \ldots, E_{j_{k-1}})$ we get $T_J = T_{j_0,\ldots,j_{k-1}} = 0$ for all $J = (j_0, \ldots j_{k-1}) \in \underline{n}^k$. ◻

Example 6.2.7. An inner product (or any bilinear form) g on V can be written as $g = \sum_{i,j} g_{i,j}\, \varepsilon^i \otimes \varepsilon^j$ using the coefficients $g_{i,j} = g(E_i, E_j) \in \mathbb{R}$. In particular, the standard inner product on $\mathbb{R}^n$ is $g = \sum_i \varepsilon^i \otimes \varepsilon^i$, where $(\varepsilon^i)_i$ is the dual of the standard basis of $\mathbb{R}^n$. Note that the numbers $g_{i,j}$ are the coefficients of the matrix G that determines g by the formula $g(X, Y) = X^t G Y$.

Remark 6.2.8. The previous proposition makes clear why the space $\operatorname{Ten}^k(V)$ of covariant k-tensors is also denoted $\bigotimes^k V^*$: any covariant k-tensor is a linear combination of tensor products of covectors.

For tensors we have the following transformation law.

Proposition 6.2.9 (Transformation law for covariant k-tensors)**.** *If $(E_i)_i$ and $(\widetilde{E}_j)_j$ are two bases of V related by the formula $\widetilde{E}_j = \sum_i a^i_j E_i$, where $(a^i_j)_{i,j} \in \mathbb{R}^{n\times n}$ is an invertible matrix, then for any tensor $T \in \mathrm{Ten}^k(V)$, the coefficients*

$$T_I = T(E_{i_0}, \ldots, E_{k-1}) \quad \textit{defined for } I = (i_0, \ldots, i_{k-1}) \in \underline{n}^k$$

are related to the coefficients

$$\widetilde{T}_J = T(\widetilde{E}_{j_0}, \ldots, \widetilde{E}_{j_{k-1}}) \quad \textit{defined for } J = (j_0, \ldots j_{k-1}) \in \underline{n}^k$$

by the formula

$$\widetilde{T}_J = \sum_{I\in \underline{n}^k} a^{i_0}_{j_0} \cdots a^{i_{k-1}}_{j_{k-1}} T_I. \tag{6.5}$$

Proof. Exercise. □

6.3 Alternating tensors

Definition 6.3.1. An **alternating k-tensor** on V is a tensor $T \in \mathrm{Ten}^k V$ such that

$$T(X_0, \ldots, X_{k-1}) = 0 \quad \text{if } X_s = X_t \text{ for some } s \neq t$$

for any vectors $X_0, \ldots, X_{k-1} \in V$. The space of alternating k-tensor on V is a subspace of $\mathrm{Ten}^k V = \bigotimes^k V^*$ denoted $\mathrm{Alt}^k V$ or $\bigwedge^k V^*$. (We will see the reason for the second notation later on.)

Note that every 0-tensor or 1-tensor is alternating, thus we have $\mathrm{Alt}^0 V = \mathrm{Ten}^0 V = \mathbb{R}$ and $\mathrm{Alt}^1 V = \mathrm{Ten}^1(V) = V^*$.

It is convenient to use the following notation. Let S_k be the group of permutations of the set $\underline{k} = \{0, \ldots, k-1\}$.

Definition 6.3.2 (Permutation of tensors and indices)**.** A permutation $\sigma \in S_k$ acts on tensors $T \in \mathrm{Ten}^k V$ by the formula

$$\sigma T(X_0, \ldots, X_{k-1}) := T(X_{\sigma(0)}, \ldots, X_{\sigma(k-1)}),$$

and acts on multi-indices $I = (i_0, \ldots, i_{k-1}) \in \underline{n}^k$ by

$$\sigma^* I = (i_{\sigma(0)}, \ldots, i_{\sigma(k-1)}) \in \underline{n}^k.$$

Exercise 6.3.3. Let $(T_I)_{I\in\underline{n}^k}$ be the coefficients of a tensor $T \in \mathrm{Ten}^k V$ with respect to some base $(E_i)_{i\in\underline{n}}$. Show that $(\sigma T)_I = T_{\sigma^* I}$ for any permutation $\sigma \in S_k$ and any k-index $I \in \underline{n}^k$.

Exercise 6.3.4. For a tensor $T \in \mathrm{Ten}^k V$, the following are equivalent:

(a) T is alternating,

(b) T is **skew symmetric**, i.e. $T(\cdots, X_s, \cdots X_t, \cdots) = -T(\cdots, X_t, \cdots X_s, \cdots)$ for any vectors $X_0, \ldots, X_{k-1}$,

(c) For any permutation $\sigma \in S_k$ we have $\sigma T = \mathrm{sgn}(\sigma) \cdot T$.

(d) The coefficients in (6.4) satisfy $T_{\sigma^* I} = \mathrm{sgn}(\sigma)\, T_I$ for any permutation $\sigma \in S_k$.

Example 6.3.5. On $\mathbb{R}^2$, denoting $(\varepsilon^i)_i$ the dual of the standard base, a general 2-tensor T is of the form

$$T = T_{0,0}\,\varepsilon^0 \otimes \varepsilon^0 + T_{0,1}\,\varepsilon^0 \otimes \varepsilon^1 + T_{1,0}\,\varepsilon^1 \otimes \varepsilon^0 + T_{1,1}\,\varepsilon^1 \otimes \varepsilon^1,$$

and it is alternating iff $T_{0,0} = T_{1,1} = 0$ and $T_{0,1} = -T_{1,0}$. Thus the alternating 2-tensors on $\mathbb{R}^2$ are those of the form $T = a(\varepsilon^0 \otimes \varepsilon^1 - \varepsilon^1 \otimes \varepsilon^0)$ with $a \in \mathbb{R}$, i.e. the multiples of the determinant function.

From any tensor $T \in \operatorname{Ten}^k V$ we can get an alternating tensor as follows.

Definition 6.3.6. The **skew-symmetrization** of a tensor $T \in \operatorname{Ten}^k V$ is the alternating tensor

$$A(T) := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot \sigma T \quad \in \operatorname{Alt}^k V,$$

A base of alternating tensors

Definition 6.3.7. As before, let $(E_i)_{i \in \underline{n}}$ be a fixed base of $V \simeq \mathbb{R}^n$ and denote $(\varepsilon^i)_i$ the corresponding dual base. The **elementary alternating k-tensors** on V are the tensors

$$\begin{aligned}\varepsilon^I := A(\varepsilon^I_\otimes) &= A(\varepsilon^{i_0} \otimes \cdots \otimes \varepsilon^{i_{k-1}}) \\ &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, \varepsilon^{i_{\sigma(0)}} \otimes \cdots \otimes \varepsilon^{i_{\sigma(k-1)}} \ \in \operatorname{Alt}^k V\end{aligned}$$

defined for each multi-index $I = (i_0, \ldots, i_{k-1}) \in \underline{n}^k$.

Note that if we evaluate on vectors $X_0, \ldots, X_{k-1}$ we get

$$\varepsilon^I(X_0, \ldots, X_{k-1}) = \det \begin{pmatrix} X_0^{i_0} & \ldots & X_{k-1}^{i_0} \\ \vdots & & \vdots \\ X_0^{i_{k-1}} & \ldots & X_{k-1}^{i_{k-1}} \end{pmatrix} \tag{6.6}$$

where $X_s^i = \varepsilon^i(X_s)$ is the j-th component of the vector X_s in the base $(E_i)_i$.

We state some basic properties:

Lemma 6.3.8. *The elementary alternating tensors satisfy:*

(a) $\varepsilon^I = 0$ *if I has a repeated index.*

(b) If $J = \sigma^* I$ *then* $\varepsilon^J = \operatorname{sgn}(\sigma)\, \varepsilon^I$.

(c) For $J = (j_0, \ldots, j_{k-1})$ *a multi-index we have*

$$\varepsilon^I(E_J) = \begin{cases} 0 & \text{if } I \text{ or } J \text{ have a repeated index or if they are} \\ & \text{not permutations of each other} \\ \operatorname{sgn}(\sigma) & \text{if } J = \sigma^* I \text{ and } I \text{ does not have a repeated index.} \end{cases}$$

Proof. Straightforward by looking at the expression (6.6). □

The repetitions among the tensors ε^I suggests that we only need to use the *increasing* multi-indices.

Definition 6.3.9. An **increasing k-index** is a multi-index $I = (i_0, \ldots, i_{k-1}) \in \underline{n}^k$ such that $i_0 < \cdots < i_{k-1}$. The set of increasing k-indices is denoted $\underline{n}^k_{\nearrow}$.

Remark 6.3.10. There are $\binom{n}{k}$ increasing multi-indices $I \in \underline{n}^k_{\nearrow}$.
Any multi-index $J \in \underline{n}^k$ with no repetitions can be uniquely written as $J = \sigma^* I$, where $\sigma \in S_k$ is a permutation and $I \in \underline{n}^k_{\nearrow}$ is an increasing multi-index.

Now we can describe a basis for $\operatorname{Alt}^k V$.

Proposition 6.3.11. *As before, let $(E_i)_{i\in\underline{n}}$ be a base for V and $(\varepsilon^i)_{i\in\underline{n}}$ the dual base. Then the set of elementary tensors ε^I, where $I \in \underline{n}^k$ is an increasing k-index, is a base for $\operatorname{Alt}^k V$. In fact, for any $T \in \operatorname{Alt}^k(V)$ we have*

$$T = \sum_{I\in\underline{n}^k_{\nearrow}} T_I\,\varepsilon^I$$

where $T_I = T(E_I) = T(E_{i_0}, \ldots, E_{i_{k-1}})$. Therefore, $\dim \operatorname{Alt}^k V = \binom{n}{k}$.

In particular, for $k = n$ the dimension is one and $\operatorname{Alt}^k V$ is spanned by $\varepsilon^{(0,\ldots,n-1)}$, which is just the determinant function (with respect to the base $(E_i)_i$). For $k > n$ the dimension is zero.

Proof. First we show that the given set spans $\operatorname{Alt}^k V$. Let $T \in \operatorname{Alt}^k V$. Writing T in the form (6.4) and using Remark 6.3.10 and the fact that T is alternating, we get

$$T = \sum_{J\in\underline{n}^k} T_J\,\varepsilon_{\otimes}^J = \sum_{\substack{\sigma\in S_k\\ I\in\underline{n}^k_{\nearrow}}} T_{\sigma^* I}\,\varepsilon_{\otimes}^{\sigma^* I} = \sum_{\substack{\sigma\in S_k\\ I\in\underline{n}^k_{\nearrow}}} \operatorname{sgn}(\sigma)\,T_I\,\varepsilon_{\otimes}^{\sigma^* I} = \sum_{I\in\underline{n}^k_{\nearrow}} T_I\,\varepsilon^I.$$

In the second step we used the fact that $T_J = T(E_J)$ vanishes when J has repetitions. Therefore we only need to sum over those J that have no repetitions, which can be uniquely expressed as $\sigma^* I$ for some permutation σ and some increasing multi-index I.

To show linear independence, suppose that for some coefficients $T_I \in \mathbb{R}$ we have

$$\sum_{I\in\underline{n}^k_{\nearrow}} T_I \cdot \varepsilon^I = 0.$$

Then we see that $T_J = 0$ for each $J \in \underline{n}^k_{\nearrow}$ by evaluating on E_J. $\square$

Wedge product of alternating tensors

Like general tensors, alternating tensors can be multiplied to obtain new alternating tensors.

Definition 6.3.12. The **wedge product** (a.k.a. **alternating product** or **exterior product**) of two alternating tensors $S \in \operatorname{Alt}^k V$, $T \in \operatorname{Alt}^\ell V$ is the alternating tensor

$$S \wedge T := \frac{1}{k!\,\ell!}\,A(S\otimes T) \;\in \operatorname{Alt}^{k+\ell} V.$$

More generally, the wedge product of m alternating tensors $T_0 \in \operatorname{Alt}^{k_0} V$, …, $T_{m-1} \in \operatorname{Alt}^{k_{m-1}} V$ is the alternating tensor

$$T_0 \wedge \cdots \wedge T_{m-1} := \frac{1}{k_0!\cdots k_{m-1}!}\,A(T_0\otimes\cdots\otimes T_{m-1}).$$

The "A" ensures that we get an alternating tensor, and the factor $\frac{1}{k_0!\cdots k_{m-1}!}$ compensates for the repeated terms that we obtain if we use the "A" alone. Which repeated terms? This will be clear in the examples.

Example 6.3.13. The wedge product of scalar c with a tensor $T \in \operatorname{Alt}^k V$ is $c \wedge T = cT$.

Example 6.3.14. The wedge product of two 1-tensors $\alpha, \beta \in \operatorname{Alt}^1 V$ is

$$\alpha \wedge \beta(X, Y) = \alpha(X)\,\beta(Y) - \alpha(Y)\,\beta(X).$$

Example 6.3.15. The wedge product of an alternating 2-tensor $\omega \in \operatorname{Alt}^2(V)$ with a 1-tensor $\sigma \in \operatorname{Alt}^1(V)$ is

$$\begin{aligned}\omega \wedge \theta(X_0, X_1, X_2) = \frac{1}{2}(&\omega(X_0, X_1)\,\theta(X_2) - \omega(X_0, X_2)\,\theta(X_1) + \omega(X_1, X_2)\,\theta(X_0) \\ &-\omega(X_1, X_0)\,\theta(X_2) + \omega(X_2, X_0)\,\theta(X_1) - \omega(X_2, X_1)\,\theta(X_0)).\end{aligned}$$

Since ω is alternating, each term of the second line is equal to the term that is above it. Thus there is a simpler formula

$$\omega \wedge \theta(X_0, X_1, X_2) = \omega(X_0, X_1)\,\theta(X_2) - \omega(X_0, X_2)\,\theta(X_1) + \omega(X_1, X_2)\,\theta(X_0).$$

Example 6.3.16 (Wedge product of covectors)**.** The wedge product of several covectors $\varphi^0, \dots, \varphi^{k-1} \in V^* = \operatorname{Alt}^1 V$ is

$$\varphi^0 \wedge \cdots \wedge \varphi^{k-1} = A(\varphi^0 \otimes \cdots \otimes \varphi^{k-1}),$$

therefore for any vectors $X_0, \dots, X_{k-1} \in V$ we have

$$\begin{aligned}\varphi_0 \wedge \cdots \wedge \varphi_{k-1}(X_0, \dots, X_{k-1}) &= \sum_{\sigma \in S^k} \operatorname{sgn}(\sigma) \cdot \varphi^0(X_{\sigma(0)}) \cdots \varphi^{k-1}(X_{\sigma(k-1)}) \\ &= \det\left((\varphi^s(X_t))_{\substack{s \in \underline{k} \\ t \in \underline{k}}} \right).\end{aligned}$$

Proposition 6.3.17. *The wedge product satisfies the following properties*

(a) The bilinear operation $\wedge : \operatorname{Alt}^k V \times \operatorname{Alt}^\ell V \to \operatorname{Alt}^{k+\ell} V$ *is associative.*

(b) In addition, it is ***anticommutative****, that is for* $S \in \operatorname{Alt}^k V$, $T \in \operatorname{Alt}^\ell V$ *we have*

$$S \wedge T = (-1)^{k\ell}\, T \wedge S.$$

(c) The wedge product of base covectors $\varepsilon^{i_0}, \dots, \varepsilon^{i_{k-1}}$ *is an elementary alternating tensor* $\varepsilon^{i_0} \wedge \cdots \wedge \varepsilon^{i_{k-1}} = \varepsilon^I$*, where* $I = (i_0, \dots, i_{k-1})$.

(d) For multi-indices $I = (i_0, \dots, i_{k-1})$, $J = (j_0, \dots, j_{\ell-1})$*, the product of the elementary alternating tensors* ε^I, ε^J *is*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}$$

where $IJ := (i_0, \dots, i_{k-1}, j_0, \dots, j_{\ell-1})$.

Proof. (a) Exercise. The proof is found in Tu's book [Tu11, Prop. 3.25].

(c) follows from the computation of a wedge product of covectors, Example 6.3.16.

(d) follows from (c) by associativity.

(b) It is sufficient to verify anticommutativity for elementary covectors $\varepsilon^I \in \operatorname{Alt}^k V$, $\varepsilon^J \in \operatorname{Alt}^l V$. To do so we compute

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ} = \operatorname{sgn}(\sigma)\varepsilon^{JI} = (-1)^{kl}\varepsilon^J \wedge \varepsilon^I$$

where σ is the permutation that exchanges the first k with the last ℓ indices. $\square$

This finally motivates the notation $\bigwedge^k V^*$ for $\operatorname{Alt}^k V$: any alternating covariant k-tensor is a linear combination of wedge products of covectors.

6.4 Differential forms on manifolds

Let M be a differentiable manifold of dimension n. We apply the definitions from the previous section to the case where V is the tangent space of M at some point $p \in M$. Hence for each p we consider the vector space of alternating covariant k-tensors on T_pM, denoted $\mathrm{Alt}^k(T_pM)$ or $\bigwedge^k(T_p^*M)$, which has dimension $\binom{n}{k}$. We denote

$$\mathrm{Alt}^k(TM) := \coprod_{p\in M} \mathrm{Alt}^k(T_pM).$$

If $\varphi = (\varphi^0, \dots, \varphi^{n-1})$ is a chart defined at p, recall that the coordinate vectors $\frac{\partial}{\partial\varphi^i}\Big|_p$ form a base of T_pM and the coordinate covectors $\mathrm{d}\varphi^j|_p$ form the dual base of T_p^*M. Therefore we have the elementary k-tuples of vectors

$$\frac{\partial}{\partial\varphi J}\bigg|_p := \left(\frac{\partial}{\partial\varphi^{j_0}}\bigg|_p, \dots, \frac{\partial}{\partial\varphi^{j_{k-1}}}\bigg|_p\right),$$

defined for any multi-index $J = (j_0, \dots, j_{k-1}) \in \underline{n}^k$, and the elementary alternating k-tensors

$$\mathrm{d}\varphi^I|_p := \mathrm{d}\varphi^{i_0}|_p \wedge \cdots \wedge \mathrm{d}\varphi^{i_{k-1}}|_p \in \mathrm{Alt}^k(\mathrm{T}_pM),$$

also defined for a multi-index $I = (i_0, \dots, i_{k-1}) \in \underline{n}^k$.

The elementary tensors $\mathrm{d}\varphi_p^I$ that correspond to *increasing* multi-indices $I \in \underline{n}^k_{\nearrow}$ (i.e. when $i_0 < \cdots < i_{k-1}$) constitute a base of $\mathrm{Alt}^k\, \mathrm{T}_pM$ by Proposition 6.3.11. Thus any alternating k-form $\omega \in \mathrm{Alt}^k(\mathrm{T}_pM)$ can be written uniquely as a linear combination $\omega = \sum_{I\in\underline{n}^k_{\nearrow}} \omega_I\, \mathrm{d}x^I|_p$, where the coefficients ω_I are defined for all multi-indices $I = (i_0, \dots, i_{k-1}) \in \underline{n}^k$ (not only the increasing ones) by evaluating ω on the corresponding k-uple of vectors:

$$\omega_I = \omega\left(\frac{\partial}{\partial\varphi^I}\right) = \omega\left(\frac{\partial}{\partial\varphi^{i_0}}\bigg|_p, \dots, \frac{\partial}{\partial\varphi^{i_{k-1}}}\bigg|_p\right).$$

The coefficient ω_I vanishes if I has repeated indices, and we have $\omega_{\sigma^* I} = (\operatorname{sgn}\sigma)\omega_I$ for any permutation $\sigma \in S_k$.

Definition 6.4.1. A **differential k-form** (or a **k-form**, for short) on a $\mathcal{C}^{r+1}$ manifold M is a function $\omega : M \to \mathrm{Alt}^k(\mathrm{T}M)$ that maps each point $p \in M$ to an alternating k-tensor $\omega|_p \in \mathrm{Alt}^k(\mathrm{T}_pM)$.

The **component functions** of a k-form ω with respect to a chart (U, φ) are the functions $\omega_J : U \to \mathbb{R}$ defined for any multi-index $J = (j_0, \dots, j_{k-1}) \in \underline{n}^k$ by

$$\omega_J|_p = \omega|_p\left(\frac{\partial}{\partial\varphi^J}\bigg|_p\right) = \omega|_p\left(\frac{\partial}{\partial\varphi^{j_0}}\bigg|_p, \dots, \frac{\partial}{\partial\varphi^{j_{k-1}}}\bigg|_p\right)$$

so that $\omega|_p = \sum_{I\in\underline{n}^k_{\nearrow}} \omega_I|_p\, \mathrm{d}\varphi^I|_p$. We say that ω is $\mathcal{C}^r$ at a point $p \in M$ if its component functions (w.r.t some chart defined at p) are $\mathcal{C}^r$ at p. (Exercise: Verify that this does not depend on the chart.) And we say that ω is $\mathcal{C}^r$ if it is $\mathcal{C}^r$ at all points.

The set of $\mathcal{C}^r$ differential k-forms on M is denoted $\Omega^k(M)$.

Note that a 0-form is a function $M \to \mathbb{R}$, and a 1-form is a covector field.

Operations involving alternating tensors such as sum, multiplication by a scalar and wedge product can be applied pointwise to differential forms:

Definition 6.4.2 (Pointwise operations of differential forms)**.**

- The **sum** of two k-forms $\alpha, \beta \in \Omega^k(M)$ is the k-form $\alpha + \beta \in \Omega^k(M)$ defined by $(\alpha + \beta)|_p := \alpha|_p + \beta|_p$ for all points $p \in M$.
- The **wedge product** of a k-form $\alpha \in \Omega^k$ and an ℓ-form $\beta \in \Omega^\ell(M)$ is the form $(k+\ell)$-form $\alpha \wedge \beta \in \Omega^{k+\ell}(M)$ defined by $(\alpha \wedge \beta)|_p := \alpha|_p \wedge \beta|_p$.

Wedge product of differential forms is clearly associative and anticommutative because the pointwise operations have these properties.

Note that the wedge product of a 0-form (i.e. a scalar field) $h \in \Omega^0(M)$ by a k-form $\omega \in \Omega^k(M)$ is simply the product of ω by the scalar field h,

$$(h \wedge \omega)|_p = (h\,\omega)|_p = h(p)\,\omega|_p.$$

The operation of pullback of covector fields generalises easily to k-forms.

Definition 6.4.3 (Pullback of k-forms.)**.** The **pullback** of a k-form $\omega \in \Omega^k(N)$ by a $\mathcal{C}^{r+1}$ map $f : M \to N$ is the k-form $f^*\omega \in \Omega^k(M)$ defined by

$$(f^*\omega)|_p(X_0, \ldots, X_{k-1}) := \omega|_{f(p)}(\,\mathrm{T}_p f(X_0), \ldots,\, \mathrm{T}_p f(X_{k-1})) \quad \text{for all } p \in M,\ X_i \in\, \mathrm{T}_p M.$$

(Exercise: Show that $f^*\omega \in \Omega^k(M)$.)

Note that for a 0-form $h \in \Omega^0(N) = C^r(N)$ we have $f^*h = h \circ f$.

Remark 6.4.4. If $\omega \in \Omega^k(M)$, $U \subset M$ open and $\iota : U \to M$ the inclusion, we denote $\omega|_U := \iota^*\omega \in \Omega^k(U)$ the so-called **restriction** of ω to U. Of course, for $p \in U$, we can identify $(\omega|_U)|_p$ and $\omega|_p$.

An important property of the operations of pullback and wedge product is that they commute.

Lemma 6.4.5. *For $f : M \to N$ a smooth map, $\omega \in \Omega^k(N)$, $\eta \in \Omega^\ell(N)$ we have:*

(i) $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$.

(ii) In any coordinate chart y^i on N,

$$f^*\left(\sum_{I \in \underline{n}^k_\nearrow} \omega_I\, \mathrm{d}y^I\right) = \sum_{I \in \underline{n}^k_\nearrow} (\omega_I \circ f)\, \mathrm{d}(y^{i_0} \circ f) \wedge \cdots \wedge \mathrm{d}(y^{i_{k-1}} \circ f).$$

Proof. Exercise. □

Example 6.4.6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2 : (r, \theta) \mapsto (r\cos\theta, r\sin\theta)$. Then the pullback of the 2-form $\mathrm{d}x \wedge \mathrm{d}y$ (where x, y are the standard coordinates on $\mathbb{R}^2$) by f is

$$\begin{aligned}
f^*(\mathrm{d}x \wedge \mathrm{d}y) &= f^*(\mathrm{d}x) \wedge f^*(\mathrm{d}y) \\
&= \mathrm{d}(x \circ f) \wedge \mathrm{d}(y \circ f) \\
&= (\cos\theta\, \mathrm{d}r - r\sin\theta\, \mathrm{d}\theta) \wedge (\sin\theta\, \mathrm{d}r + r\cos\theta\, \mathrm{d}\theta) \\
&= r\cos^2\theta\ \mathrm{d}r \wedge \mathrm{d}\theta - r\sin^2\theta\ \mathrm{d}\theta \wedge \mathrm{d}r \\
&= r\ \mathrm{d}r \wedge \mathrm{d}\theta
\end{aligned}$$

In general we have the following relation for the pullback of an n-form under a smooth map between n-dimensional manifolds:

Lemma 6.4.7. *Let $F : M \to N$ be a smooth map between smooth n-manifolds. Let (U, φ^i) be a chart on M and (V, y^i) be a chart on N. Let $u \in C^\infty(V)$. Then on $U \cap F^{-1}(V)$:* [1]

$$F^*(u \, \mathrm{d}y^0 \wedge \cdots \wedge \mathrm{d}y^{n-1}) = (u \circ F) \det \left(\frac{\partial F^j}{\partial \varphi^i} \right) \mathrm{d}\varphi^0 \wedge \cdots \wedge \mathrm{d}\varphi^{n-1}.$$

Here, $F^j := y^j \circ F$.

Proof. It suffices to show that the equality holds when we evaluate both sides on $\left(\frac{\partial}{\partial \varphi^0}, \dots, \frac{\partial}{\partial \varphi^{n-1}} \right)$. Note that by Lemma 6.4.5 we can rewrite the left-hand side as $(u \circ F) \, \mathrm{d}F^0 \wedge \cdots \wedge \mathrm{d}F^{n-1}$. But then the result follows immediately from

$$\mathrm{d}F^0 \wedge \cdots \wedge \mathrm{d}F^{n-1} \left(\frac{\partial}{\partial \varphi^0}, \dots, \frac{\partial}{\partial \varphi^{n-1}} \right) = \det \left(\mathrm{d}F^i \left(\frac{\partial}{\partial \varphi^j} \right) \right),$$

which follows from Example 6.3.16. □

For $M = N$, $F = \mathrm{id}_M$, $u \equiv 1$ we obtain the change of coordinates formula

$$\mathrm{d}y^0 \wedge \cdots \wedge \mathrm{d}y^{n-1} = \det \left(\frac{\partial y^j}{\partial \varphi^i} \right) \mathrm{d}\varphi^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}.$$

[1] In the expression below $\frac{\partial F^j}{\partial \varphi^i} = \frac{\partial (y^j \circ F \circ \varphi^{-1})}{\partial \varphi^i}$ so the matrix whose determinant we take on the right hand side is the Jacobian of the coordinate representation of F in the given charts.

7 Integration on manifolds

7.1 Oriented manifolds

We will need an oriented n-manifold M to define the integral $\int_M \omega$ of an n-form $\omega \in \Omega^n(M)$. The sign of the result will change if we switch the orientation.

7.1.1 Orientations on a vector space

Before dealing with manifolds, we define orientations on a vector space $V \simeq \mathbb{R}^n$.

Definition 7.1.1. An **orientation** on a vector space $V \simeq \mathbb{R}^n$ is a function $\mathcal{O}$ that assigns to each base $B = (B_i)_{0 \leq i < n}$ of V a number $\mathcal{O}(B) \in \{1, -1\}$ called the **sign** of B with respect to $\mathcal{O}$, such that

$$\mathcal{O}(B') = \mathcal{O}(B) \cdot \operatorname{sgn} \det(C) \quad \text{for every pair of bases } B, B' = BC \text{ of } V, \tag{7.1}$$

where $C = (C^i{}_j)_{i,j} \in \mathbb{R}^{n \times n}$ is an invertible matrix. The equation $B' = BC$ means that $B'_j = \sum_i B_i C^i{}_j$.

The sign $\mathcal{O}(B)$ of a base B with respect to an orientation $\mathcal{O}$ is also denoted $\operatorname{sgn}_{\mathcal{O}}(B)$. We write $\operatorname{sgn}(B)$ instead of $\operatorname{sgn}_{\mathcal{O}}(B)$ when we don't want to give a name to the orientation.

Example 7.1.2. The **standard orientation** on $\mathbb{R}^n$ is the orientation $\mathcal{O}_{\text{std}}$ given by $\mathcal{O}_{\text{std}}(B) = \operatorname{sgn}(\det B)$ for any base B of $\mathbb{R}^n$. (To define $\det B$ we identify the base B with the matrix whose columns are the vectors of B.)

Remark 7.1.3.
(a) An orientation $\mathcal{O}$ on a vector space $V \simeq \mathbb{R}^n$ is determined once we know the value $\mathcal{O}(B)$ for a single base B, since any other base B' can be written as $B' = BC$, with $C \in \mathbb{R}^{n \times n}$ invertible.
(b) Given a base B of a vector space $V \simeq \mathbb{R}^n$, there is a unique orientation $\mathcal{O}$ satisfying $\mathcal{O}(B) = 1$, and a unique orientation $\mathcal{O}'$ (equal to $-\mathcal{O}$) satisfying $\mathcal{O}'(B) = -1$. Thus V has exactly two orientations $\mathcal{O}$, $-\mathcal{O}$.
(c) If $\dim(V) = 0$, there is just one base of V, namely the empty base $B_\emptyset$. Thus an orientation $\mathcal{O}$ on V can be identified with the single number $\mathcal{O}(B_\emptyset) = \pm 1$. The condition (7.1) imposes no restriction because any matrix $C \in \mathbb{R}^{0 \times 0}$ has $\det C = 1$ since being C an empty matrix, it is equal to the identity matrix.

Definition 7.1.4. The **sign** of an isomorphism $T : V \to W$ between oriented vector spaces $V, W \simeq \mathbb{R}^n$ is the unique number $\operatorname{sgn}(T) \in \{\pm 1\}$ that satisfies

$$\operatorname{sgn}(T(B)) = \operatorname{sgn}(T) \cdot \operatorname{sgn}(B) \quad \text{for each base } B = (B_i)_{i \in \underline{n}} \text{ of } V,$$

where $T(B)$ is the base of W given by $(T(B))_i = T(B_i)$. We say that T is **orientation preserving** or **orientation reversing** if $\operatorname{sgn} T = 1$ or $\operatorname{sgn} T = -1$, respectively. Note that $\operatorname{sgn}(S \circ T) = \operatorname{sgn}(S) \cdot \operatorname{sgn}(T)$ if $S : W \to Z$ is a second isomorphism between oriented vector spaces.

7.1.2 Orientations on manifolds

The notion of orientation is applied to manifolds as follows.

Definition 7.1.5. A **(tangent) orientation field** on an n-dimensional differentiable manifold M is a function $\mathcal{O}$ that assigns to each point $p \in M$ an orientation $\mathcal{O}_p$ of the tangent space $\mathrm{T}_p M$. Such a field is considered **continuous** at a point p_0 if for some chart (U, φ) defined at p_0, the function

$$p \in U \mapsto \mathcal{O}_p(\varphi) := \mathcal{O}_p \left(\left. \frac{\partial}{\partial \varphi^0} \right|_p, \dots, \left. \frac{\partial}{\partial \varphi^{n-1}} \right|_p \right) \in \{\pm 1\}$$

is constant (either positive or negative) on a neighborhood of p_0. (We will see below that this fact does not depend on the chart φ.)

An **orientation** on M is a tangent orientation field $\mathcal{O}$ that is continuous at all points of M. The pair $(M, \mathcal{O})$ is called an **oriented manifold**. A differentiable manifold is **orientable** if it admits an orientation.

The number $\mathcal{O}_p(\varphi) \in \{\pm 1\}$ is called the **sign** of the chart φ with respect to the orientation $\mathcal{O}$ at the point p. If this sign is constant (i.e. independent of p) we denote it $\mathcal{O}(\varphi)$.

Note that $\mathcal{O}_p(\varphi)$ is the sign of the linear isomorphism $\mathrm{D}_p\varphi : (\mathrm{T}_p M, \mathcal{O}_p) \to (\mathbb{R}^n, \mathcal{O}_{std})$. Thus if ψ is a second chart defined at p, since $\mathrm{D}_p\psi = \mathrm{D}_{\varphi(p)}(\psi \circ \varphi^{-1}) \circ \mathrm{D}_p\varphi$, we have

$$\begin{aligned} \mathcal{O}_p(\psi) &= \operatorname{sgn}\left(\mathrm{D}_{\varphi(p)}(\psi \circ \varphi^{-1}) \right) \cdot \mathcal{O}_p(\varphi) \\ &= \operatorname{sgn} \det \mathrm{D}_{\varphi(p)}(\psi \circ \varphi^{-1}) \cdot \mathcal{O}_p(\varphi). \end{aligned} \tag{7.2}$$

Since $p \mapsto \operatorname{sgn} \det \mathrm{D}_{\varphi(p)}(\psi \circ \varphi^{-1})$ is locally constant, we conclude that if $\mathcal{O}_p(\varphi)$ is constant on a neighborhood of p_0, then so is $\mathcal{O}_p(\psi)$. This proves that the continuity of $\mathcal{O}$ does not depend on the chart, as stated before.

Remark 7.1.6. On an oriented manifold the sign of any chart φ is locally constant. In particular, if the domain of φ is connected, then φ has constant sign.

This suggests a practical method for defining an orientation on a manifold.

Definition 7.1.7. An **orientation-signed atlas** is an atlas $\mathcal{A}$ of a differentiable manifold M, together with a sign function $s : \mathcal{A} \to \{\pm 1\}$ that satisfies for all charts φ, $\psi \in \mathcal{A}$

$$s(\psi) = \operatorname{sgn} \det \mathrm{D}_{\varphi(p)}(\psi \circ \varphi^{-1}) \cdot s(\varphi) \quad \text{for all points } p \in \operatorname{Dom} \varphi \cap \operatorname{Dom} \psi. \tag{7.3}$$

Proposition 7.1.8. *If $(\mathcal{A}, s)$ is an orientation-signed atlas, then there is a unique orientation $\mathcal{O}$ on M such that $\mathcal{O}(\varphi) \equiv s(\varphi)$ for all charts $\varphi \in \mathcal{A}$.*

In particular, an atlas $\mathcal{A}$ whose transition maps have positive jacobian $\det \mathrm{D}(\varphi \circ \psi^{-1}) > 0$ everywhere (thus $\mathcal{A}$ is orientation-signed by the sign function $s \equiv 1$) defines an orientation $\mathcal{O}$ such that all charts of $\mathcal{A}$ are positive.

Proof. Let us say that a (possibly discontinuous) orientation field $\mathcal{O}$ on M *pleases* a chart $\varphi \in \mathcal{A}$ at a point $p \in \operatorname{Dom} \varphi$ if $\mathcal{O}_p(\varphi) = s(\varphi)$. This condition determines $\mathcal{O}_p$. We are searching for an orientation field $\mathcal{O}$ that pleases every chart $\varphi \in \mathcal{A}$ at all points $p \in \operatorname{Dom} \varphi$. Uniqueness of $\mathcal{O}$ is clear since the chart domains cover M.

Suppose $\mathcal{O}$ pleases φ at some point p, and let ψ be a second chart of $\mathcal{A}$ defined at p. Then $\mathcal{O}$ also pleases ψ at p because by (7.2) and (7.3) we have

$$\begin{aligned}\mathcal{O}_p(\psi) &= \operatorname{sgn}\det \mathrm{D}_{\varphi(p)}(\psi \circ \varphi^{-1}) \cdot \mathcal{O}_p(\varphi) \\ &= \operatorname{sgn}\det \mathrm{D}_{\varphi(p)}(\psi \circ \varphi^{-1}) \cdot s(\varphi) \\ &= s(\psi).\end{aligned}$$

This compatibility implies that there is a unique orientation field $\mathcal{O}$ that pleases all charts of $\mathcal{A}$ at all points. The field $\mathcal{O}$ is clearly continuous on the domain of each chart $\varphi \in \mathcal{A}$, hence it is continuous on M. □

This allows us to determine whether a manifold is orientable or not by taking an atlas consisting of connected charts, and analyzing the signs of the differentials of the transition maps.

Remark 7.1.9. (a) A manifold covered by a single chart is orientable.
In particular every open set $U \subseteq \mathbb{R}^n$ has a **standard orientation** that makes the identity chart positive.

(b) A manifold M covered by two connected charts (U, φ), (V, ψ) is orientable iff the function $p \in U \cap V \mapsto \operatorname{sgn}\det \mathrm{D}|_{\varphi(p)}(\psi \circ \varphi^{-1}$ is constant. (This happens, for example, if $U \cap V$ is connected.) In particular, the sphere $\mathbb{S}^n$ of dimension $n \geq 2$ is orientable since the domains of the stereographic projections with respect to the north and south poles cover M and have connected intersection.

(c) A one-component orientable manifold admits exactly two orientations $\mathcal{O}$, $-\mathcal{O}$.

(d) A manifold M is orientable iff each connected component of M is orientable.

Example 7.1.10 (See the exercises)**.**

(a) An orientation $\mathcal{O}$ on an n-manifold can be restricted to any open subset $U \subseteq M$, so that $(U, \mathcal{O}|_U)$ is an oriented n-manifold.

(b) A product $M_0 \times \cdots \times M_{k-1}$ of several oriented manifolds is naturally oriented.

(c) The torus $\mathbb{T}^n$ is orientable.

(d) The Möbius band is not orientable.

(e) The projective plane $\mathbb{R}P^2$ is not orientable.

Proposition 7.1.11. *Let M be an n-dimensional $\mathcal{C}^{k+1}$ manifold. Then the sign* $\operatorname{sgn}\omega$ *of any nonvanishing n-form on M, defined by*

$$(\operatorname{sgn}\omega)|_p(X_0, \ldots, X_{n-1}) := \operatorname{sgn}(\omega|_p(X_0, \ldots, X_{n-1}))$$

for $p \in M$ and $X_0, \ldots, X_{n-1} \in \mathrm{T}_pM$, is an orientation on M.

Reciprocally, every orientation on M is the sign of some nonvanishing n-form.

In consequence, M is orientable if and only if it has a nonvanishing n-form.

For example, the standard orientation on an open set $U \subseteq \mathbb{R}^n$ is the sign of the standard n-form $\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$.

Proof. We leave as an exercise the first part: showing that $\operatorname{sgn}(\omega)$ is an orientation if ω does not vanish.

Let $\mathcal{O}$ be an orientation on M. We will construct a nonvanishing n-form $\omega \in \Omega^n(M)$ such that $\operatorname{sgn}(\omega) = \mathcal{O}$.

Let $\mathcal{A}$ be an atlas of M consisting of connected charts. Therefore each chart $\varphi \in \mathcal{A}$ has a constant sign $\mathcal{O}(\varphi)$. For each $\varphi \in \mathcal{A}$ we define a nonvanishing n-form $\omega_\varphi = \mathcal{O}(\varphi)\, \mathrm{d}\varphi^0 \wedge \cdots \wedge \mathrm{d}\varphi^{n-1}$ on the open set $U_\varphi = \operatorname{Dom}\varphi$. Note that

$\mathrm{sgn}(\omega_\varphi) = \mathcal{O}|_{U_\varphi}$. Using a partition of unity $(\eta_\varphi)_{\varphi\in\mathcal{A}}$ subordinate to the cover $(U_\varphi)_{\varphi\in\mathcal{A}}$ we construct an n-form $\omega = \sum_{\varphi\in\mathcal{A}} \eta_\varphi\,\omega_\varphi$. We claim that ω is nonvanishing and satisfies $\mathrm{sgn}(\omega|_p) = \mathcal{O}_p$ for all points $p \in M$. Indeed, for each base B of T_pM, in the right hand side of

$$\omega|_p(B) = \sum_{\varphi\in\mathcal{A}} \eta_\varphi(p)\cdot\omega_\varphi|_p(B)$$

the numbers $\omega_\varphi|_p(B)$ have sign $\mathcal{O}_p(B)$ for all $\varphi \in \mathcal{A}$, and the numbers $\eta_\varphi(p)$ satisfy $\eta_\varphi(p) \geq 0$ and are not all zero. Therefore $\omega|_p(B)$ is nonzero and has sign $\mathcal{O}_p(B)$. □

Proposition 7.1.12. *Let S be an embedded hypersurface in an oriented $n+1$-dimensional manifold $(M, \mathcal{O})$. Then a* ***transverse vector field*** *Y on S (i.e. a continuous map $Y : S \to \mathrm{T}M$ such that $Y|_p \in \mathrm{T}_pM \setminus \mathrm{T}_pS$ for all $p \in S$) induces an orientation $\mathcal{O}^Y$ on S given by*

$$\mathcal{O}^Y_p(X_0, \dots, X_{n-1}) := \mathcal{O}_p(Y, X_0, \dots, X_{n-1}) \text{ for every base } (X_0, \dots, X_{n-1}) \text{ of } \mathrm{T}_pS.$$

Example 7.1.13. On $\mathbb{S}^2$ we can define a nowhere vanishing 2-form ω as follows: Let $N = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ the outward pointing radial vector field on $\mathbb{R}^3$. Then for $p \in \mathbb{S}^2$, $X, Y \in T_p\mathbb{S}^2$ define

$$\omega_p(X, Y) = \det(N_p, X, Y),$$

where on the right hand side we view X, Y as elements of $T_p\mathbb{R}^3$ (so actually we mean ι_*X, ι_*Y where ι is the inclusion of $\mathbb{S}^2$ into $\mathbb{R}^3$).

7.2 Definition of the integral

For M any smooth manifold and $\omega \in \Omega^k(M)$ we define the support of ω as

$$\mathrm{supp}\,\omega := \overline{\{p \in M \mid \omega_p \neq 0\}}.$$

We denote by $\Omega^k_c(M)$ the subspace of k-forms on M with compact support.

Integration of n-forms on $\mathbb{R}^n$. We first define the integral of a compactly supported continuous n-form ω on an open set $U \subseteq \mathbb{R}^n$. Any such form can be written uniquely as $\omega = h\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{n-1}$ where $x^0, \dots, x^{n-1}$ are the standard coordinates and $h : U \to \mathbb{R}$ is a compactly supported continuous function.

Definition 7.2.1 (Integration of n-forms on $\mathbb{R}^n$)**.** The **integral** of a compactly supported continuous n-form $\omega = h\,\mathrm{d}x^0 \wedge\cdots\wedge \mathrm{d}x^{n-1}$ on an open set $U \subseteq \mathbb{R}^n$ is

$$\int_U \omega = \int_U h\,\mathrm{d}x^0 \wedge\cdots\wedge \mathrm{d}x^{n-1} := \int_U h \tag{7.4}$$

where the integral on the right is the Riemann integral of h. (It is well defined since any compactly supported continuous function $h : U \to \mathbb{R}$ is Riemann integrable.)

Integration of n-forms supported in a single chart.

Definition 7.2.2. Let M be an oriented differentiable n-manifold, and let ω be a continuous n-form on M whose support is compact and contained in the image of a local parametrization $\varphi : U \to M$ that has constant sign $\mathrm{sgn}\,\varphi = \pm 1$ w.r.t. the orientation of M. Then we define the **integral** of ω as

$$\int_M \omega := \mathrm{sgn}\,\varphi \int_U \varphi^*\omega. \tag{7.5}$$

Proposition 7.2.3. *The result does not depend on the parametrization φ.*

Proof. We consider a second local parametrization $\psi : V \to M$. Note that both sets $\varphi(U)$ and $\psi(V)$ contain $\operatorname{supp}\omega$, and by shrinking the domains U, V, we may assume that $\varphi(U) = \psi(V)$.[1] We write $\psi = \varphi \circ \sigma$, where $\sigma = \varphi^{-1} \circ \psi : \widetilde{V} \to \widetilde{U}$ is a diffeomorphism. Note that $\operatorname{sgn}\psi = \operatorname{sgn}\varphi \cdot \operatorname{sgn}(\det \mathrm{D}\sigma)$.

We write $\varphi^*\omega = h\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$ and $\psi^*\omega = g\,\mathrm{d}y^0 \wedge \cdots \wedge \mathrm{d}y^{n-1}$. Recall that we have $g(y) = h(\sigma(y)) \det(\mathrm{D}_y\sigma)$ for all $y \in V$ by Proposition 6.4.7.

Using the change of variables theorem, we get

$$\begin{aligned}
\operatorname{sgn}\varphi \int_U \varphi^*\omega &= \operatorname{sgn}\varphi \int_U h(x)\,\mathrm{d}x^0 \cdots \mathrm{d}x^{n-1} \\
&= \operatorname{sgn}\varphi \int_V h(\sigma(y))\,|\det \mathrm{D}_x\sigma|\,\mathrm{d}y^0 \cdots \mathrm{d}y^{n-1} \\
&= \operatorname{sgn}\psi \int_V h(\sigma(y))\,(\det \mathrm{D}_x\sigma)\,\mathrm{d}y^0 \cdots \mathrm{d}y^{n-1} \\
&= \operatorname{sgn}\psi \int_V g(y)\,\mathrm{d}y^0 \cdots \mathrm{d}y^{n-1} \\
&= \operatorname{sgn}\psi \int_V \psi^*(\omega).
\end{aligned}$$

□

Integration in general. For an arbitrary compactly supported n-form on an oriented n-manifold we can use a partition of unity to reduce to the previous setting:

Definition 7.2.4. Let M be an oriented differentiable n-manifold, and let ω be a continuous, compactly supported n-form on M. We define $\int_M \omega$, the **integral of** ω **over** M, as follows. Take any finite collection of constant-signed local parametrizations $\varphi_i : \widetilde{U}_i \to U_i$ whose images cover $\operatorname{supp}\omega$, and let $(\chi_i)_i$ be a partition of unity of $U = \bigcup_i U_i$ subordinate to the cover $(U_i)_i$. Then set

$$\int_M \omega := \sum_i \int_M \chi_i\,\omega$$

where $\int_M \chi_i\,\omega = \operatorname{sgn}(\varphi_i) \int_{\widetilde{U}_i} \varphi_i^*(\chi_i\,\omega)$ as defined in (7.5).

Note that a collection of local parametrizations φ_i with the above properties exists: it suffices to take an atlas of M consisting of connected (and hence constant-signed) local parametrizations of M, and then reduce to a finite subset using the compactness of $\operatorname{supp}\omega$. What remains to be shown is:

Proposition 7.2.5. *The above definition is independent of the choice of charts and partition of unity.*

Proof. Consider another finite family of constant-signed local parametrizations $\psi_j : \widetilde{V}_j \to V_j$ whose images cover of supp ω by oriented charts $\psi_j \in \mathcal{A}$. Let ξ_j be a partition of unity subordinate to $(V_j)_j$. Then

$$\sum_j \int_M \xi_j\omega = \sum_j \int_M \sum_i \chi_i\,\xi_j\,\omega = \sum_{i,j} \int_M \chi_i\,\xi_j\,\omega$$

[1] To *shrink the domains* means the following: we note that $\operatorname{supp}\omega$ is contained in the open set $W = \varphi(U) \cap \psi(V)$. Thus by restricting the parametrizations φ and ψ to the sets $\widetilde{U} = \varphi^{-1}(W)$ and $\widetilde{V} = \psi^{-1}(W)$ respectively, we get two new parametrizations $\widetilde{\varphi} := \varphi|_{\widetilde{U}}$ and $\widetilde{\psi} := \psi|_{\widetilde{V}}$ such that $\widetilde{\varphi}(\widetilde{U}) = W = \widetilde{\psi}(\widetilde{V})$.

where the second and third expression are integrals of n-forms supported in a single chart as defined previously. In the same way we have

$$\sum_i \int_M \chi_i \, \omega = \sum_i \int_M \sum_j \xi_j \, \chi_i \, \omega = \sum_{i,j} \int_M \xi_j \, \chi_i \, \omega.$$

The right hand sides of the last two relations are the same and therefore $\sum_j \int_M \xi_j \omega = \sum_i \int_M \chi_i \omega$. □

Proposition 7.2.6 (Properties of the integral)**.** *Let M be an oriented differentiable n-manifold and let ω, η be two continuous, compactly supported n-forms on M.*

(a) Linearity: If $a, b \in \mathbb{R}$, then

$$\int_M (a\,\omega + b\,\eta) = a \int_M \omega + b \int_M \eta.$$

(b) Positivity: If $\operatorname{sgn}(\omega|_p)$ coincides with the orientation of M at every point $p \in M$ where $\omega|_p \neq 0$, then $\int_M \omega \geq 0$, and the inequality is strict unless ω is identically zero.

(c) Diffeomorphism invariance: If $f : N \to M$ is an diffeomorphism of constant sign $\operatorname{sgn}(f) = \pm 1$ (i.e. f is either orientation preserving or orientation reversing), then

$$\int_N f^* \omega = \operatorname{sgn} f \cdot \int_M \omega.$$

(d) Orientation reversal: If $-M$ denotes M with the reversed orientation, then

$$\int_{-M} \omega = - \int_M \omega.$$

We leave the proof as exercise.

7.3 Manifolds with boundary

In the same way that an n-manifold is modeled on $\mathbb{R}^n$ (i.e. is locally isomorphic to $\mathbb{R}^n$), a *manifold with boundary*, to be defined below, is modeled on the **closed upper half space**

$$\mathbb{H}^n := \{x = (x^0, \dots, x^{n-1}) \in \mathbb{R}^n \mid x^{n-1} \geq 0\} \quad \text{where } n \geq 1.$$

This model space is partitioned into two sets: the **interior** $\operatorname{Int} \mathbb{H}^n := \{x \in \mathbb{H}^n \mid x^{n-1} > 0\}$ and the **boundary** $\partial \mathbb{H}^n := \{x \in \mathbb{H}^n \mid x^{n-1} = 0\}$. We identify $\partial \mathbb{H}^n \equiv \mathbb{R}^{n-1}$ using the bijection $\mathbb{R}^{n-1} \to \partial \mathbb{H}^n : (x^0, \dots, x^{n-2}) \mapsto (x^0, \dots, x^{n-2}, 0)$.

To work with functions defined on $\mathbb{H}^n$, we use the following definition:

Definition 7.3.1. A function $f : A \to \mathbb{R}^n$ defined on any set $A \subseteq \mathbb{R}^m$ is $\mathcal{C}^k$ if it can be extended to a $\mathcal{C}^k$ function defined on an open neighborhood of A.

Note that a $\mathcal{C}^k$ function defined on an open subset of $\mathbb{H}^n$ has its partial derivatives of order $\leq k$ well defined (independent of the extension).

Example 7.3.2. The map $g : [0, \infty) \to \mathbb{R} : y \mapsto \sqrt{y}$ is *not* smooth in this sense, since $g'(y) \to \infty$ as $y \to 0$, so g cannot be extended smoothly to an open subset containing $[0, \infty)$.

Definition 7.3.3. An C^r **manifold with boundary** of dimension n is a second countable Hausdorff space M, together with a maximal C^r atlas $\mathcal{A}$ consisting of charts $\varphi : U \to \widetilde{U}$ (where $U \subseteq M$ and $\widetilde{U} \subseteq \mathbb{H}^n$ are open sets) whose transition maps $\psi \circ \varphi^{-1}$ are C^r for $\varphi, \psi \in \mathcal{A}$.

The **boundary** of M, denoted ∂M, is the set of points of M that are in the inverse image of $\partial\mathbb{H}^n$ by some chart of the atlas.

The **interior** of M, denoted $\operatorname{Int} M$, is the set of points of M that are in the inverse image of $\operatorname{Int}\mathbb{H}^n$ by some chart of the atlas.

Remark 7.3.4. The "interior" and "boundary" of a topological manifold M in the sense above should not to be confused with the interior and boundary of M viewed as a subset of some other space (e.g. $\mathbb{R}^n$). For example, $[0,1] \times \{0\} \subset \mathbb{R}^2$ with the standard topology is a topological manifold with boundary and its boundary (in the manifold sense) is $\{(0,0),(1,0)\}$.

A map $f : M \to N$ between $\mathcal{C}^k$ manifolds with boundary is $\mathcal{C}^k$ iff its local expressions are $\mathcal{C}^k$ in the sense of Def. 7.3.1.

Proposition 7.3.5. *Let M be a $\mathcal{C}^r$ differentiable n-manifold with boundary ($n \geq 1$). Then $\operatorname{Int} M$ and ∂M are complementary subsets of M. Moreover, endowed with the subspace topology, these sets have natural structures of $\mathcal{C}^r$ manifolds (without boundary) of dimension n and $n-1$ respectively, and the inclusion maps into M are $\mathcal{C}^r$.*

Proof. It is clear that $\operatorname{Int} M \cup \partial M = M$. We leave as an exercise proving that $\operatorname{Int} M \cap \partial M = \emptyset$ and that ∂M is a closed subset of M.

Each chart $\varphi : U \to \widetilde{U}$ of M can be split into an "interior part" $\operatorname{Int}\varphi := \varphi|_{U \cap \operatorname{Int} M}$ and a "boundary part" $\partial\varphi := \varphi|_{\partial M}$. The interior part $\operatorname{Int}\varphi$ has image $\widetilde{U} \cap \operatorname{Int}\mathbb{H}^n$, which is an open subset of $\mathbb{R}^n$. Thus the interior parts of the charts of M constitute a $\mathcal{C}^r$ atlas for $\operatorname{Int} M$. Similarly, the boundary part $\partial\varphi$ has image $\widetilde{U} \cap \partial\mathbb{H}^n$, and since $\partial\mathbb{H}^n \equiv \mathbb{R}^{n-1}$, the boundary parts of the charts form a $\mathcal{C}^r$ atlas of ∂M of dimension $n-1$.

To see that the inclusion map $\iota : \operatorname{Int} M \to M$ is $\mathcal{C}^r$, let $p \in \operatorname{Int} M$ and let φ be a chart of M defined at p. Then the local expression of ι w.r.t. the charts $\operatorname{Int}\varphi$, φ is the inclusion map of $\widetilde{U} \cap \operatorname{Int}\mathbb{H}^n$ into $\widetilde{U}$, and this map is clearly $\mathcal{C}^r$. Thus ι is $\mathcal{C}^r$. The proof that the inclusion map $\partial M \to M$ is $\mathcal{C}^r$ is similar. ☐

Example 7.3.6. The closed interval $[0,1]$ is a smooth manifold with boundary with the standard topology and the smooth structure given by the two charts

$$[0,1) \to [0,1) : x \mapsto x, \qquad (0,1] \to [0,1) : x \mapsto 1-x.$$

Example 7.3.7. Consider the closed unit ball $B := \{x \in \mathbb{R}^n, |x| \leq 1\}$, where $|x| := \sqrt{\sum_i x_i^2}$, with the subspace topology. Let

$$U_i^{\pm} := \{x \in B \mid \pm x_i > 0\}.$$

On U_i^+ we define a ∂-chart $\varphi_i^+ : U_i^+ \to \mathbb{H}^n$ that maps

$$(x_0, \ldots, x_{n-1}) \mapsto (x_0, \ldots, x_{i-1}, -x_i + \sqrt{1 - \sum_{j \neq i} x_j^2}, x_{i+1}, \ldots, x_{n-1}).$$

Similarly we can define ∂-charts φ_i^- on the sets U_i^- (just change the sign of the i-th component).

It is a simple computation to check that the maps $\varphi_i^{\pm}$ form a smooth ∂-atlas of B. Note also that $\partial B = \mathbb{S}^{n-1}$.

Extension of known concepts to manifolds with boundary All the notions that we studied on differentiable manifolds can be extended, with few changes, to differentiable manifolds with boundary.

(a) C^r maps between manifolds with boundary are already defined above.

(b) Partitions of unity exist on manifolds with boundary; they are defined in the same way as on manifolds without boundary.

(c) The tangent space T_pM at a point $p \in M$ of a differentiable n-manifold with boundary M is defined also in the same way as for manifolds without boundary, and we have $\mathrm{T}_pM \simeq \mathbb{R}^n$.

A tangent vector $X \in \mathrm{T}_pM$ at a boundary point $p \in \partial M$ is classified as **inwards pointing**, **tangent to the boundary**, or **outwards pointing** according to the sign (positive, zero, negative respectively) of the last coordinate X^{n-1} in the expression $X = \sum_i X^i \left.\frac{\partial}{\partial \varphi^i}\right|_p$ with respect to a ∂-chart φ. (Exercise: This sign does not depend on the chart.)

(d) A vector field $X \in \mathfrak{X}(M)$ on a differentiable manifold with boundary M defines a flow Φ_X, whose domain is open if X is tangent to the boundary of M. If X is inwards-pointing along ∂M, then the future flow has open domain, and is complete if X has compact support.

(Exercise: Every $\mathcal{C}^{k+1}$ manifold has a strictly inwards-pointing, $\mathcal{C}^k$ vector field.)

(e) Differential forms, orientations and integration can also be extended to manifolds with boundary without significant changes.

(Exercise: Every orientation on $\operatorname{Int} M$ extends to a unique orientation on M.)

(f) The definitions of immersion and embedding are also easily extended. The inclusion maps of $\operatorname{Int} M$ and ∂M into M are embeddings.

Definition 7.3.8 (Induced orientation on the boundary)**.** Let M be a differentiable $(n+1)$-manifold with boundary. Every orientation $\mathcal{O}$ on M induces an orientation $\partial\mathcal{O}$ on the boundary ∂M, called the **boundary** of $\mathcal{O}$, defined as follows. For a point $p \in \partial M$,

$$(\partial\mathcal{O})|_p(X_0, \ldots, X_{n-1}) = \mathcal{O}|_p(Y, X_0, \ldots, X_{n-1})$$

for each base $(X^0, \ldots, X^{n-1})$ of $\mathrm{T}_p\partial M$, where $Y \in \mathrm{T}_pM$ is an outward-pointing tangent vector.

The definition of $(\partial\mathcal{O})|_p$ is independent of the choice of Y because if $Y' \in \mathrm{T}_pM$ is any other outward pointing tangent vector, then $Y' = kY + \sum_i c^i X_i$ for some coefficients $c^i \in \mathbb{R}$, $k > 0$, therefore the operation of replacing Y by Y' does not change the sign of the base $(Y, X_0, \ldots, X_{n-1})$ of T_pM.

Exercise 7.3.9. Show that the orientation field $\partial\mathcal{O}$ is continuous.

If M is an oriented 1-manifold, then ∂M has dimension zero, and the boundary orientation at a point $p \in \partial M$ is equal to the sign of an outward-pointing vector at p.

Example 7.3.10. If $(M, \mathcal{O})$ is a closed interval $[a, b] \subseteq \mathbb{R}$ endowed with the standard orientation of $\mathbb{R}$, then $\partial\mathcal{O}(a) = -1$ and $\partial\mathcal{O}(b) = +1$.

Example 7.3.11. $\mathbb{S}^2$ has an induced orientation as the boundary of the unit ball in $\mathbb{R}^3$. This orientation is given by the 2-form in Example 7.1.13.

Example 7.3.12 (Boundary orientation of the closed half-space *)**.** Consider $\mathbb{H}^n$ with the standard orientation of $\mathbb{R}^n$. Identifying $\partial\mathbb{H}^n \to \mathbb{R}^{n-1}, (x^0, \ldots, x^{n-2}, 0) \mapsto (x^0, \ldots, x^{n-2})$, the induced orientation on $\partial\mathbb{H}^n$ corresponds to the standard orientation on $\mathbb{R}^{n-1}$ iff n is even. Hence with respect to the induced orientation on $\partial\mathbb{H}^n$, the $(n-1)$-form

$$\omega = (-1)^n \, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-2} \in \Omega^{n-1}$$

is positive, i.e. $\omega|_p(B_0, \ldots, B_{n-2}) > 0$ if $(B_0, \ldots, B_{n-2})$ is a positive base of $\mathrm{T}_p(\partial\mathbb{H}^n)$.

7.4 Differential forms as integrands

An important property of a continuous k-form ω on a manifold is that ω is determined by the value of its k-dimensional integrals.

Proposition 7.4.1. *Let ω, θ be two continuous k-forms on a differentiable manifold M. Then $\omega = \theta$ if and only if $\int_N f^*\omega = \int_N f^*\theta$ for every differentiable map $f : N \to M$ whose domain N is a compact oriented k-manifold with boundary.*

Proof: Exercise.

8 Exterior derivative and Stokes' theorem*

At the end of last chapter we saw that a (continuous) k-form ω on a manifold M is determined by the value of its k-dimensional integrals. In this chapter we will show that if the k-form ω is $\mathcal{C}^1$, then it has a naturally associated $(k+1)$-form called the *exterior derivative* of ω, denoted $\mathrm{d}\omega$, whose integrals are determined by those of ω by the *Stokes formula*:

Proposition 8.0.1 (Stokes formula)**.** *Let ω be a k-form on a manifold M. If ω is $\mathcal{C}^1$, then*

$$\int_N f^* \mathrm{d}\omega = \int_{\partial N} f^*\omega$$

for any $\mathcal{C}^2$ map $f : N \to M$, where N is an oriented $(k+1)$-manifold with boundary such that $f^*\omega$ has compact support.

We could take the Stokes formula as the definition of $\mathrm{d}\omega$ since by Proposition 7.4.1, any continuous $k+1$-form is determined by the value of its integrals. However, we will not define $\mathrm{d}\omega$ in this way. Instead, we'll give a less intuitive but more practical definition of $\mathrm{d}\omega$ that tells us directly how to compute $\mathrm{d}\omega$ in terms of its component functions in some local chart. Then we will prove that $\mathrm{d}\omega$ satisfies the Stokes formula.

8.1 Exterior derivative

For simpler notation, here we restrict to smooth manifolds and differential forms.

Given a chart (U, φ) of a smooth n-manifold M, we define a map

$$\mathrm{d}_\varphi : \Omega^k(U) \to \Omega^{k+1}(U) : \omega = \sum_I{}' \omega_I \, \mathrm{d}\varphi^I \mapsto \mathrm{d}_\varphi\omega := \sum_I{}' \, \mathrm{d}\omega_I \wedge \mathrm{d}\varphi^I. \tag{8.1}$$

Here, each index I in a primed sum $\sum_I{}'$ is an increasing multi-index[1], $I = (i_0, \ldots, i_{k-1})$, and $\mathrm{d}\omega_I$ is the differential of the component function ω_I as defined previously. The definition of d_φ is local. We want to define a "global" operator d, i.e. $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$, by covering M with charts and setting $(\mathrm{d}\omega)_p := (\mathrm{d}_U\omega|_U)_p$ for some chart (U, φ) containing p. For this we have to make sure that for two charts (U, φ), (V, ψ), the forms $\mathrm{d}_\varphi(\omega|_U)$ and $\mathrm{d}_\psi(\omega|_V)$ coincide on $U \cap V$. This is part of the proof of the following theorem:

Theorem 8.1.1. *Let M be a smooth manifold. There exists a unique linear operator*

$$\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$$

(i.e. one operator for each $k \geq 0$), called the ***exterior derivative*** *operator, satisfying the following conditions:*

(i) If $f \in \Omega^0(M) = C^\infty(M)$ then $\mathrm{d}f$ is the differential of f as defined previously, i.e. $\mathrm{d}f(X) = Xf$.

[1] For $k = 0$ we use the convention that $\mathrm{d}\varphi^\emptyset$ is the constant function 1.

(ii) For $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta.$$

(iii) $\mathrm{d}^2 = 0$.

This operator d *is given in local coordinates by* (8.1).

Proof. Let us first suppose that M is covered by a single chart (U, φ). Then any operator d satisfying conditions (i)-(iii) must be equal to d_φ. Indeed, for a form $\omega \in \Omega^k(M)$, writing $\omega = \sum_I{}' \,\mathrm{d}\varphi^I$, we have

$$\mathrm{d}\omega = \sum_I{}' \,\mathrm{d}\omega_I \wedge \mathrm{d}\varphi^I + \omega_I \wedge \mathrm{d}(\mathrm{d}\varphi^I) = \mathrm{d}_\varphi \omega \tag{8.2}$$

because

$$\mathrm{d}(\mathrm{d}\varphi^I) = \mathrm{d}(\mathrm{d}\varphi^{i_0} \wedge \cdots \wedge \mathrm{d}\varphi^{i_{k-1}}) = \sum_{s \in \underline{k}} (-1)^s \,\mathrm{d}\varphi^{i_0} \wedge \cdots \wedge \mathrm{d}(\mathrm{d}\varphi^{i_s}) \wedge \cdots \wedge \mathrm{d}\varphi^{i_{k-1}} = 0.$$

A few straightforward calculations show that d_φ as defined in (8.1) satisfies conditions (i)-(iii); see e.g. [Lee13, Prop. 14.23].

Note that if ψ is a second chart with the same domain U as φ, then the operator d_φ and d_ψ are equal, because they both satisfy conditions (i)-(iii). Also, for any open subset $W \subseteq U$, the map $\varphi|_W$ is a chart on W and it is clear from the formula (8.1) that for any k-form $\omega \in \Omega^k(U)$ we have $(\mathrm{d}_\varphi \omega)|_W = \mathrm{d}_{\varphi|_W}(\omega|_W)$.

Now consider the general case, i.e. M is perhaps not covered by a single chart. Suppose that d is an operator satisfying (i)-(iii). We claim that this operator is *local* in the sense that if two k-forms ω, $\widetilde{\omega} \in \Omega^k(M)$ coincide on a neighborhood V of a point $p \in M$, then $(\mathrm{d}\omega)_p = (\mathrm{d}\widetilde{\omega})_p$. To see this, let α be a bump function with support in V and which is constant equal to 1 on a (smaller) neighborhood of p. Let $\eta = \omega - \widetilde{\omega}$. Note that $\alpha\,\eta = 0$ on M, so $\mathrm{d}(\alpha\,\eta) = 0$ by the linearity of d. Then using the other properties of d, we see that

$$0 = (\mathrm{d}(\alpha\,\eta))_p = (\mathrm{d}\alpha)_p \wedge \eta + \alpha(p)(\mathrm{d}\eta)_p = (\mathrm{d}\eta)_p.$$

Here we used the fact that $(\mathrm{d}\alpha)_p = 0$ since α is constant in a neighborhood of p.

We continue in the setting where we suppose the existence of an operator d satisfying (i)-(iii). Let us prove that d is unique by showing that it is given by the local formula (8.1), i.e. for any chart (U, φ) and any k-form $\omega \in \Omega^k(M)$ we have $(\mathrm{d}\omega)|_U = \mathrm{d}_\varphi(\omega|_U)$. To see this, take any point $p \in U$. Write $\omega|_U = \sum_I{}' \,\omega_I \,\mathrm{d}\varphi^I$, and let $\widetilde{\omega}_I$ and $\widetilde{\varphi}^i$ be smooth functions defined on the whole manifold M that coincide with ω_I and φ^i resp. on a neighborhood of p. Let

$$\widetilde{\omega} := \sum_I{}' \,\widetilde{\omega}_I \,\mathrm{d}\widetilde{\varphi}^I \in \Omega^k(M).$$

Then by the locality of d and properties (i)-(iii), computing as in (8.2) we get

$$\begin{aligned}(\mathrm{d}\omega)_p = (\mathrm{d}\widetilde{\omega})_p &= \left(\sum_I{}' \,\mathrm{d}\widetilde{\omega}_I \wedge \mathrm{d}\widetilde{\varphi}^I + \omega_I \wedge \mathrm{d}(\mathrm{d}\widetilde{\varphi}^I)\right)_p \\ &= \sum_I{}' \,(\mathrm{d}\omega_I)_p \wedge \mathrm{d}\varphi^I_p = (\mathrm{d}_\varphi(\omega|_U))_p.\end{aligned}$$

Since this holds for each point $p \in U$, we conclude that $(\mathrm{d}\omega)|_U = \mathrm{d}_\varphi(\omega|_U)$ as claimed.

Finally, let us show the existence of the operator d satisfying (i)-(iii). For a form $\omega \in \Omega^k(M)$, we define the form $\mathrm{d}\omega \in \Omega^{k+1}(M)$ as follows: for any chart (U, φ), let

$(\mathrm{d}\omega)|_U = \mathrm{d}_\varphi(\omega|_U)$. The form $\mathrm{d}\omega$ is well defined because if (U,φ), (V,ψ) are two charts, then on the intersection $W = U \cap V$ the operators d_φ and d_ψ coincide as explained before. More precisely, we have

$$(\mathrm{d}_\varphi(\omega|_U))|_W = \mathrm{d}_{\varphi|_W}(\omega|_W) = \mathrm{d}_{\psi|_W}(\omega|_W) = (\mathrm{d}_\psi|_W(\omega|_U))|_W.$$

Thus we have defined an operator $\mathrm{d} : \Omega^k M \to \Omega^{k+1} M$. The fact that this operator satisfies conditions (i)-(iii) (and linearity) follows from the fact that the local operators $\mathrm{d}|_\varphi$ also do. For example, to verify the product property (ii), we take two forms $\omega \in \Omega^k M$, $\eta \in \Omega^\ell M$ and we note that for any chart (U,φ) we have

$$\begin{aligned}
(\mathrm{d}(\omega \wedge \eta))|_U &= \mathrm{d}|_\varphi(\omega \wedge \eta)|_U \\
&= (\mathrm{d}|_\varphi \omega|_U) \wedge \eta|_U + (-1)^k \omega|_U \wedge (\mathrm{d}|_\varphi \eta|_U) \\
&= (\mathrm{d}\omega)|_U \wedge \eta|_U + (-1)^k \omega|_U \wedge (\mathrm{d}\eta)|_U \\
&= (\mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta)|_U.
\end{aligned}$$

The other properties are proved similarly. □

In the exercises we prove that the exterior derivative operator commutes with pullbacks :

Lemma 8.1.2. *Let $F : M \to N$ be a smooth map between smooth manifolds. Then for all $\omega \in \Omega^k(M)$ we have*

$$F^*(\mathrm{d}\omega) = \mathrm{d}(F^*\omega).$$

8.2 Stokes' Theorem

The concepts developed in the last few sections blend together nicely to yield the following theorem:

Theorem 8.2.1 (Stokes)**.** *Let M be an oriented n-manifold with boundary and let $\omega \in \Omega_c^{n-1}(M)$. Then*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

Here ∂M has the induced orientation and ω on the right hand side is understood to be restricted to ∂M (i.e. we actually mean the form on ∂M given by $\iota^*\omega$ where $\iota : \partial M \to M$ is the inclusion, also written $\omega|_{\partial M}$).

Remark 8.2.2. Stokes' theorem generalizes the fundamental theorem of calculus: For $\omega = f \in C^0([a,b])$ where $[a,b] \subset \mathbb{R}$ is an interval (with the standard orientation) we obtain

$$\int_{[a,b]} f'(x)\,\mathrm{d}x = \int_{[a,b]} \mathrm{d}f = \int_{\partial([a,b])} f = f(b) - f(a).$$

where we recall that the induced boundary orientation on $\partial([a,b])$ is $+1$ at b and -1 at a.

Proof of Stokes' theorem (sketch). Covering M with an oriented atlas and using a partition of unity, the result can be obtained from the one for $M = \mathbb{H}^n$. In this case $\omega \in \Omega_c^{n-1}(\mathbb{H}^n)$ can be written

$$\omega = \sum_{0 \le i < n} \omega_i \,\mathrm{d}x^0 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^{n-1}$$

where the hat means that we omit the corresponding term and x^i are the standard coordinates on $\mathbb{R}^n$. Since ω has compact support, for some $R > 0$ we have

$$\operatorname{supp} \omega \subset (-R, R) \times \cdots \times (-R, R) \times [0, R).$$

Note that

$$\mathrm{d}\omega = \sum_{0 \le i < n} (-1)^i \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^0 \wedge \cdots\cdots \wedge \mathrm{d}x^{n-1}.$$

Therefore

$$\begin{aligned}\int_{\mathbb{H}^n} \mathrm{d}\omega &= \int_{\mathbb{H}^n} \sum_{0 \le i < n} (-1)^i \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^0 \cdots \mathrm{d}x^{n-1} = \\ &= \sum_{0 \le i < n} (-1)^i \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i} \, \mathrm{d}x^0 \cdots \mathrm{d}x^{n-1}\end{aligned}$$

and using the fundamental theorem of calculus we see that all the terms in the sum where $i \neq n-1$ are 0 because $\omega_i(x) = 0$ if $x^i = \pm R$. Hence the integral above becomes

$$\begin{aligned}\int_{\mathbb{H}^n} \mathrm{d}\omega &= (-1)^{n-1} \int_{-R}^R \cdots \int_{-R}^R \left(\underbrace{\omega_n(x^0, \ldots, x^{n-2}, R)}_{=0} - \omega_{n-1}(x^0, \ldots, x^{n-2}, 0) \right) \mathrm{d}x^0 \cdots \mathrm{d}x^{n-2} \\ &= (-1)^n \int_{-R}^R \cdots \int_{-R}^R \omega_{n-1}(x^0, \ldots, x^{n-2}, 0) \, \mathrm{d}x^0 \cdots \mathrm{d}x^{n-2} = \\ &= \int_{\partial \mathbb{H}^n} \omega_{n-1}(x^0, \ldots, x^{n-2}, 0) \, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-2}\end{aligned}$$

The factor $(-1)^n$ disappears because $\partial \mathbb{H}^n$ has the boundary orientation, so $(-1)^n \, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-2}$ is a positive n-form (see Example 7.3.12). Note that the restriction of ω to $\partial \mathbb{H}^n$ is (for $\iota : \partial \mathbb{H}^n \to \mathbb{H}^n$ the inclusion)

$$\iota^* \omega = (\omega_{n-1} \circ \iota) \, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-2}$$

since all the other terms of ω contain the factor $\mathrm{d}x^{n-1}$, which vanishes on $\partial \mathbb{H}^n$. Hence the last line above is $\int_{\partial \mathbb{H}^n} \iota^* \omega$, which we write as $\int_{\partial \mathbb{H}^n} \omega$ by the common abuse of notation of not writing the restriction. $\square$

8.2.1 Classical vector calculus

Example 8.2.3. Let us relate d with some well-known differential operators on $\mathbb{R}^3$: We can identify $\Omega^1(\mathbb{R}^3)$ with smooth functions $\mathbb{R}^3 \to \mathbb{R}^3$ via

$$f_1 \, \mathrm{d}\varphi^1 + f_2 \, \mathrm{d}\varphi^2 + f_3 \, \mathrm{d}\varphi^3 \mapsto (f_1, f_2, f_3)$$

where φ^i are the standard coordinates on $\mathbb{R}^3$ and $f_i \in C^\infty(\mathbb{R}^3)$. Similarly, we can identify $\Omega^2(\mathbb{R}^3)$ with smooth functions $\mathbb{R}^3 \to \mathbb{R}^3$ via

$$f_1 \, \mathrm{d}\varphi^2 \wedge \mathrm{d}\varphi^3 + f_2 \, \mathrm{d}\varphi^3 \wedge \mathrm{d}\varphi^1 + f_3 \, \mathrm{d}\varphi^1 \wedge \mathrm{d}\varphi^2 \mapsto (f_1, f_2, f_3).$$

Finally $\Omega^3(\mathbb{R}^3)$ can be identified with $C^\infty(\mathbb{R}^3)$ via

$$f \, \mathrm{d}\varphi^1 \wedge \mathrm{d}\varphi^2 \wedge \mathrm{d}\varphi^3 \mapsto f.$$

Using these identifications, $\mathrm{d} : \Omega^0(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$ corresponds to the gradient; $\mathrm{d} : \Omega^1(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ corresponds to the curl (aka rotation); and $\mathrm{d} : \Omega^2(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$ corresponds to the divergence. The property $\mathrm{d}^2 = 0$ then translates into

$$\text{curl grad} = 0, \qquad \text{div curl} = 0.$$

Example 8.2.4 (Classical Stokes Theorem)**.** Let $M \subset \mathbb{R}^3$ be a smooth embedded submanifold with boundary: This means, just as in the case of ordinary manifolds, that M has the subspace topology and a smooth structure such that the inclusion is a smooth immersion.

Suppose $\dim M = 2$ (we call this a hypersurface). Then ∂M has dimension one. Consider a 1-form $\omega = f_1 dx^1 + f_2 dx^2 + f_3 dx^3 \in \Omega_c^1(\mathbb{R}^3)$. Then $d\omega$ is a 2-form on $\mathbb{R}^3$, which restricts to a 2-form on M, and similarly ω restricts to a 1-form on ∂M (abusing notation we denote the restrictions with the same symbol). By Stokes $\int_M d\omega = \int_{\partial M} \omega$. Setting $F := (f_1, f_2, f_3)$ and $ds := (dx^1, dx^2, dx^3)$, we can write ω as the formal scalar product, $\omega = F \cdot ds$ and similarly, setting $dS = (dx^2 \wedge dx^3, dx^3 \wedge dx^1, dx^1 \wedge dx^2)$ we have by Example 8.2.3: $d\omega = \operatorname{curl} F \cdot dS$. Hence we can write

$$\int_M \operatorname{curl} F \cdot dS = \int_{\partial M} F \cdot ds.$$

The left hand side is the so-called *surface integral* of the vector field $\operatorname{curl} F$ over the surface M. (See [Lee13], Chapter 10, *Integration on Riemannian manifolds*, for more information about the induced volume form on hypersurfaces in $\mathbb{R}^3$.)