Potential Theory

2nd part

Outlines

Spherical systems:

- Circular speed, circular velocity, circular frequency, escape speed, potential energy

Examples of spherical models:

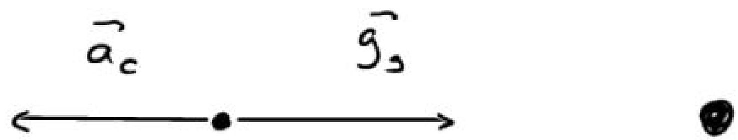
- “Potential based” models
- “Density based” models

Axisymmetric models for disk galaxies

- “Potential based” models
- Potential of flattened systems
- Potential of infinite thin (razor-thin) disks
- Potential of spheroidal shells (homoeoids)
- Potential of infinite thin (razor-thin) disks from homoeoids

Spherical systems : circular speed, circular velocity

Speed of a test particle in a cirular orbit in the potential $\phi(r)$ at a radius r :



$\vec{a}_c$: centripetal acceleration $\qquad \dfrac{V_c^2}{r}$

$\vec{g}_s$: gravity acceleration (spec force) $\qquad -\dfrac{GM(r)}{r^2} = -\dfrac{\partial\phi}{\partial r}$

$$V_c^2 = \frac{GM(r)}{r}$$

$$V_c^2 = r\,\frac{\partial\phi}{\partial r}$$

$[V_c^2] : \dfrac{erg}{g}$

as ϕ

$\equiv$ specific energy

$$GM(r) = r^2\,\frac{\partial\phi}{\partial r}$$

Velocity composition

<u>Note</u>: V_0^2 scale with the mass ($M(r)$) : it is thus the "important" quantity (spec. energy)

Multi-components system : ex: bulge + stellar halo + DM halo

$$\begin{cases} \rho_B(r), & M_B(r), & \phi_B(r) & \rightarrow & V_{c,B}(r) \\ \rho_H(r), & M_H(r), & \phi_H(r) & \rightarrow & V_{c,H}(r) \\ \rho_{DM}(r), & M_{DM}(r), & \phi_{DM}(r) & \rightarrow & V_{c,DM}(r) \end{cases}$$

$$V_{c,tot}^2 = \frac{G M_{tot}(r)}{r} = \frac{G}{r} \sum_i M(r)$$

$$\boxed{V_{c,tot}^2 = \sum_i V_{c,i}^2}$$

$V_c^2 \sim$ energy : extensive quantity

Period of the circular orbit

$$T(r) = \frac{2\pi r}{V_c(r)} = 2\pi\sqrt{\frac{r^3}{GM(r)}} = 2\pi\sqrt{\frac{r}{\frac{\partial \phi}{\partial r}}}$$

Circular frequency (angular frequency)

$$\Omega(r) = \frac{2\pi}{T(r)} = \sqrt{\frac{GM(r)}{r^3}} = \sqrt{\frac{1}{r}\frac{\partial \phi}{\partial r}}$$

Escape speed V_e

if $\frac{1}{2}V_e^2 > \phi(r) = E > 0$

the partcle may escape the system

$$V_e(r) = \sqrt{2|\phi(r)|}$$

Potential energy

from $$W = -\int \rho(\vec{x})\, \vec{x} \cdot \vec{\nabla}\phi(\vec{x})\, d^3\vec{x}$$

$$\boxed{W = -4\pi G \int_0^\infty \rho(r)\, M(r)\, r\, dr}$$

Spherical systems : useful relations

	$\rho(r)$	$\Phi(r)$	$M(r)$	$\frac{d\Phi}{dr}$
$\rho(r)$	$\rho(r)$	$\frac{1}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right)$	$\frac{1}{4\pi r^2}\frac{dM(r)}{dr}$	$\frac{1}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right)$
$\Phi(r)$	$-\frac{G\,M(r)}{r} - 4\pi G\int_r^\infty dr'\,r'\,\rho(r')$	$\Phi(r)$	$-G\int_r^\infty dr'\,\frac{M(r')}{r'^2}$	$-\int_r^\infty dr'\,\frac{d\Phi}{dr}$
$M(r)$	$4\pi\int_0^r dr'\,r'^2\,\rho(r')$	$\frac{r^2}{G}\frac{d\Phi}{dr}$	$M(r)$	$\frac{r^2}{G}\frac{d\Phi}{dr}$
$\frac{d\Phi}{dr}$	$\frac{4\pi G}{r^2}\int_0^r dr'\,r'^2\,\rho(r')$	$\frac{d\Phi}{dr}$	$\frac{GM(r)}{r^2}$	$\frac{d\Phi}{dr}$

Poisson in spherical coordinates

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi\,G\,\rho(r)$$

Potential in spherical coordinates

$$\Phi(r) = -\frac{G\,M(r)}{r} - 4\pi G\int_r^\infty \rho(r')r'dr'$$

Mass inside a radius r

$$M(r) = 4\pi\int_0^r dr'\,r'^2\,\rho(r')$$

Gradient of the potential in spherical coordinates

$$\frac{d\Phi(r)}{dr} = \frac{G\,M(r)}{r^2}$$

Spherical Systems

Examples of Spherical models

“Potential based” models

Point mass

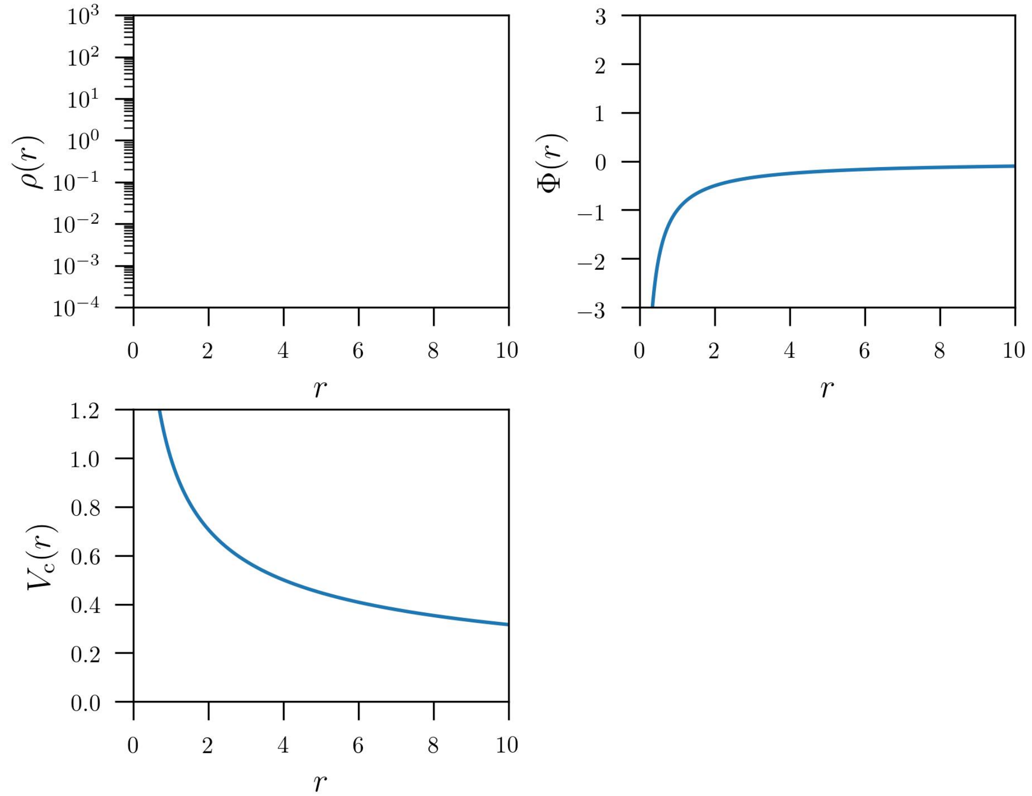
$$\Phi(r) = -\frac{GM}{r}$$

$$\rho(r) = \frac{M\delta(0)}{4\pi r^2}$$

$$M(r) = M$$

$$V_c^2(r) = \frac{GM}{r}$$

$$T(r) = 2\pi\sqrt{\frac{r^3}{GM}}$$



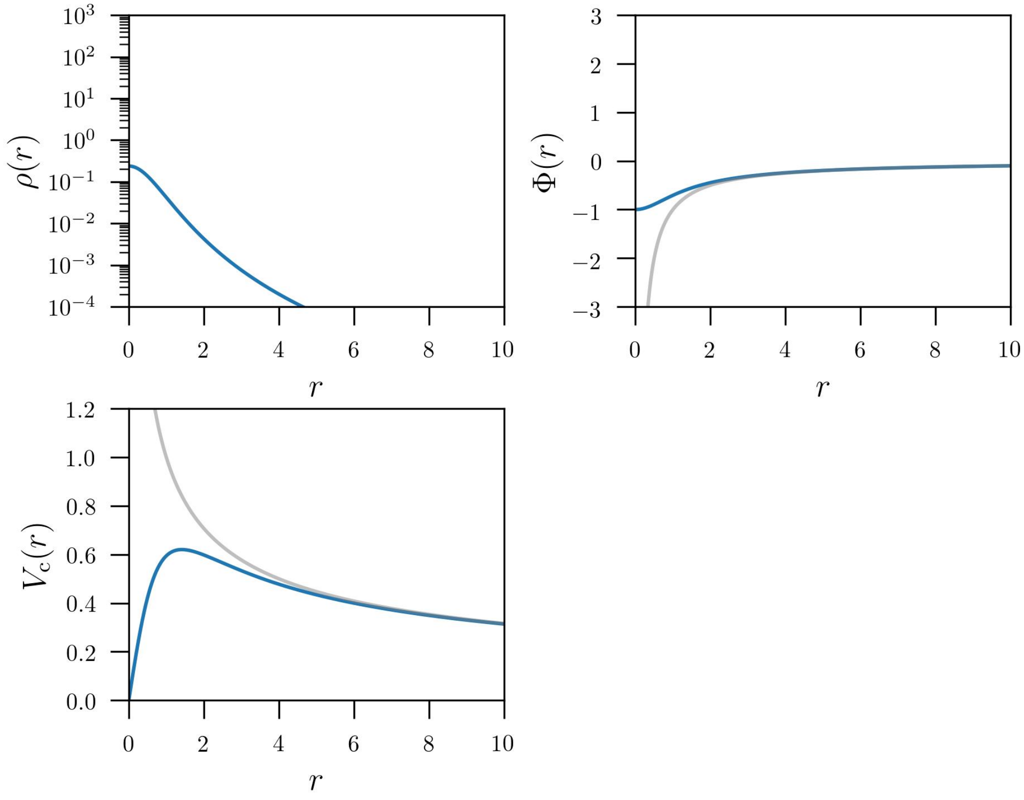
Plummer model

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + b^2}}$$

$$\rho(r) = \frac{3M}{4\pi b^3}\left(1 + \frac{r^2}{b^2}\right)^{-5/2}$$

$$M(r) = \frac{Mr^3}{(r^2 + b^2)^{3/2}}$$

$$V_c^2(r) = \frac{GMr^2}{(r^2 + b^2)^{3/2}}$$



- Globular clusters, dwarf spheroidal galaxies

Isochrone potential

$$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}$$

$$\rho(r) = M\frac{3(b + \sqrt{b^2 + r^2})(b^2 + r^2) - r^2(b + 3\sqrt{b^2 + r^2})}{4\pi(b + \sqrt{b^2 + r^2})^3(b^2 + r^2)^{3/2}}$$

$$M(r) = \frac{Mr^3}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$

$$V_c^2(r) = \frac{GMr^2}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$

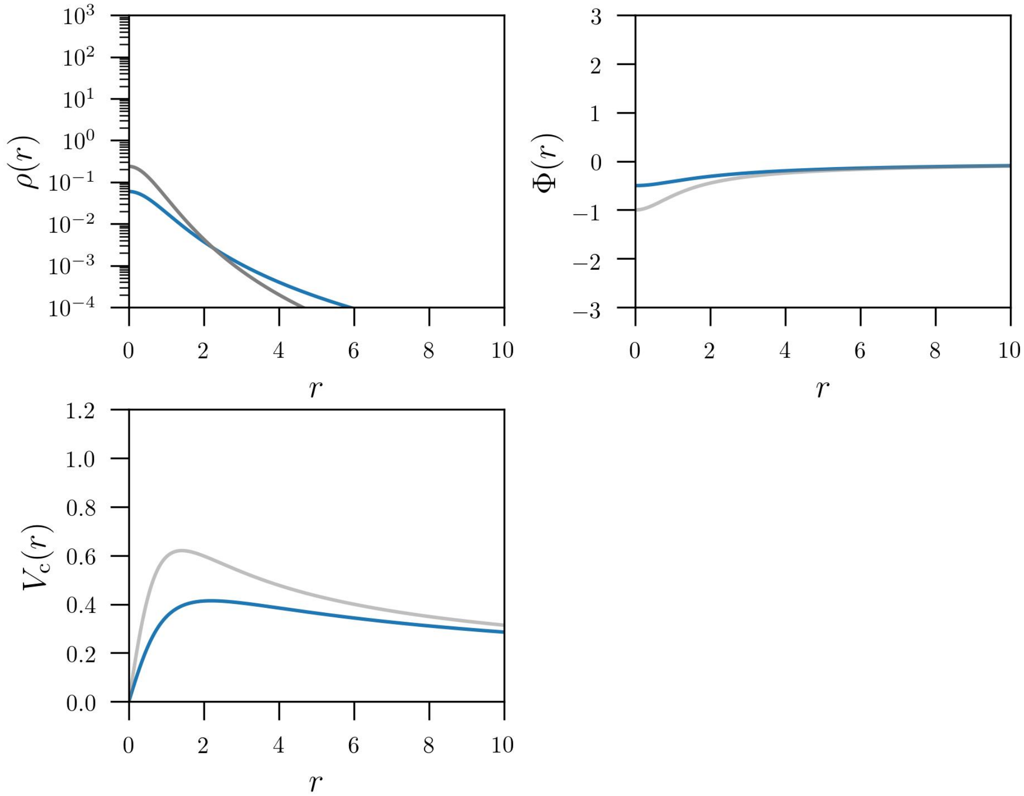
Isochrone potential

$$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}$$

$$\rho(r) = M\,\frac{3(b + \sqrt{b^2 + r^2})(b^2 + r^2) - r^2(}{4\pi(b + \sqrt{b^2 + r^2})^3(b^2 -}$$

$$M(r) = \frac{Mr^3}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$

$$V_c^2(r) = \frac{GMr^2}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$



Orbits are analytical !

Examples of Spherical models

"Density based" models

Homogeneous sphere

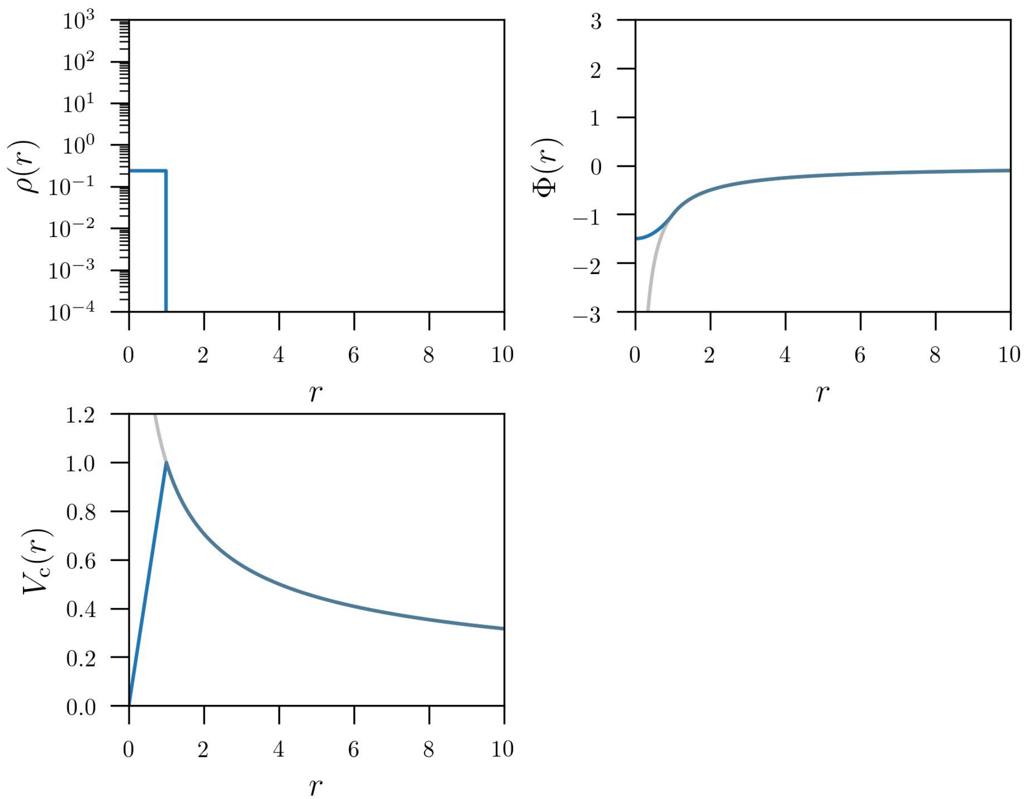
$$\rho(r) = \begin{cases} \rho & r < R \\ 0 & r > R \end{cases}$$

$$M(r) = \begin{cases} \frac{4}{3}\pi r^3\,\rho_0 & r < R \\ \frac{4}{3}\pi R^3\,\rho & r > R \end{cases}$$

$$\Phi(r) = \begin{cases} -2\pi G\rho\left(R^2 - \frac{1}{3}r^2\right) & r < R \\ -4\pi G\rho R^3/(3r) & r > R \end{cases}$$

$$V_c^2(r) = \begin{cases} \frac{4}{3}\pi G\rho_0 r^2 & r < R \\ \frac{4}{3}\pi G\rho_0\frac{R^3}{r} & r > R \end{cases}$$

$$T(r) = \begin{cases} \sqrt{\frac{3\pi}{G\rho_0}} & r < R \\ \sqrt{\frac{3\pi}{G\rho_0 R^3}}\,r^{3/2} & r > R \end{cases}$$



$$\frac{d^2r}{dt^2} = -\frac{d\Phi(r)}{dr} = -\frac{GM(r)}{r^2} = -\frac{4}{3}\pi\rho_0 r = -\omega^2 r$$

Harmonic oscillator !

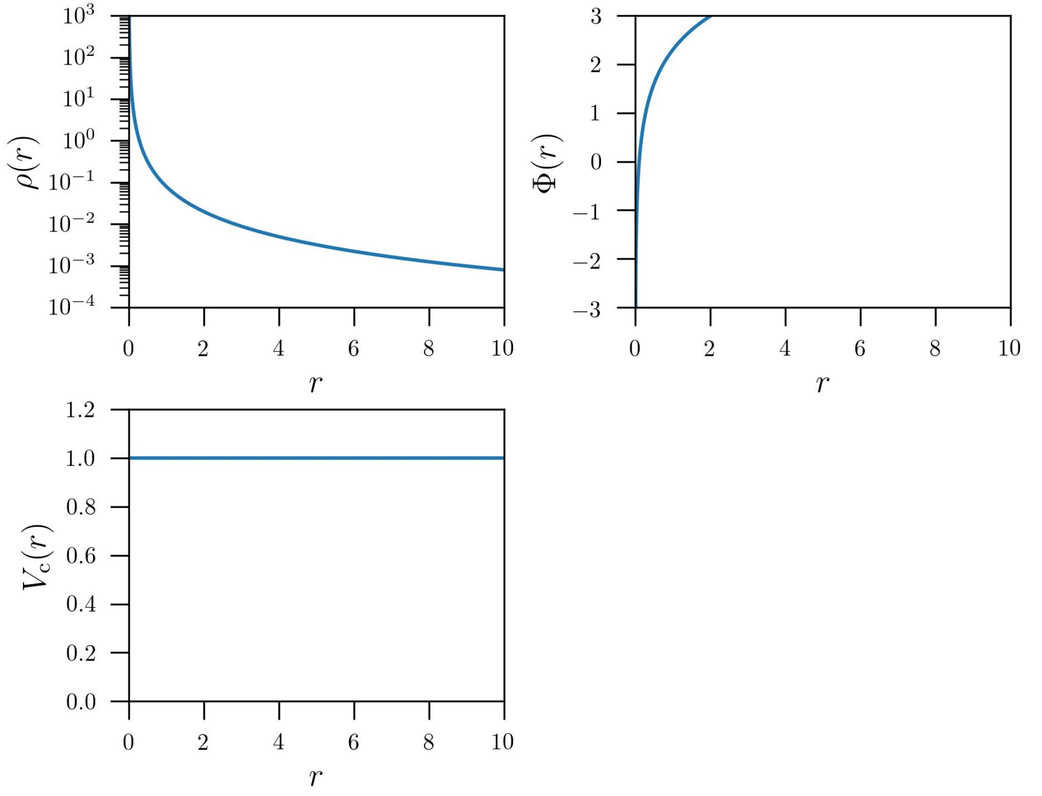
Isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \ln\left(\frac{r}{a}\right)$$

$$M(r) = 4\pi \rho_0 a^2 \, r$$

$$V_c^2(r) = 4\pi G \rho_0 a^2$$



- often used for gravitational lens models
- But !
  - diverge towards the centre !
  - infinite mass !

Pseudo-isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{a^2 + r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \left(\frac{1}{2} \ln(a^2 + r^2) + \frac{a}{r} \arctan\left(\frac{r}{a}\right) \right)$$

$$M(r) = 4\pi r \rho_0 a^2 \left(1 - \frac{a}{r} \arctan\left(\frac{r}{a}\right)\right)$$

$$V_c^2(r) = 4\pi G \rho_0 a^2 \left(1 - \frac{a}{r} \arctan\left(\frac{r}{a}\right)\right)$$

- Avoid the central divergence of the isothermal sphere
  - However, the mass is still not bounded

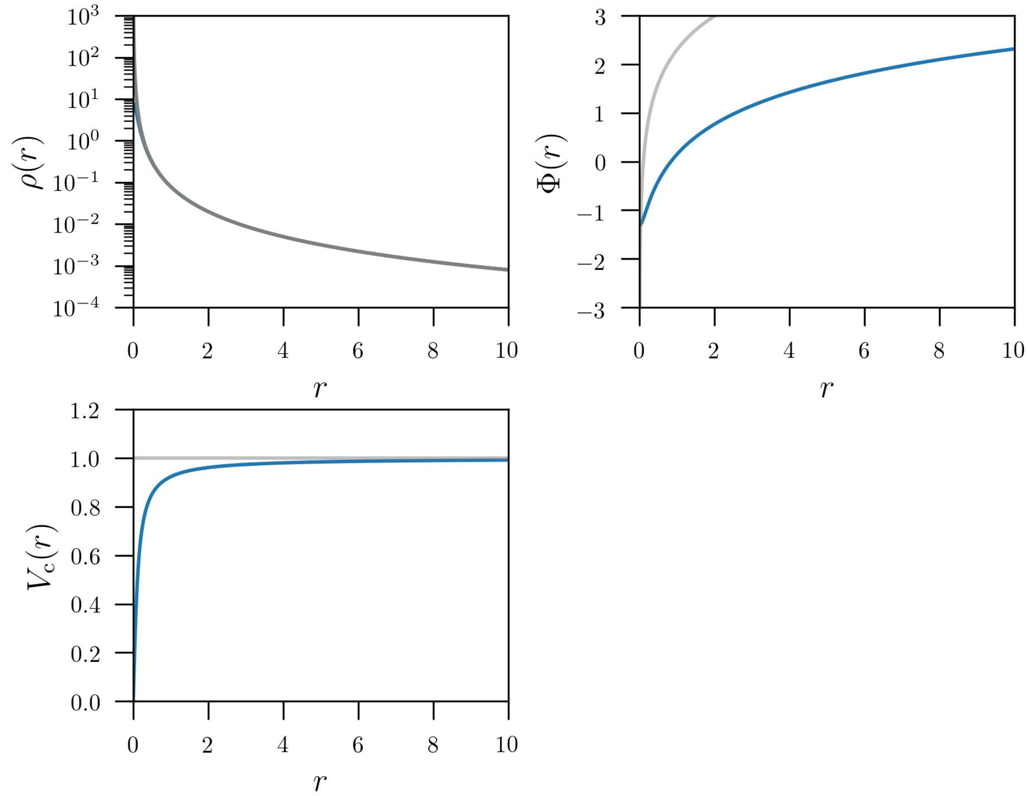
Pseudo-isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{a^2 + r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \left(\frac{1}{2} \ln(a^2 + r^2) + \frac{a}{r} \, \mathrm{arcta} \right.$$

$$M(r) = 4\pi r \rho_0 a^2 \left(1 - \frac{a}{r} \arctan\left(\frac{r}{a}\right)\right)$$

$$V_c^2(r) = 4\pi G \rho_0 a^2 \left(1 - \frac{a}{r} \arctan\left(\frac{r}{a}\right)\right)$$



- Avoid the central divergence of the isothermal sphere
  - However, the mass is still not bounded

Generic two power density models

$$\rho(r) = \frac{\rho_0}{(r/a)^\alpha (1 + r/a)^{\beta-\alpha}}$$

- diverges at the center if $\alpha \neq 0$

$$M(r) = 4\pi\rho_0 a^3 \int_0^{r/a} s \frac{s^{2-\alpha}}{(1+s)^{\beta-\alpha}}$$

model name	inner slope α	outer slope β
Plummer	0	5
Dehnen	any	4
Hernquist	1	4
Jaffe	2	4
NFW	1	3

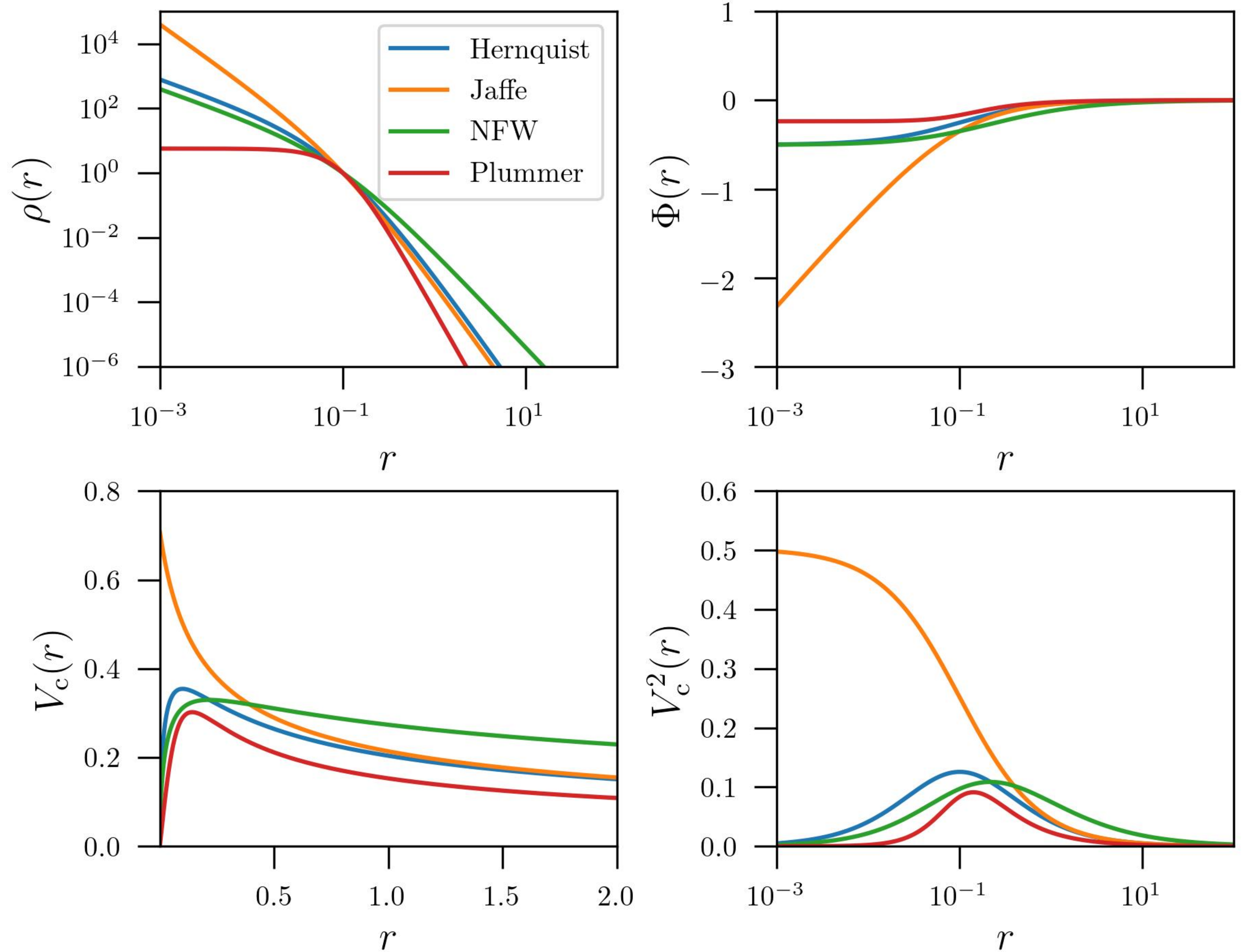
- globular clusters
- bulges, elliptic. gal.
- elliptic. galaxies
- dark haloes

Generic two power density model

$$M(r) = 4\pi\rho_0 a^3 \times \begin{cases} \frac{r/a}{1+r/a} & \text{(Jaffe)} \\ \frac{(r/a)^2}{2(1+r/a)^2} & \text{(Hernquist)} \\ \ln(1+r/a) - \frac{r/a}{1+r/a} & \text{(NFW)} \end{cases}$$

- diverges !!

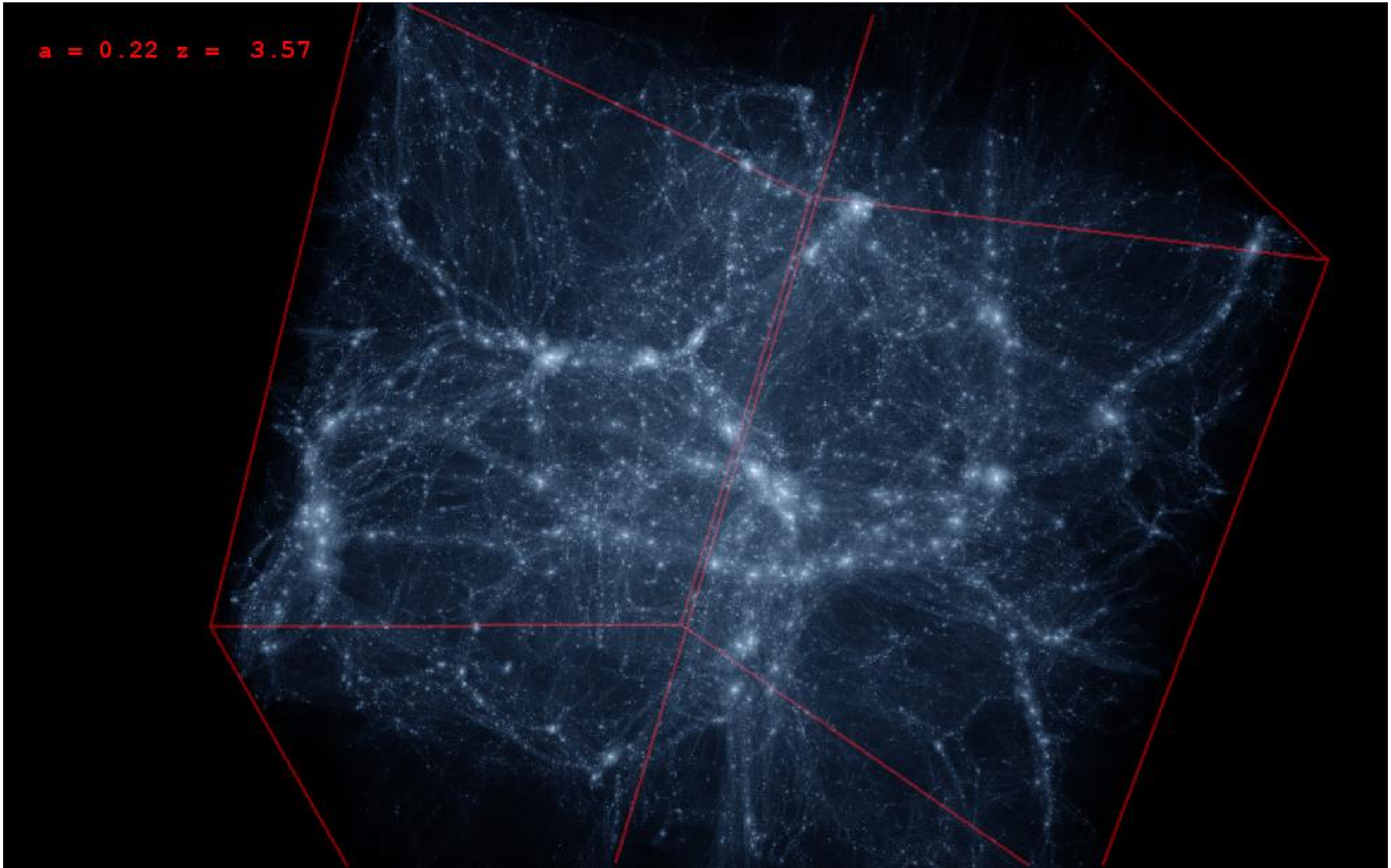
$$\Phi(r) = -4\pi G\rho_0 a^2 \times \begin{cases} \ln(1+a/r) & \text{(Jaffe)} \\ \frac{1}{2(1+r/a)} & \text{(Hernquist)} \\ \frac{\ln(1+r/a)}{r/a} & \text{(NFW)} \end{cases}$$



Hernquist
Jaffe
NFW
Plummer
$\rho(r)$
$\Phi(r)$
$V_c(r)$
$V_c^2(r)$
r
10^4
10^2
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-3}
10^{-1}
10^1
1
0
−1
−2
−3
0.8
0.6
0.4
0.2
0.0
0.5
1.0
1.5
2.0
0.5
0.3
0.1

NFW (Navarro, Frenk & White 1995, 1996)

- Density profile that fit dark matter haloes formed in LCDM numerical simulations



NFW (Navarro, Frenk & White 1995, 1996)

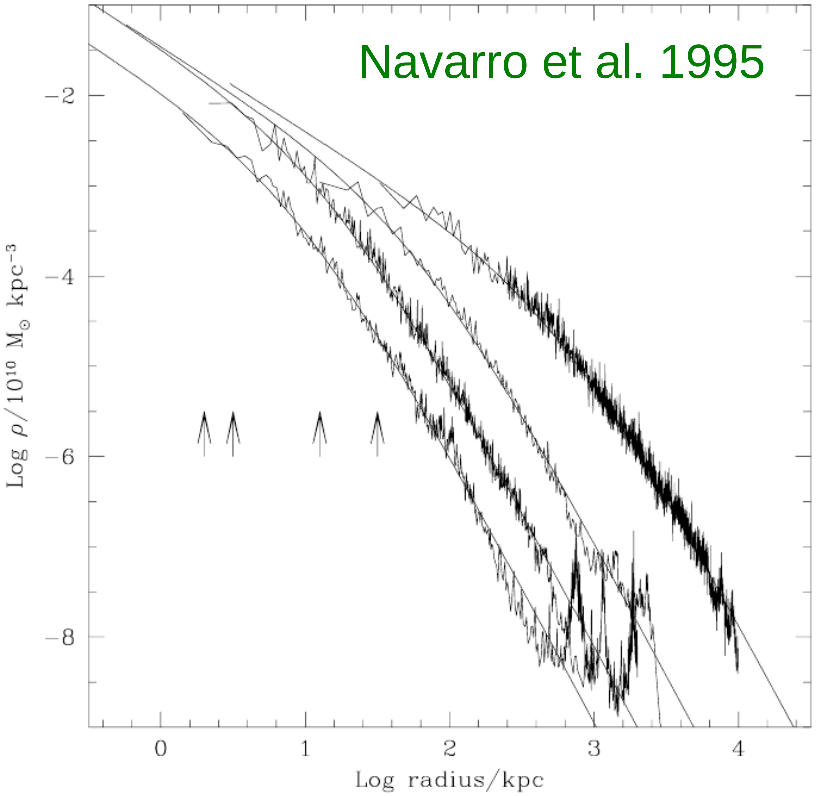


Fig. 3.— Density profiles of four halos spanning four orders of magnitude in mass. The arrows indicate the gravitational softening, h_g, of each simulation. Also shown are fits from eq.3. The fits are good over two decades in radius, approximately from h_g out to the virial radius of each system.

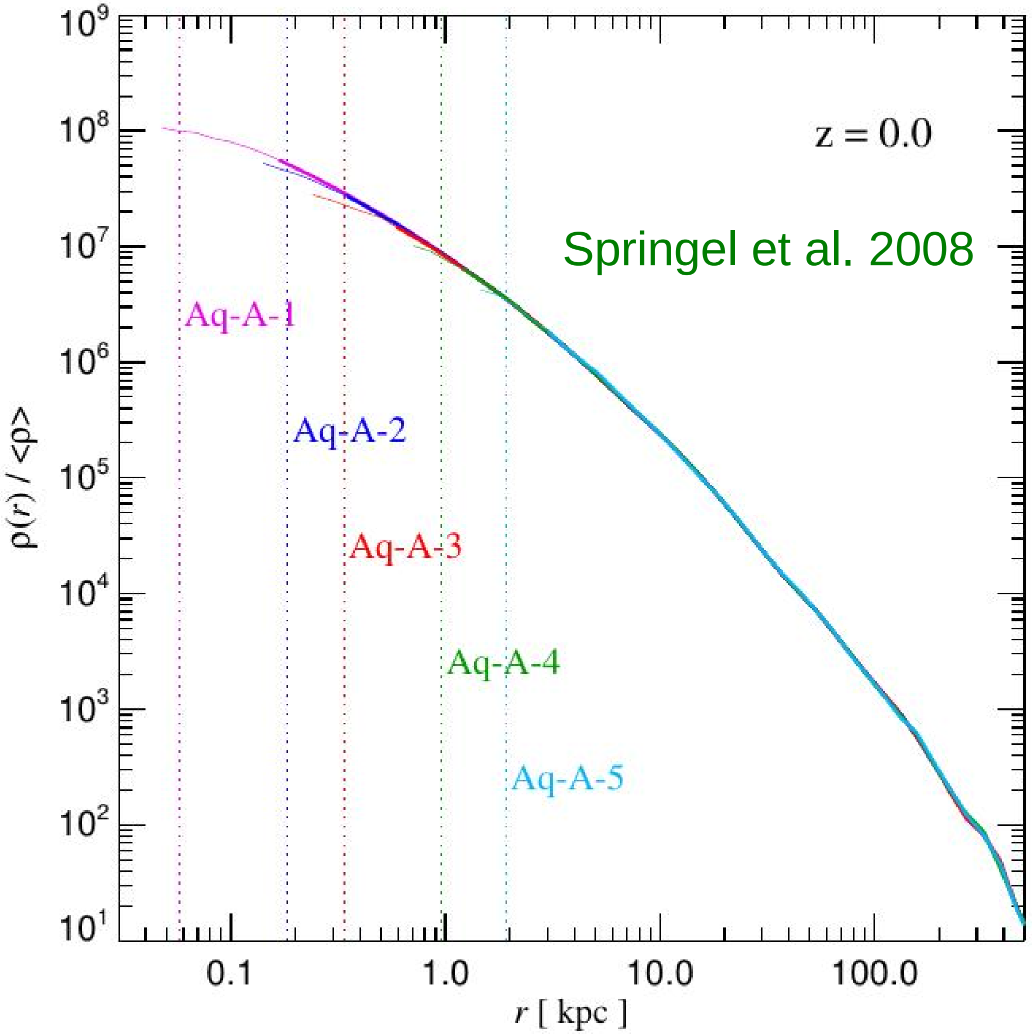
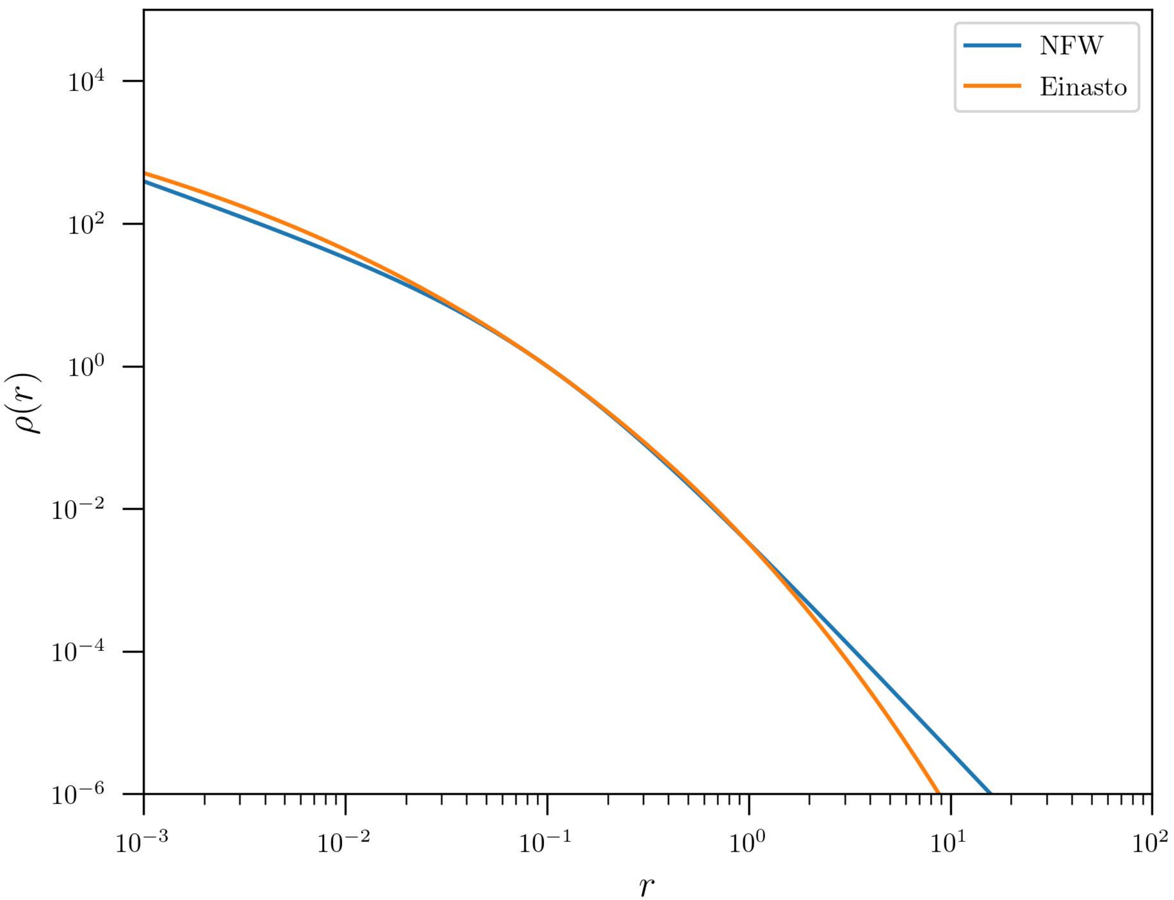


Figure 4. Spherically averaged density profile of the Aq-A halo at $z = 0$, at different numerical resolutions. Each of the pro-

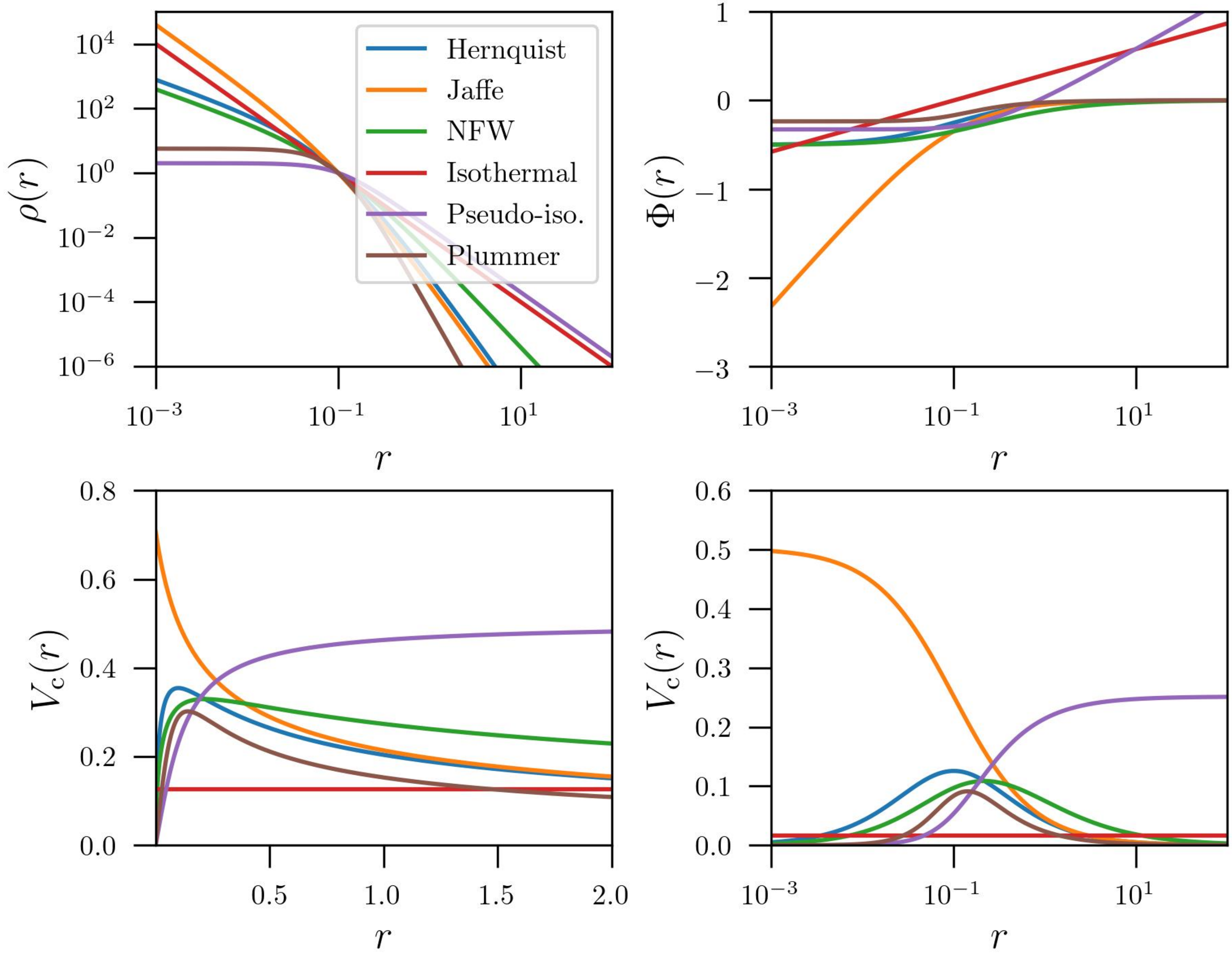
Einasto model

$$\rho(r) = \rho_0 \exp\left[-(r/a)^{1/m}\right] \quad (m \cong 6)$$



- Alternative to NFW

Spherical systems model comparison



Axisymmetric models for disk galaxies

$$\rho(\vec{x}) = \rho(R, |z|)$$

$$R = \sqrt{x^2 + y^2}$$

Examples of axisymmetric models

“Potential based” models

Kuzmin disk

Kuzmin 1956

$$\Phi_{\mathrm{K}}(R,z) = -\frac{GM}{\sqrt{R^2+(a+|z|)^2}} = -\frac{GM}{\sqrt{R^2+z^2+a^2+2\,a\,|z|}}$$

Comparison with Plummer:

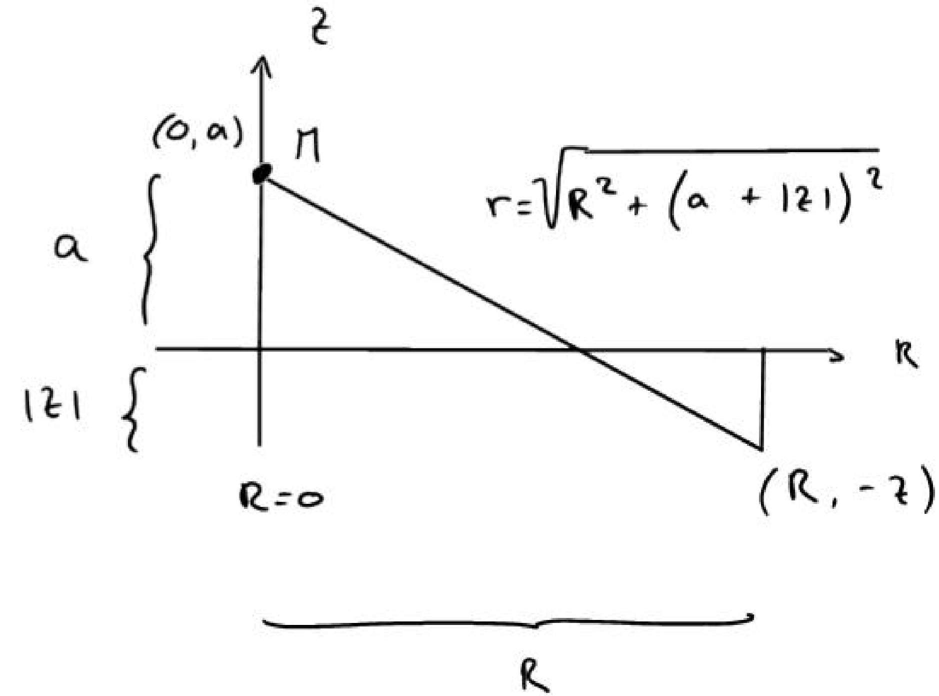
$$\Phi_{\mathrm{P}}(R,z) = -\frac{GM}{\sqrt{R^2+z^2+a^2}}$$

Equivalent to the following configuration

Potential due to

a mass M at $(0, a)$

$$\Rightarrow \quad -\frac{GM}{r} = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}}$$



Kuzmin disk

Kuzmin 1956

$$\Phi_{\mathrm{K}}(R,z) = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}}$$

Plummer based model

$$\Sigma_{\mathrm{K}}(R) = \frac{aM}{2\pi(R^2 + a^2)^{3/2}}$$

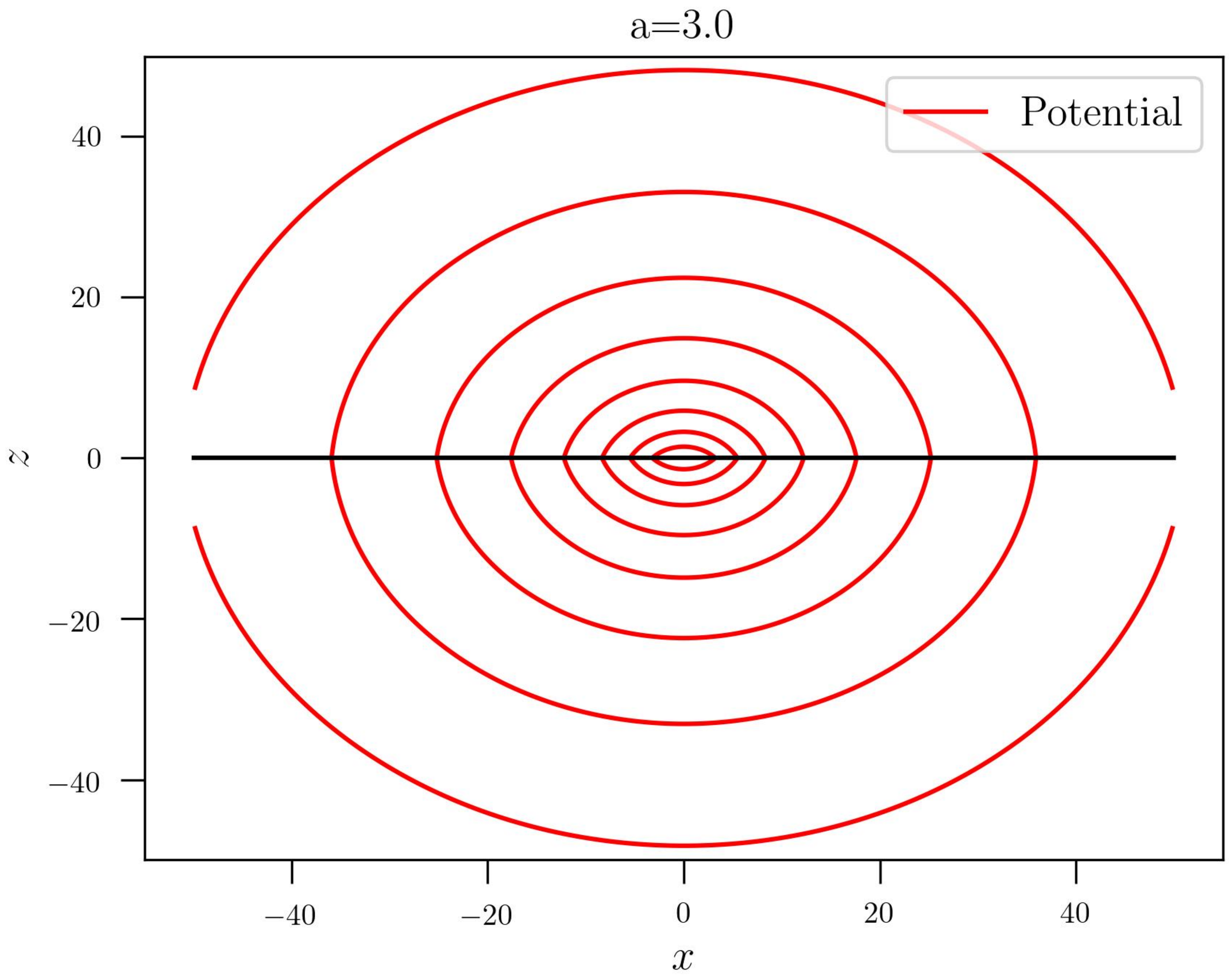
Infinitely thin disk

$$V_{c,\mathrm{K}}^2(R) = \frac{G\,M\,R^2}{\left(R^2 + a^2\right)^{3/2}}$$

Equivalent to the Plummer model

$$V_{c,\mathrm{P}}^2(r) = \frac{GMr^2}{(r^2 + b^2)^{3/2}}$$

Kuzmin disk



Miyamoto-Nagai potential

Miyamoto & Nagai 1975

$$\Phi_{\rm MN}(R,z) = -\frac{GM}{\sqrt{R^2+(a+\sqrt{z^2+b^2})^2}}$$

b=0 → Kuzmin

$$\rho_{\rm MN}(R,z) = \left(\frac{b^2M}{4\pi}\right)\frac{aR^2+(a+3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})^2}{[R^2+(a+\sqrt{z^2+b^2})^2]^{5/2}(z^2+b^2)^{3/2}}$$

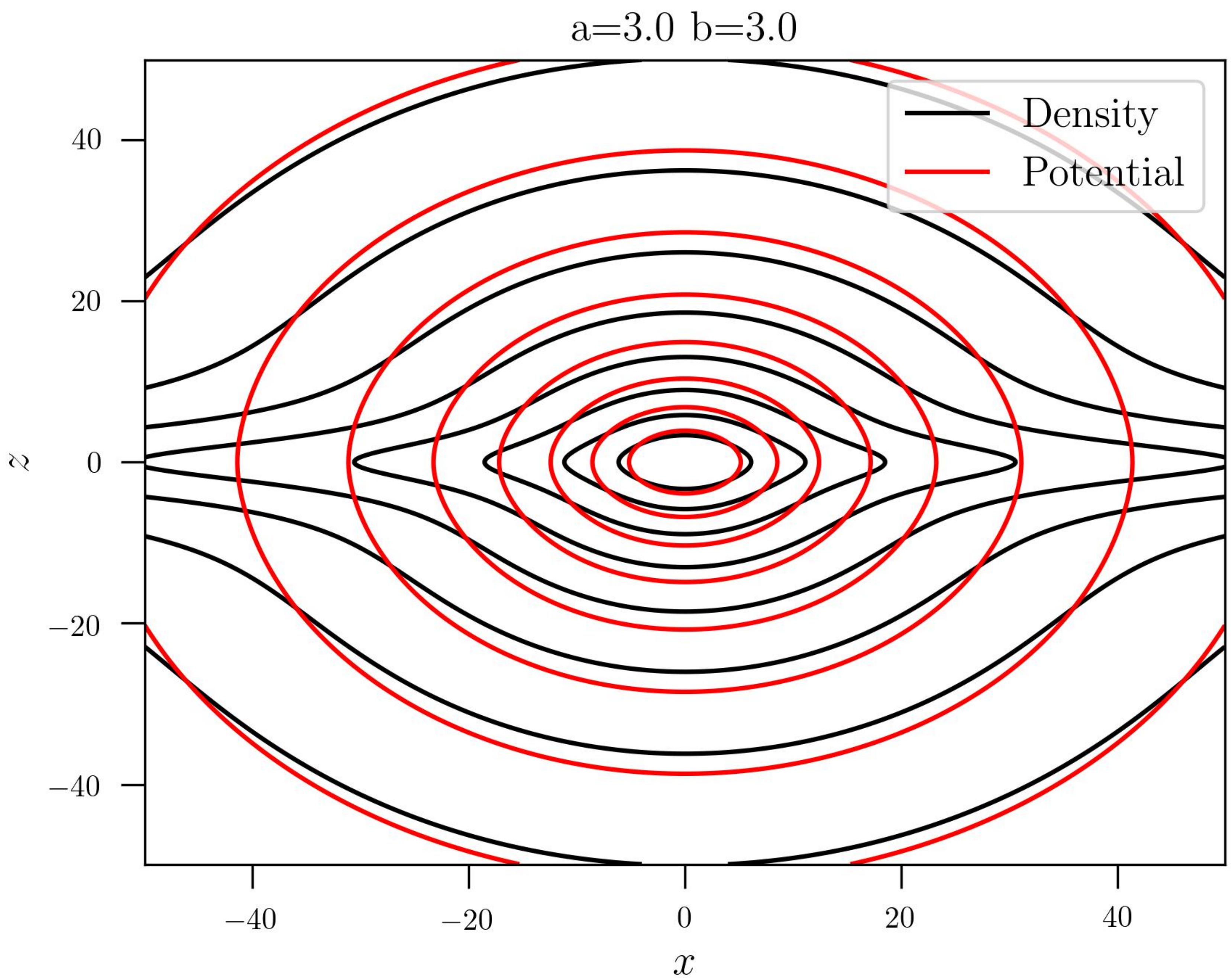
$$V_{c,\rm MN}^2(R) = \frac{G\,M\,R^2}{\left(R^2+(a+b)^2\right)^{3/2}}$$

Equivalent to the Plummer model

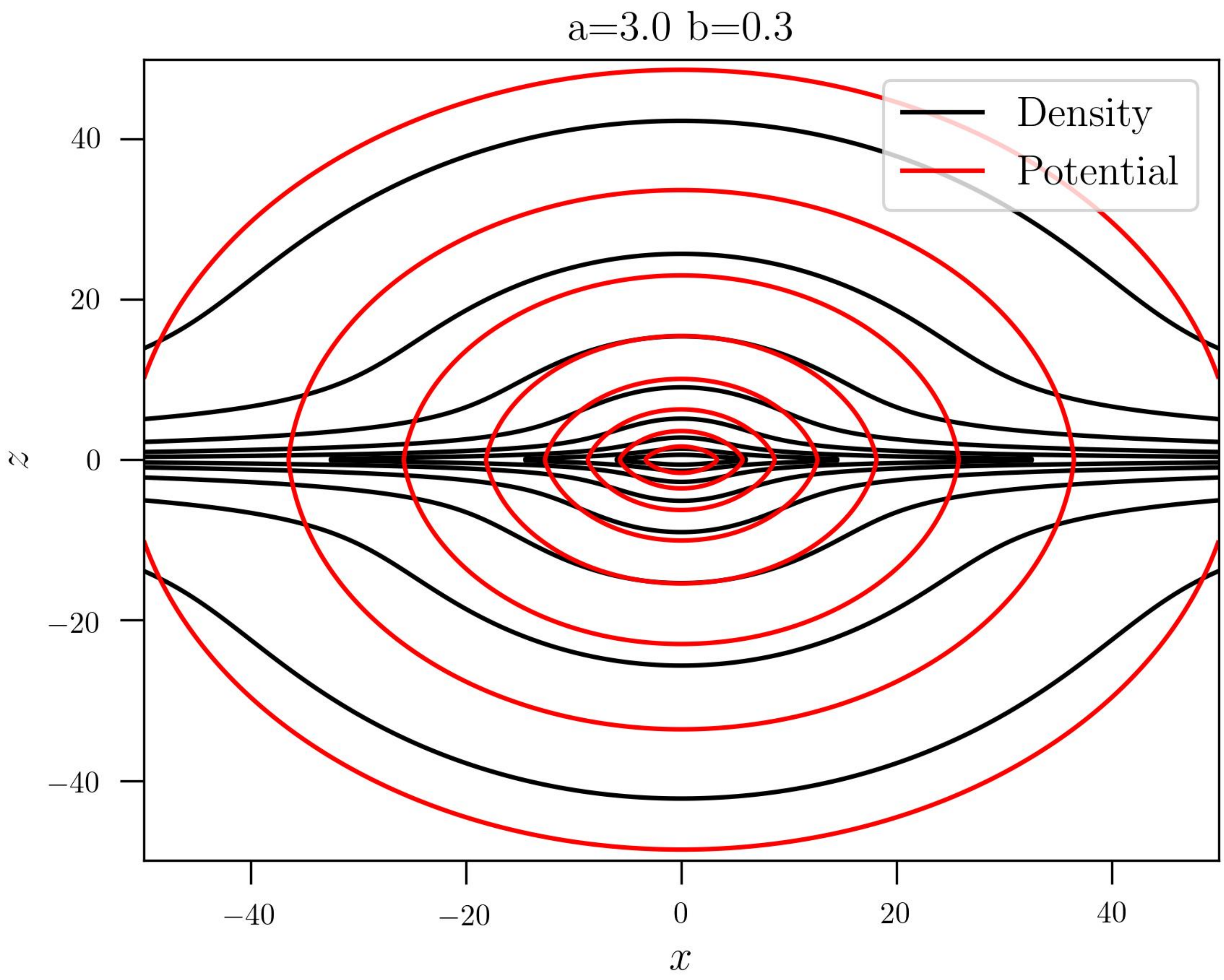
$$V_{c,\rm P}^2(r) = \frac{GMr^2}{(r^2+b^2)^{3/2}}$$

Better parametrisation :
Revaz & Pfenniger 2004

Miyamoto-Nagai potential



Miyamoto-Nagai potential



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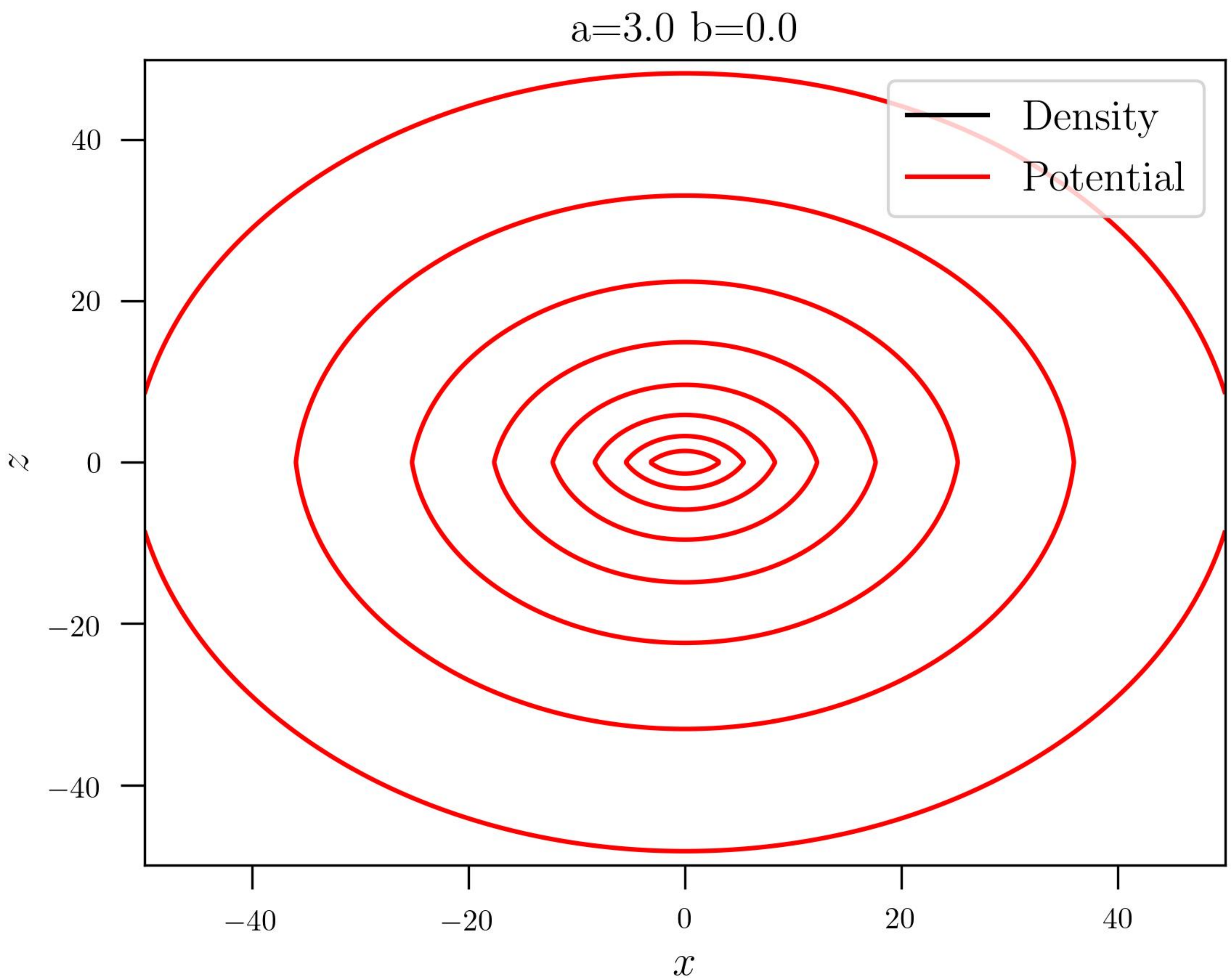
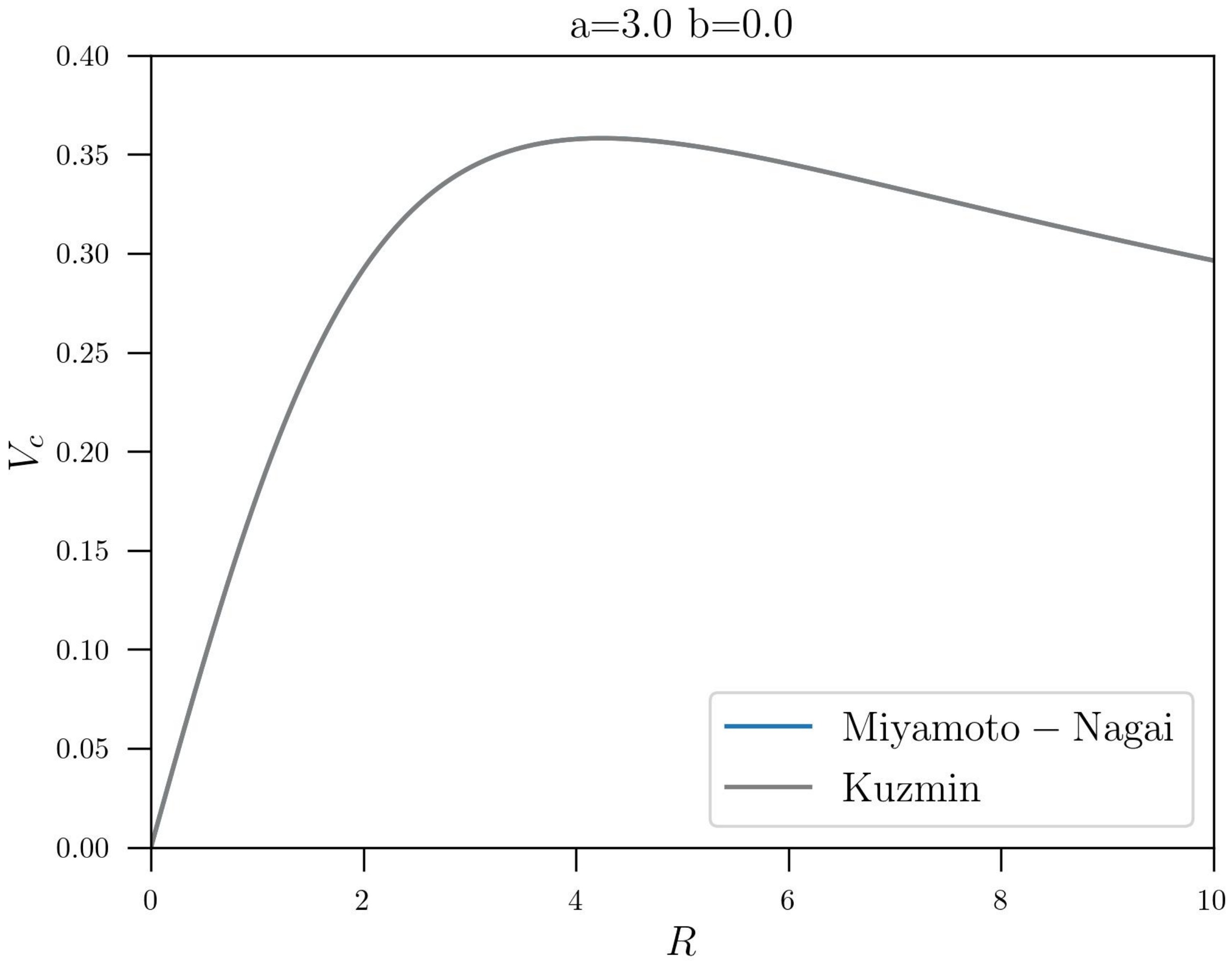


Miyamoto-Nagai potential

Miyamoto & Nagai 1975

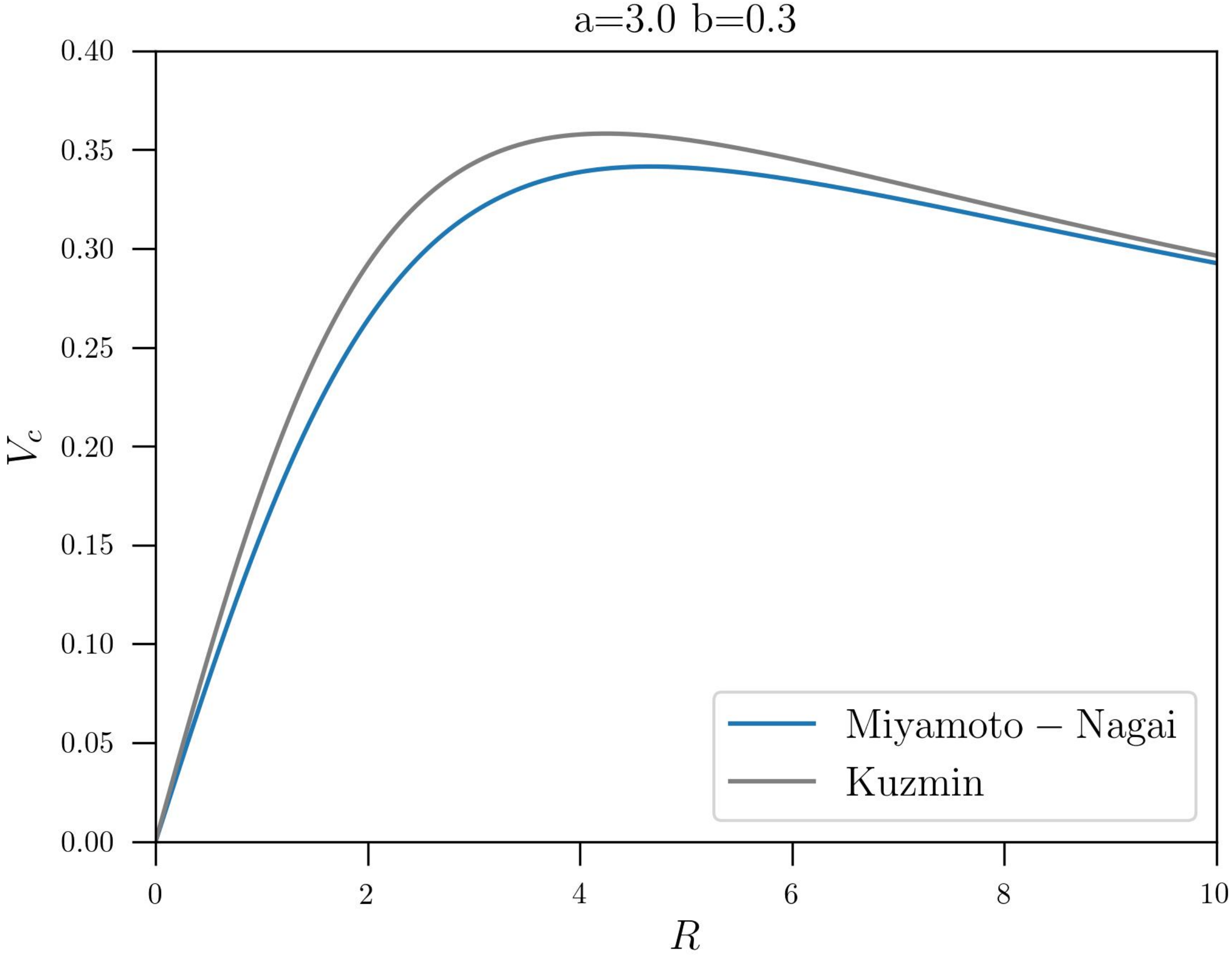
Circular velocity rotation curve



Miyamoto-Nagai potential

Miyamoto & Nagai 1975

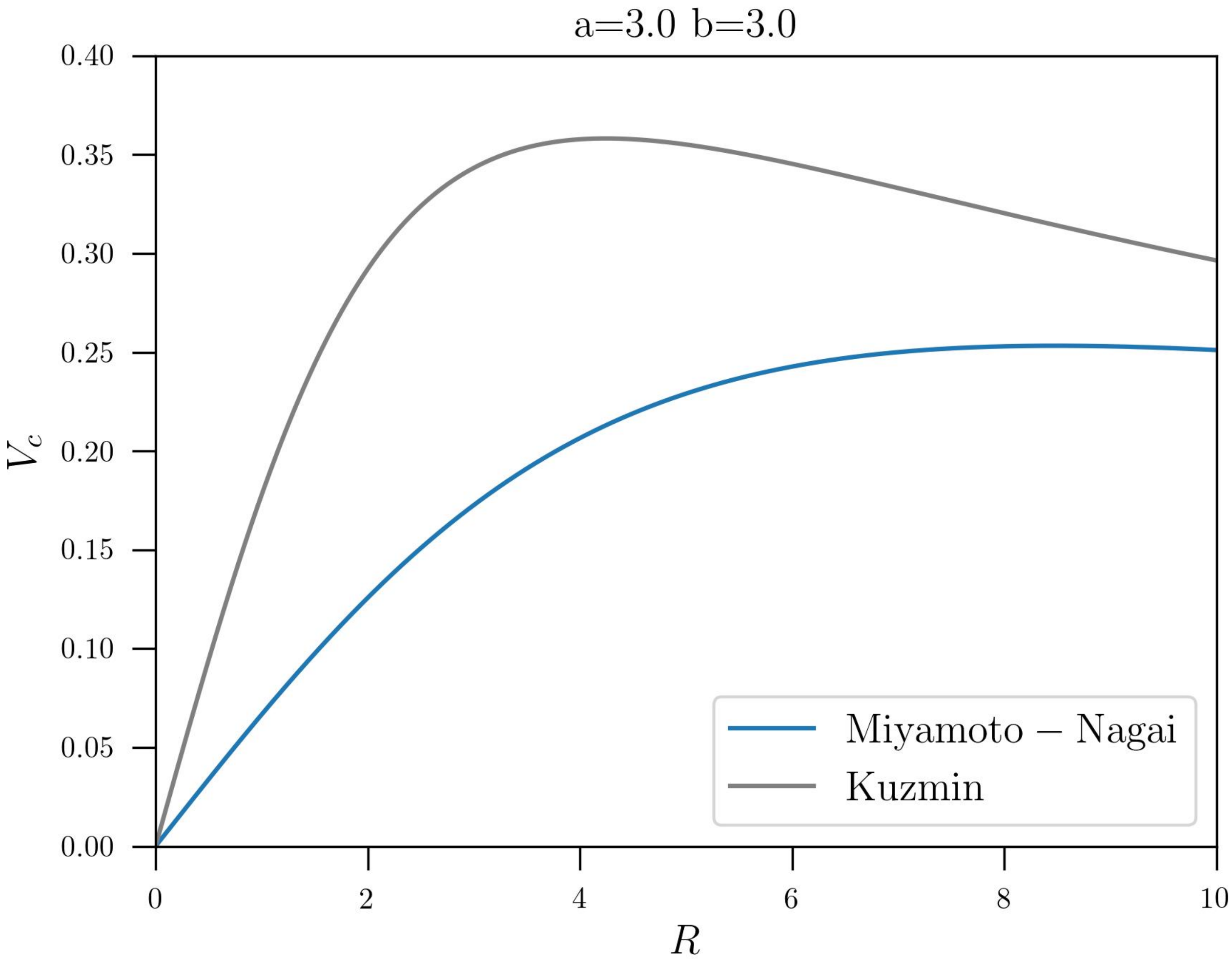
Circular velocity rotation curve



Miyamoto-Nagai potential

Miyamoto & Nagai 1975

Circular velocity rotation curve



Logarithmic potential

$$\Phi_{\log}(R, z) = \frac{1}{2} V_0^2 \ln \left(R_c^2 + R^2 + \frac{z^2}{q^2} \right)$$

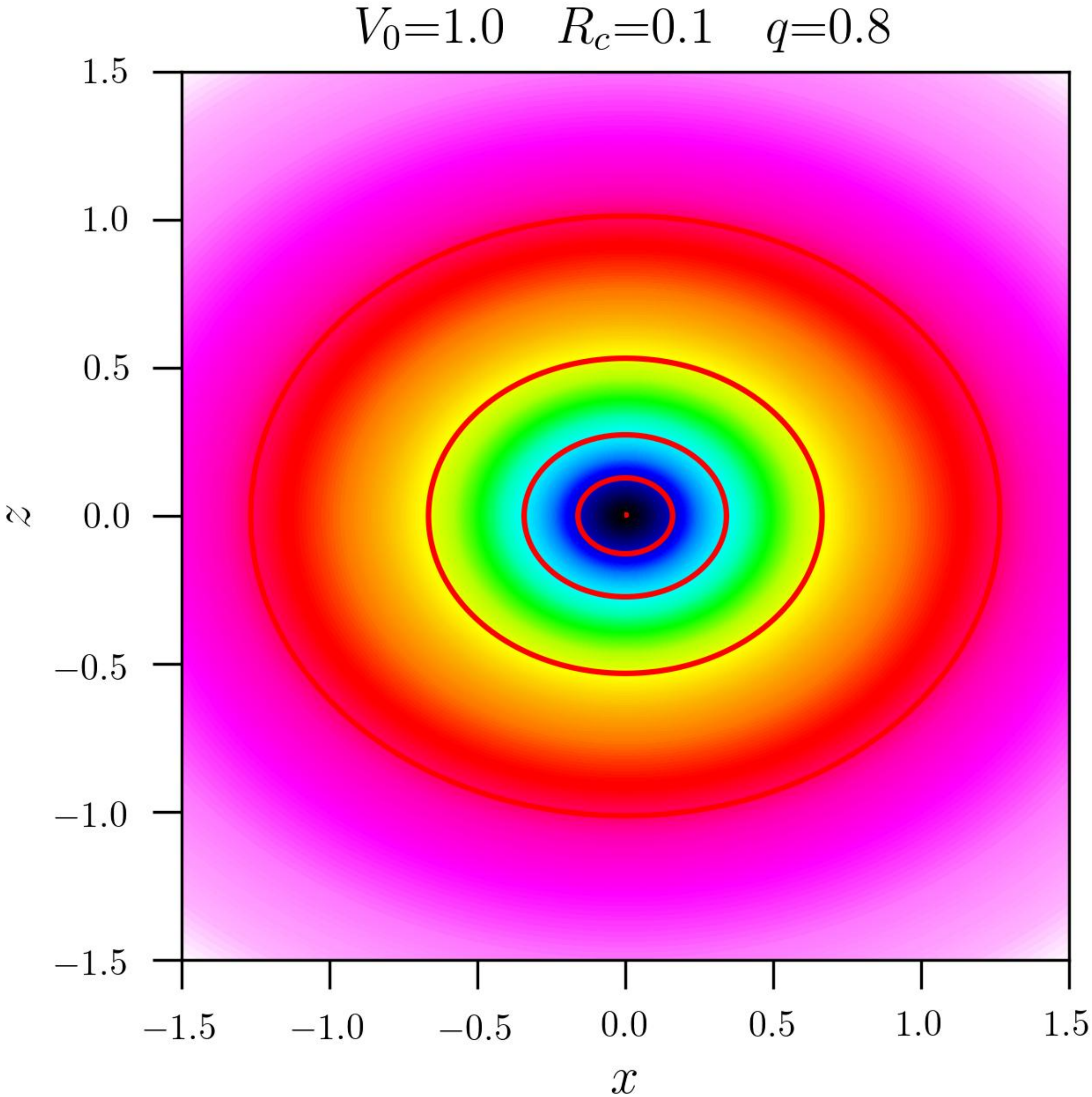
$$\rho_{\log}(R, z) = \frac{V_0^2}{4\pi G q^2} \frac{(2q^2 + 1)R_c^2 + R^2 + (2 - 1/q^2) z^2}{\left(R_c^2 + R^2 + (z^2/q^2)\right)^2}$$

- negative for $q < 1\sqrt{(2)} \cong 0.707$

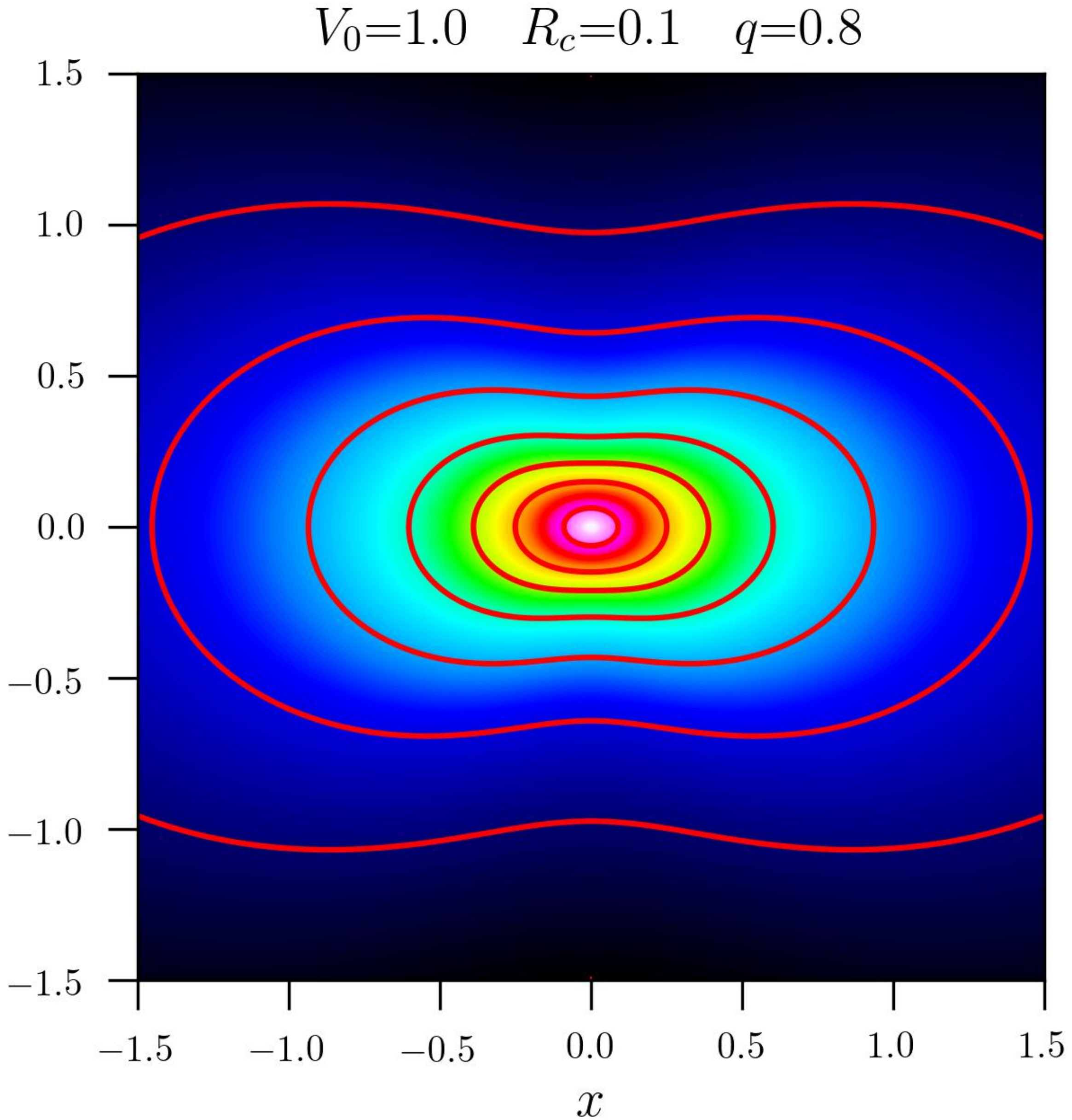
$$V_{c,\log}^2(R) = V_0^2 \frac{R^2}{R_c^2 + R^2}$$

- does not depends on q
- flat rotation curve at large radius

Logarithmic potential

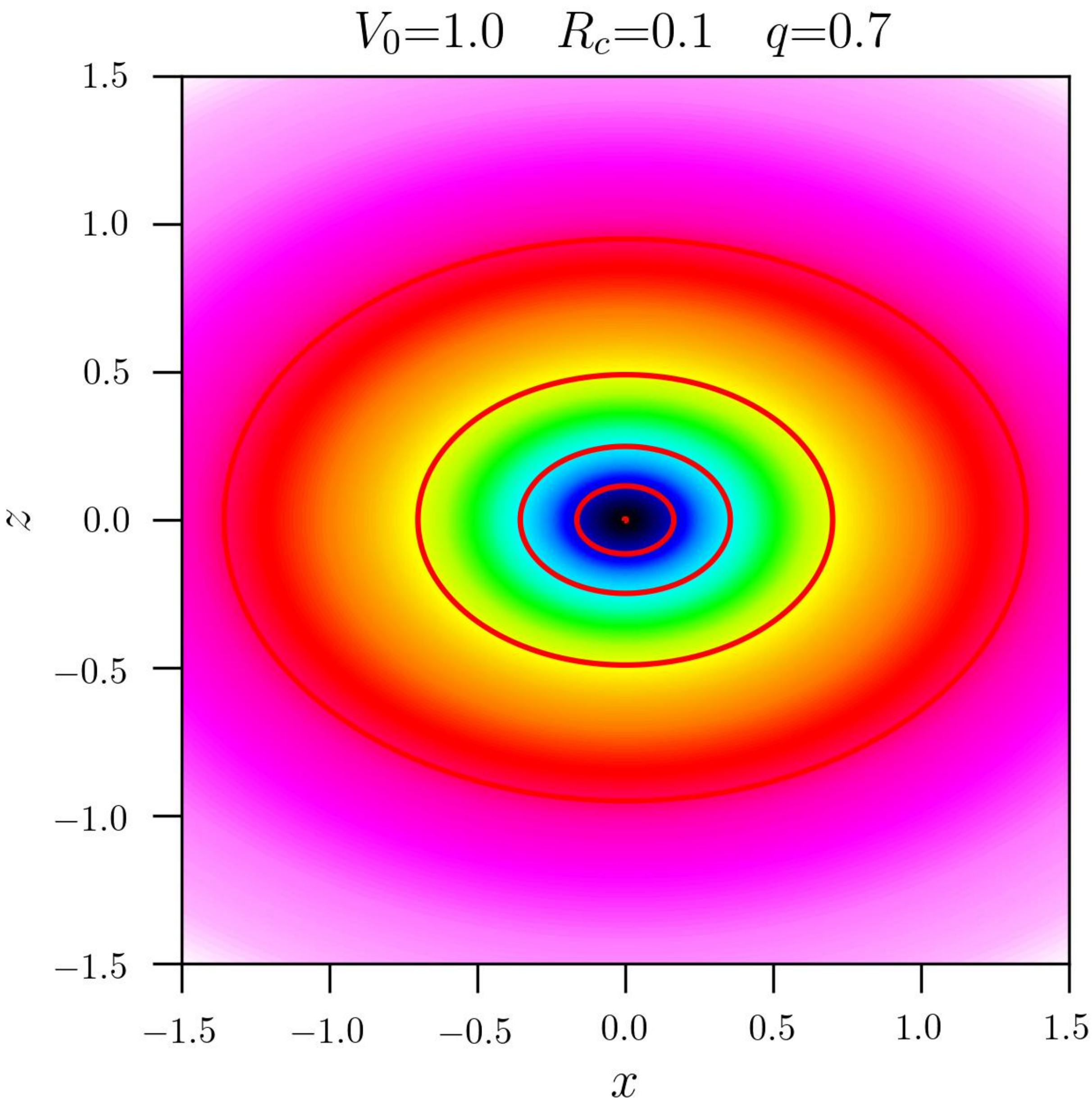


Logarithmic potential

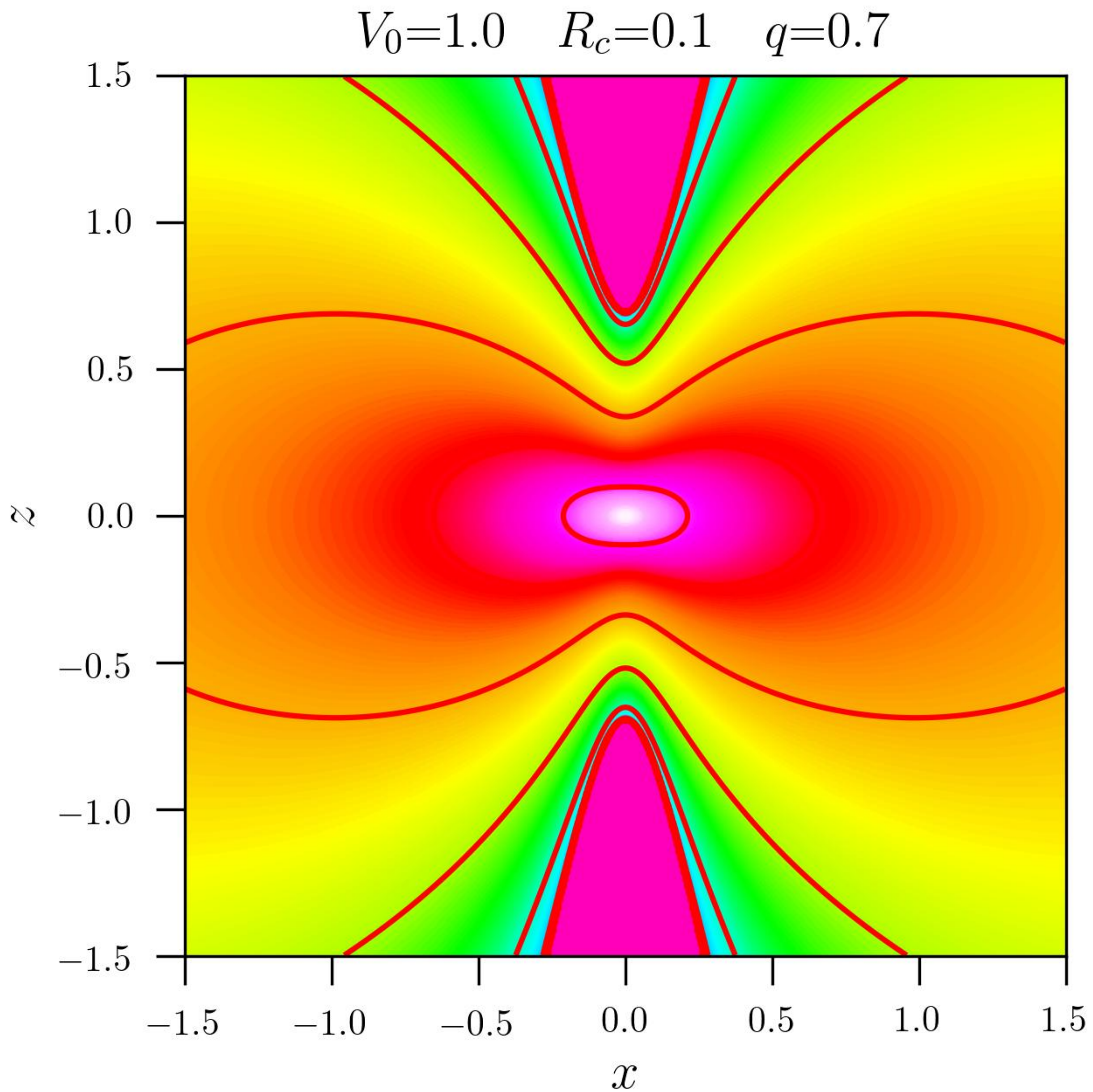


density

Logarithmic potential

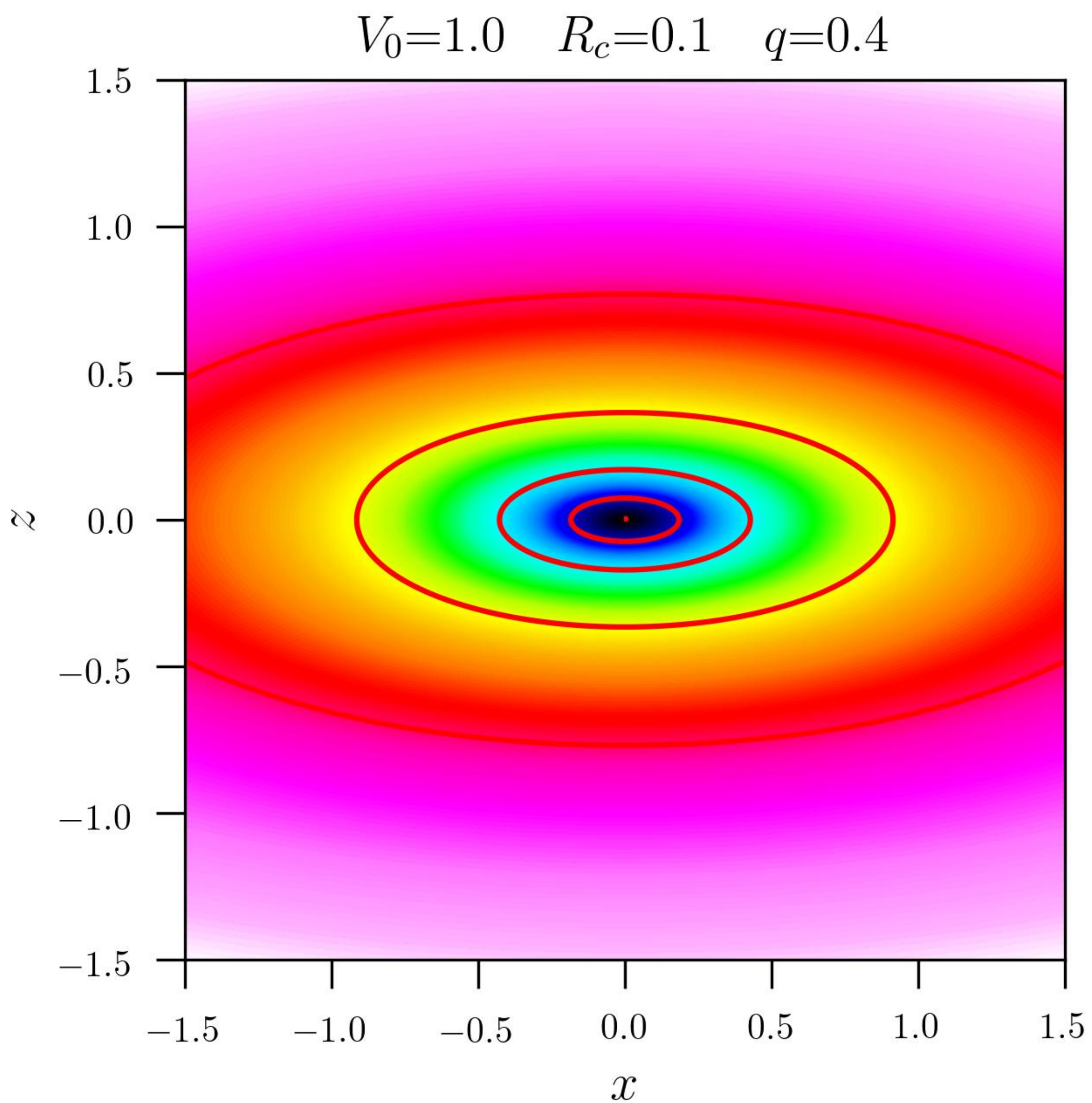


Logarithmic potential

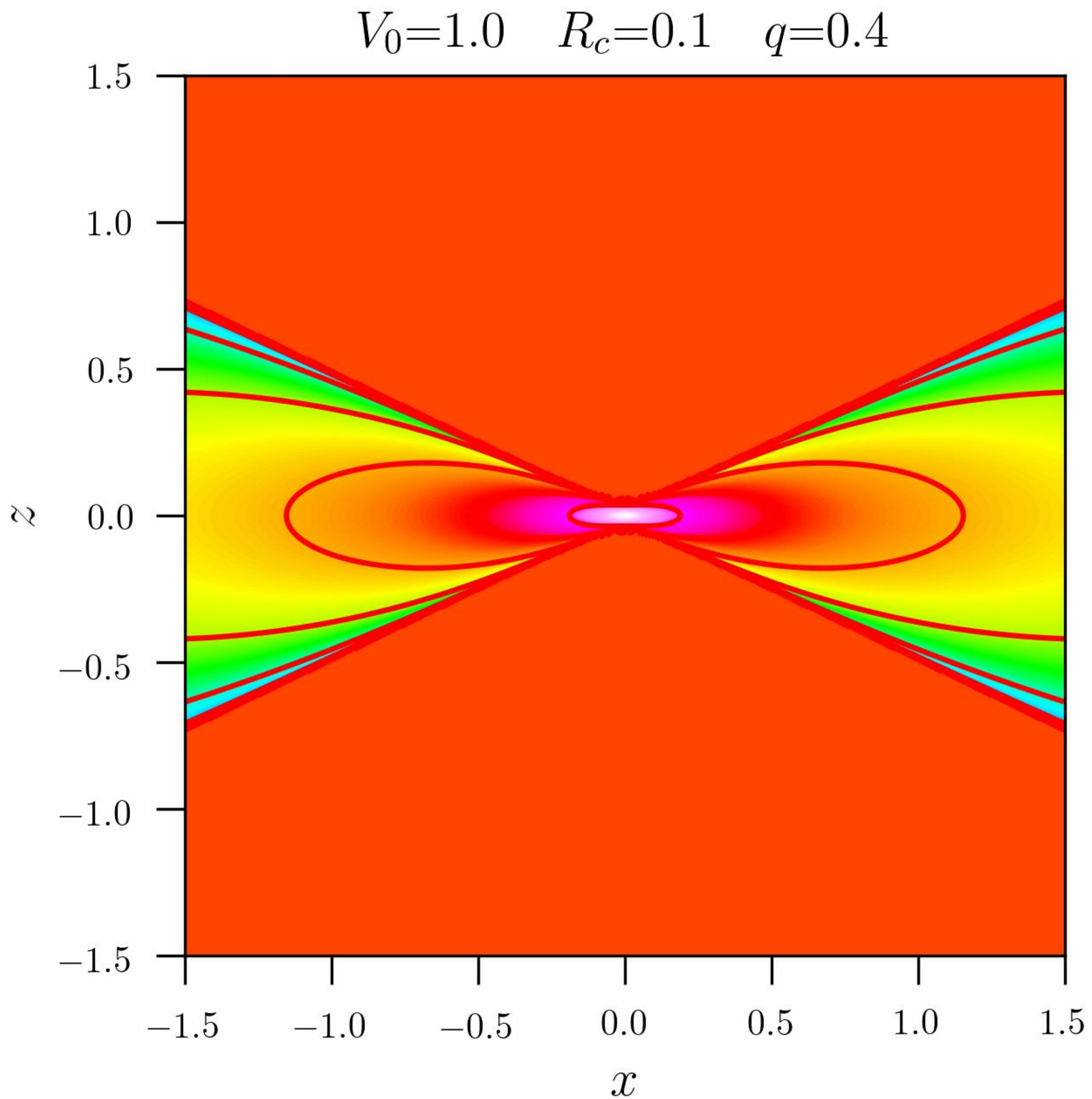


density

Logarithmic potential



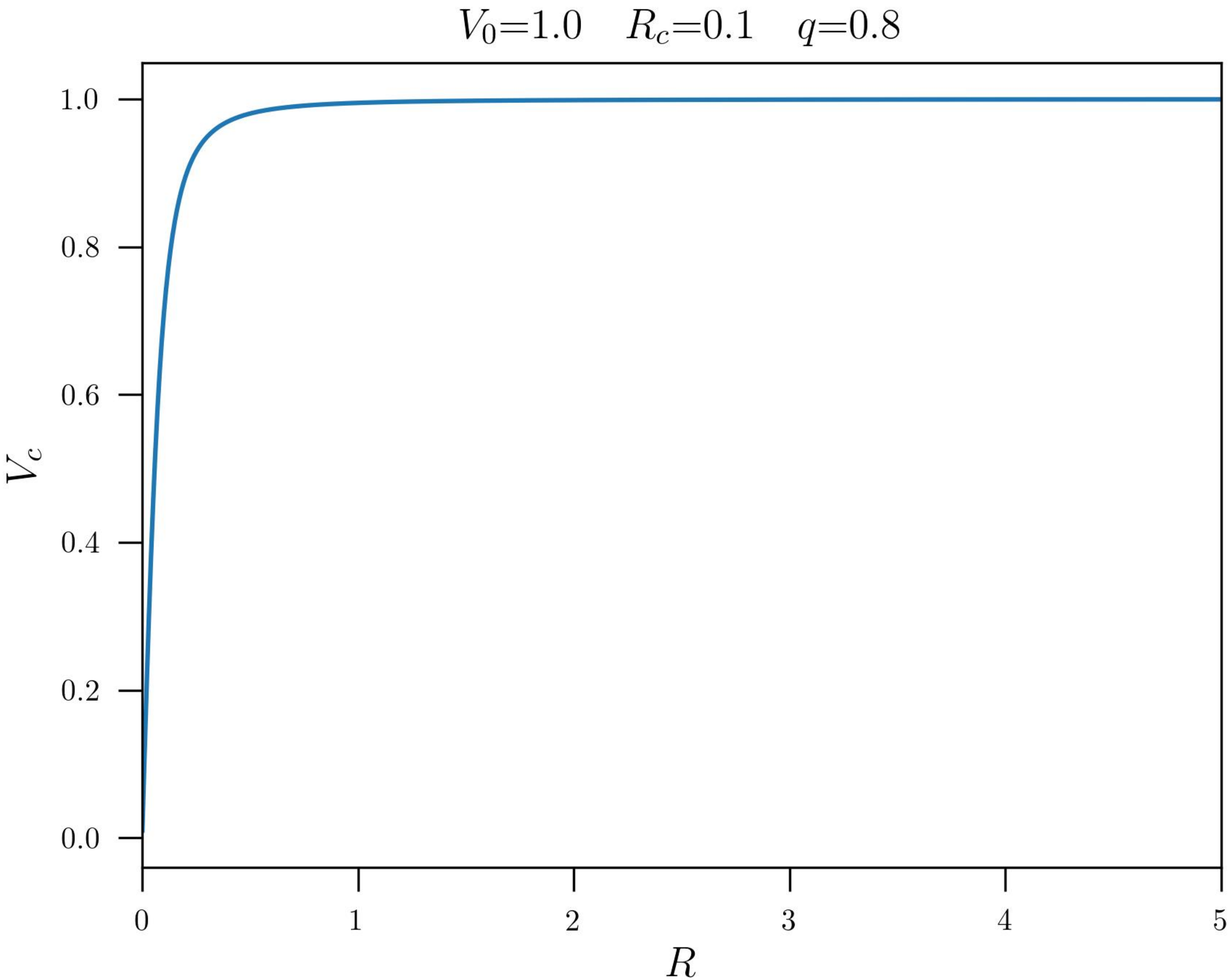
Logarithmic potential



density

Logarithmic potential

Circular velocity rotation curve



The potential of flattened systems

Poisson Equation for very flattened axisymmetric systems

Aim : get $\phi(R, z)$ from $\rho(R, z)$

Poisson equation in cylindrical coord. for axisymmetric systems $\frac{\partial}{\partial \phi}\phi = 0$

$$\nabla^2 \phi(R, z) = \frac{1}{R}\frac{\partial}{\partial R}\left(R \frac{\partial \phi}{\partial R}\right) + \frac{\partial^2 \phi}{\partial z^2} = 4 \pi G \rho(R, z)$$

What is the behaviour of the Poisson equation when the system get flatter and flatter ?

Example : Miyamoto-Nagai disk $b \to 0$

1) $$\rho_{MN}(R, z=0) \overset{b\to 0}{=} \frac{b^2 M}{4\pi} \frac{aR^2 + a^3}{(R^2+a^2)^{5/2}} \frac{1}{b^3} \sim \frac{1}{b} \longrightarrow \infty$$

2) $$\left.\frac{\partial \phi_{MN}}{\partial R}\right|_{z=0} = \left.\frac{\partial \phi_K}{\partial R}\right|_{z=0} = \frac{GM}{(R^2+a^2)^{3/2}}$$: does not diverge

$$\Rightarrow \frac{1}{R}\frac{\partial}{\partial R}\left(R \frac{\partial \phi}{\partial R}\right)$$: does not diverge

$\Rightarrow$ becomes negligible compared to ρ

Near $z=0$ the Poisson equation becomes

$$\boxed{\frac{\partial^2 \phi}{\partial z^2} = 4\pi G \rho(R, z)}$$

The vertical variation of ϕ depends only on the density ρ at that radius

Solutions of the Poisson equation

$$\phi(R,z) = \underbrace{\phi_0(R,0)}_{\text{"zero point"}} + \underbrace{\phi_z(R,z)}_{\text{"vertical dep."}}$$

1) $\phi_z(R,z) = 4\pi G \int_0^z dz' \int_0^{z'} dz'' \rho(R,z'')$

2) $\phi_0(R,0)$ is obtained by assuming a "razor - thin" disk

$$\rho(R,z) \rightarrow \Sigma(R)$$

We need a machinery to find $\phi_0(R,0)$ from $\Sigma(R)$

The potential of infinite thin (razor-thin) disks

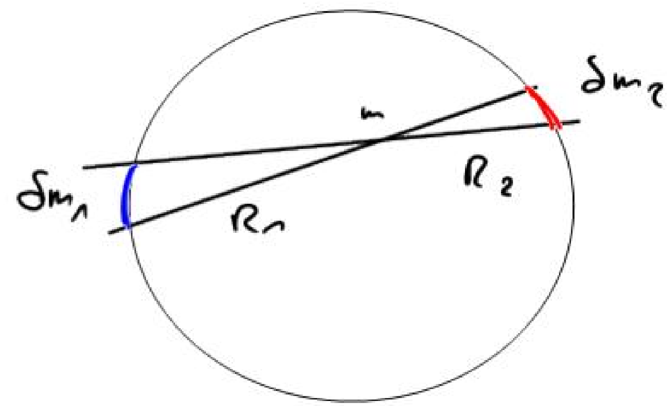
Potential of zero-thickness (razor-thin) disks

<u>Idea</u>: Sum the contribution of a set of rings as we did for spherical models, summing shells

$$\Sigma(R) = \frac{M\,\delta(R-R_0)}{2\pi R_0} \qquad \text{as } M = 2\pi \int_0^\infty \frac{M}{2\pi R_0}\,\delta(R-R_0)\,R\,dR$$

Potential of a ring

<u>no Newton theorem</u> ! ☹

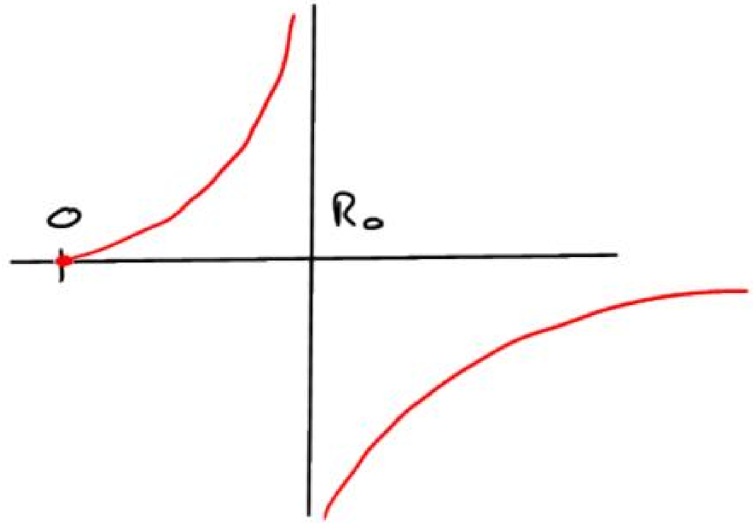


- $\delta m_1 = \Sigma \cdot R_1 \, d\Theta \, dR$
- $\delta m_2 = \Sigma \cdot R_2 \, d\Theta \, dR$

$$\delta F_1 = \frac{G m \Sigma \, d\Theta \, dR}{R_1} \neq \frac{G m \Sigma \, d\Theta \, dR}{R_1} = \delta F_2$$

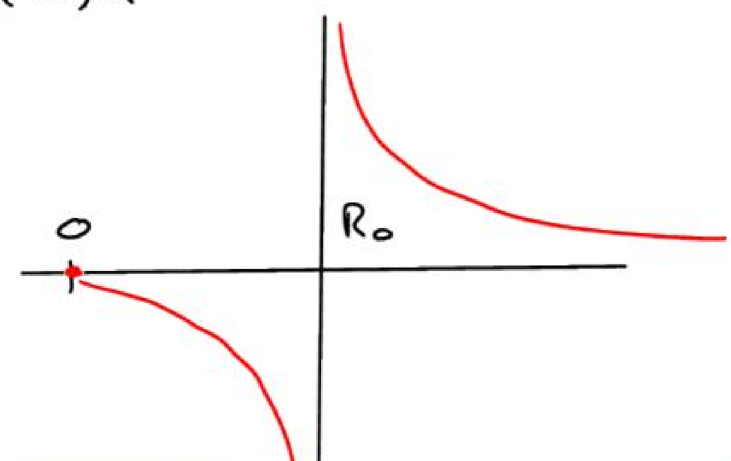
Estimation of the gravitational field / circular velocity

$g(R)$



$\frac{1}{R^2}$ $\quad R \gg R_0$

$$V_c^2 = R \frac{\partial \phi}{\partial R} = - g(R) R$$



$\frac{1}{R^2}$ $\quad R \gg R_0$

$V_c^2 \ll 0$

$V_c \sim \frac{1}{R}$

Potential / force of a ring

Lass & Blitzer 1982

$$\phi(R, z) = -\frac{2\,G M\,K(k)}{\pi\sqrt{(R_0+R)^2+z^2}} \qquad \left(k^2 = \frac{4 R_0 R}{(R+R_0)^2+z^2} \right)$$

$$g(R, z) = -\frac{G M}{R\,\pi\sqrt{(R_0+R)^2+z^2}}\left[k\,\frac{R^2-R_0^2-z^2}{4(1-k)R R_0}\,E(k) + K(k) \right]$$

with
- $K(m)$: complete elliptic integral of first kind

$$K(m) = \int_0^{\pi/2} \left[1 - m^2 \sin(t)^2\right]^{-1/2} dt$$

- $E(m)$: complete elliptic integral of second kind

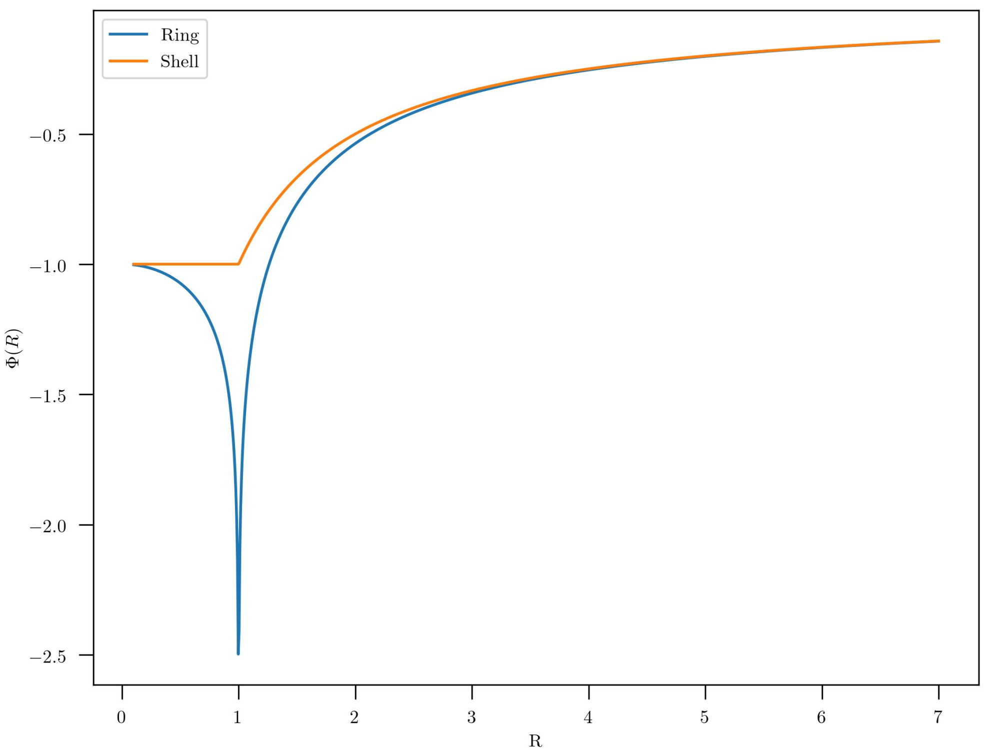
$$E(m) = \int_0^{\pi/2} \left[1 - m^2 \sin(t)^2\right]^{+1/2} dt$$

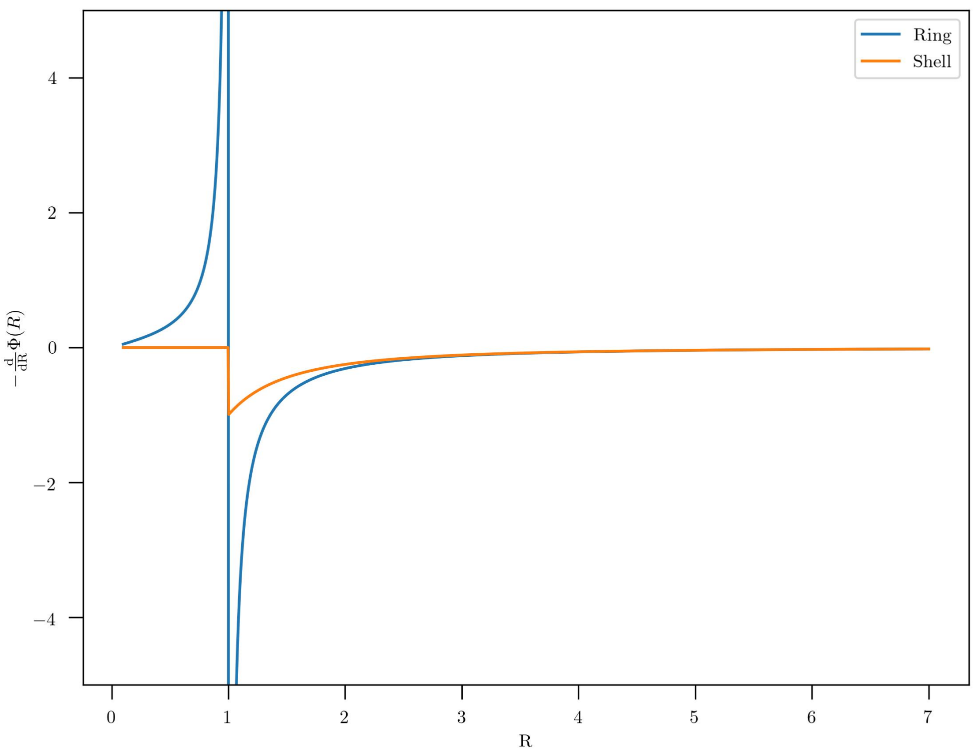
Potential / force of a ring

Lass & Blitzer 1982

$$\phi(R, z) = - \frac{2\,G M\, K(k)}{\pi \sqrt{(R_0 + R)^2 + z^2}} \qquad \left(k^2 = \frac{4 R_0 R}{(R + R_0)^2 + z^2} \right)$$

$$g(R, z) = - \frac{G M}{R\, \pi \sqrt{(R_0 + R)^2 + z^2}} \left[k \, \frac{R^2 - R_0^2 - z^2}{4(1-k) R R_0} \, E(k) + K(k) \right]$$





Potential of a razor-thin disk of surface density $\Sigma(R)$

Sum of rings

$$\phi(R,z) = \int_0^\infty \delta_{R'} \phi(R)$$

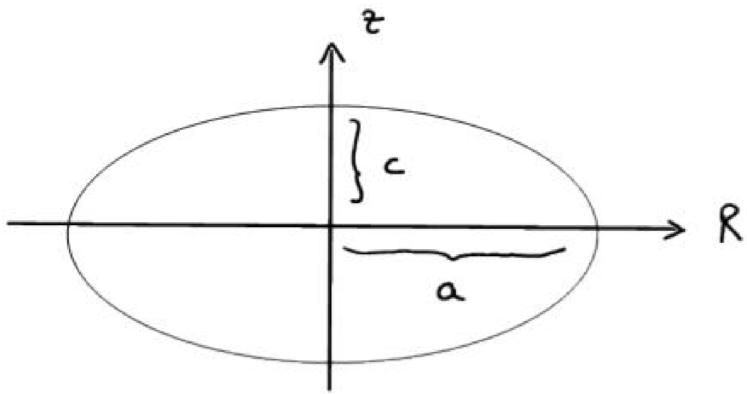
$$= \int_0^\infty - \frac{2 G \delta M' K(k)}{\pi \sqrt{(R'+R)^2 + z^2}}$$

with $\delta M' = 2\pi \Sigma(R') R' dR'$

$$\boxed{\phi(R,z) = -4G \int_0^\infty dR' \frac{\Sigma(R') R'}{\pi \sqrt{(R'+R)^2 + z^2}} K(k)}$$

The potential of spheroidal shells (homoeoids)

Spheroids ≡ ellipse of revolution (axisymmetric system)



Equation of an ellipse

$$\frac{R^2}{a^2} + \frac{z^2}{c^2} = 1$$

we assume a constant density ρ

ellipticity : $e^2 = 1 - \frac{c^2}{a^2}$

q tq $c = q \cdot a$

Mass $V = \frac{4}{3}\pi a^2 c \Rightarrow$

$$M(a) = \frac{4}{3}\pi q a^3 \rho$$

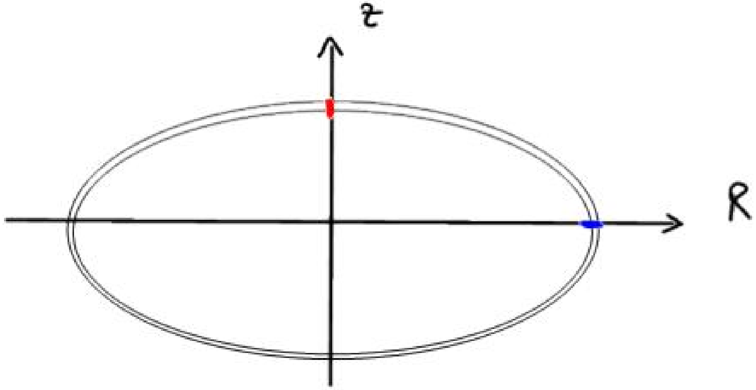
Surface density $\Sigma(R) = \int_{-\infty}^{\infty} dz\, \rho(R,z) = \rho \int_{-z(R)}^{z(R)} dz = 2\rho\, z(R)$

from the ellipse equation $z^2(R) = c^2\left(1 - \frac{R^2}{a^2}\right)$ $z(R) = q\sqrt{a^2 - R^2}$

$$\Sigma(R,a) = 2\rho\, q\sqrt{a^2 - R^2}$$

$$\Sigma_0(a) = \Sigma(0,a) = 2\rho q a$$

Homoeoid : Shell of a spheroid of constant density



(i) inner $\quad \dfrac{R^2}{a^2} + \dfrac{z^2}{c^2} = 1$

(o) outer $\quad \dfrac{R^2}{a^2} + \dfrac{z^2}{c^2} = (1 + \delta\beta)^2$

For $z = 0$

(i) $R = a$

(o) $R = a + a\,\delta\beta$

$\Big\}\ \Delta R = a\,\delta\beta$

For $R = 0$

(i) $z = c$

(o) $z = c + c\,\delta\beta$

$\Big\}\ \Delta z = c\,\delta\beta$

$\Longrightarrow$ the thickness of the shell varies

Mass

$M(a)$ = mass of a spheroid

$$\delta M(a) = \frac{dM}{da}\,\delta a = 4\pi q \rho a^2\,\delta a = 2\pi a\,\Sigma_0(a)\,\delta a$$

$$\boxed{\delta M(a) = 2\pi a\,\Sigma_0(a)\,\delta a}$$

Surface density

$\Sigma(a)$ = surface density of a spheroid

$$\delta\Sigma(a) = \frac{d\Sigma}{da}\,\delta a = \frac{2\rho q a}{\sqrt{a^2 - R^2}}\,\delta a = \frac{\Sigma_0(a)}{\sqrt{a^2 - R^2}}\,\delta a$$

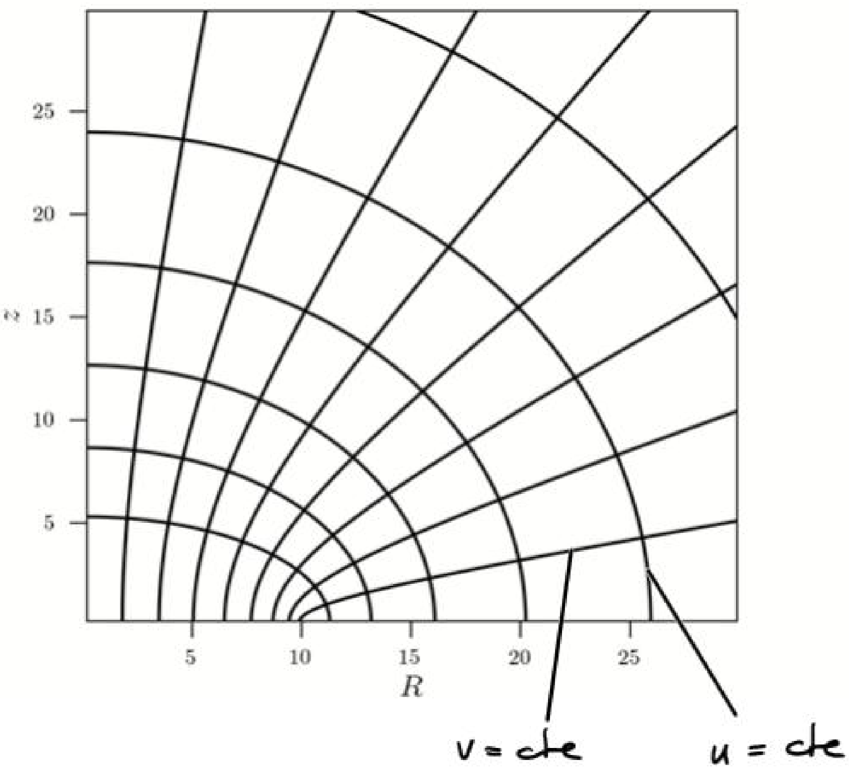
$$\boxed{\delta\Sigma(a) = \frac{\Sigma_0(a)}{\sqrt{a^2 - R^2}}\,\delta a}$$

Potential of homoeoids

Spheroidal coordinates

positive constant

$$(R, \phi, z) \rightarrow (u, \phi, v) \quad \begin{cases} R = \Delta \cosh u \, \sin v \\ z = \Delta \sinh u \, \cos v \end{cases}$$



For a constant u (removing v)

$$\frac{R^2}{\Delta^2 \cosh^2 u} + \frac{z^2}{\Delta^2 \sinh^2 u} = 1$$

Ellipticity $\quad e^2 = 1 - \frac{\sinh^2 u}{\cosh^2 u} = 1 - \tanh^2 u$

$$\begin{cases} u = 0 & e = 1 \\ u = \infty & e = 0 \end{cases}$$

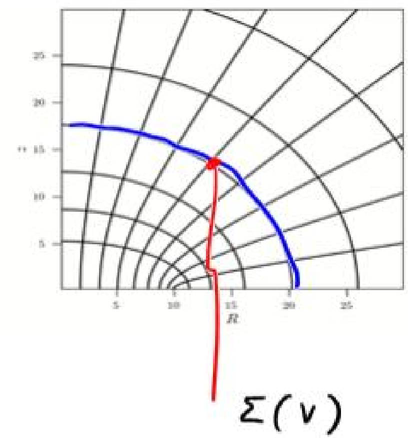
$$\begin{cases} a = \Delta \cosh u \\ c = \Delta \sinh u \end{cases}$$

It is possible to demonstrate that

1) the surface density of an homoeoid of "radius" u_0 and mass δM

$a = \Delta \cosh u$

$$\Sigma(v) = \frac{\delta M}{4\pi a^2 \sqrt{1 - e^2 \sin^2 v}}$$



2) its corresponding potential is

$$\phi(u) = -\frac{G\delta M}{ae} \begin{cases} \arcsin(e) & u \leq u_0 \\ \arcsin\left(\frac{1}{\cosh(u)}\right) & u \geq u_0 \end{cases}$$

Potential of an homoeoid

Assume $\phi = \phi(u)$ and try to solve $\nabla^2\phi = 0$

for $\phi = \phi(u)$

$$\nabla^2\phi = \frac{1}{\Delta^2(\sinh^2 u + \cos^2 v)}\left[\frac{1}{\cosh u}\frac{\partial}{\partial u}\left(\cosh u \frac{\partial \phi}{\partial u}\right)\right] = 0$$

$$\frac{\partial}{\partial u}\left(\cosh u \frac{\partial \phi}{\partial u}\right) = 0$$

$$\left(\begin{array}{l} \arcsin' \to \frac{1}{\sqrt{1-u^2}} \\ \cosh' \to \sinh u \end{array} \right)$$

solutions

1) $\phi = \phi_0 = $ cte

2) $\phi = -A \arcsin\left(\frac{1}{\cosh u}\right) + B$

for $u \to \infty$, using $R = \Delta \cosh u \cdot \sin v$

$$z = \Delta \sinh u \cos v$$

and $\cosh^2 u = \sinh^2 u \quad (u = \infty)$

we get $r^2 = R^2 + z^2 = \Delta^2(\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v)$

$$= \Delta^2 \cosh^2 u$$

$$\Rightarrow \frac{1}{\cosh(u)} \to \frac{\Delta}{r}$$

so, $-A \arcsin\left(\frac{1}{\cosh(u)}\right) + B \cong -A \arcsin\left(\frac{\Delta}{r}\right) + B \cong -\frac{\Delta}{r} + B$

$$\Rightarrow A = \frac{G\delta M}{\Delta} \qquad B = 0$$

So, we get the potential:

$$\phi(u) = -\frac{G\delta M}{\Delta} \begin{cases} \arcsin\left(\frac{1}{\cosh(u_0)}\right) & u < u_0 \\ \arcsin\left(\frac{1}{\cosh(u)}\right) & u > u_0 \end{cases}$$

u_0 is the surface of an ellipsoid of semi-major/minor axis

$$\begin{cases} a = \Delta \cosh u_0 \\ c = \Delta \sinh u_0 \end{cases} \Rightarrow e = \sqrt{1 - \frac{\cosh^2 u_0}{\sinh^2 u_0}} = \frac{1}{\cosh u_0}$$

and $ae = \Delta$

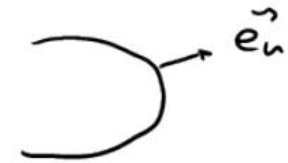
$$\phi(u) = -\frac{G\delta M}{ae} \begin{cases} \arcsin(e) & u < u_0 \\ \arcsin\left(\frac{1}{\cosh(u)}\right) & u > u_0 \end{cases}$$

Demonstration

What is the density at the surface u_0?

Gauss theorem:

$$\int \vec{\nabla}\phi \, d^2\vec{s} = 4\pi G M$$



$$\delta M = \frac{\vec{\nabla}\phi \cdot \vec{e}_n \, ds^2}{4\pi G}$$

$$\Sigma(u) = \frac{\delta M}{ds^2} = \frac{\vec{\nabla}\phi \cdot \vec{e}_n}{4\pi G}$$

In elliptical coord:

$$\vec{\nabla}\phi(u) = \frac{1}{\Delta\sqrt{\sinh^2 u + \cos^2 v}} \frac{\partial \phi}{\partial u} \vec{e}_n$$

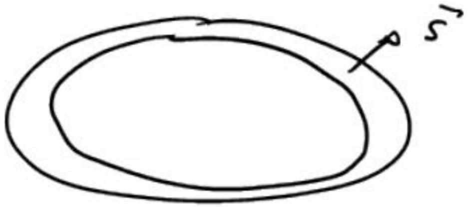
so

$$\Sigma(u) = \frac{1}{\Delta\sqrt{\sinh^2 u + \cos^2 v}} \left.\frac{\partial \phi}{\partial u}\right|_{u=u_0} \frac{1}{4\pi G}$$

$$\boxed{\Sigma(u) = \frac{\delta M}{4\pi a^2 \sqrt{1 - e^2 \sin^2 v}}}$$

Link between $\Sigma(u)$ and the surf. density of an homoeoid

$$\beta^2 = \beta^2(R,z) = \frac{R^2}{a^2} + \frac{z^2}{c^2}$$



$|\vec{s}| = s =$ thickness of the shell

$$\vec{s} = s \cdot \vec{e_n} = s \cdot \frac{\vec{\nabla}\beta}{|\vec{\nabla}\beta|}$$

$$\delta\beta = \vec{s} \cdot \vec{\nabla}\beta \qquad \beta(R,z) = \beta(R_0,z_0) + \vec{\nabla}\beta \cdot \vec{s}$$

$\frac{1}{2}(\ )^{-\frac{1}{2}}\ 2\frac{R}{a^2}$

$$= s\,|\vec{\nabla}\beta|$$

$$s = \frac{\delta\beta}{|\vec{\nabla}\beta|} \longrightarrow \begin{cases} \vec{\nabla}\beta = \frac{\partial\beta}{\partial R}\vec{e_R} + \frac{\partial\beta}{\partial z}\vec{e_z} = \frac{1}{\beta}\frac{R}{a^2}\vec{e_R} + \frac{1}{\beta}\frac{z}{c^2}\vec{e_z} \\ |\nabla\beta| = \sqrt{\frac{R^2}{a^4} + \frac{z^2}{c^4}\,\beta^{-2}} \end{cases}$$

$$s = \left(\frac{R^2}{a^4} + \frac{z^2}{c^4}\right)^{-\frac{1}{2}} \beta\,\delta\beta$$

$$\Sigma = \rho \cdot s = \rho\left(\frac{R^2}{a^4} + \frac{z^2}{c^4}\right)^{-\frac{1}{2}} \beta\,\delta\beta$$

we introduce v, such that : $R = \beta a \sin(v)$ $z = \beta c \cos(v)$

$$e = \sqrt{1 - \frac{c^2}{a^2}} \qquad e^2 = 1 - \frac{c^2}{a^2}$$

$$e^2 a^2 = a^2 - c^2$$

$$\left(\frac{R^2}{a^4} + \frac{z^2}{c^4} \right)^{-\frac{1}{2}} \beta \int d\beta$$

$$R^2 = \beta^2 a^2 \sin^2 v$$

$$z^2 = \beta^2 c^2 \cos^2 v$$

$$\frac{R^2}{a^4} + \frac{z^2}{c^4}$$

$$\beta^2 \frac{1}{a^2} \sin^2 v + \beta^2 \frac{1}{c^2} \cos^2 v$$

$$\beta^2 \left(\frac{\sin^2 v}{a^2} + \frac{\cos^2 v}{c^2} \right)$$

$$c^2 = a^2 (1 - e^2)$$

$$\beta^2 \left(\frac{\sin^2 v}{a^2} + \frac{1}{c^2} - \frac{\sin^2 v}{c^2} \right)$$

$$c = a \sqrt{1 - e^2}$$

$$\frac{\beta^2}{a^2 c^2} \left(c^2 \sin^2 v + a^2 - a^2 \sin^2 v \right)$$

$$\frac{\beta^2}{a^2c^2}\left(\sin^2 v\,(c^2-a^2)+a^2\right)$$

$$\frac{\beta^2}{a^{2\not{4}}(1-e^2)}\left(\not{a}^{\not{4}}-e^2\not{a}^{\not{4}}\sin^2 v\right)$$

$$\Sigma = \rho\left(\frac{R^2}{a^4}+\frac{t^2}{c^4}\right)^{-\frac{1}{2}}\beta\,\delta\beta$$

$$= \frac{a\sqrt{1-e^2}}{\not{\beta}\sqrt{1-e^2\sin^2 v}}\,\rho\,\not{\beta}\,\delta\beta = \frac{a\sqrt{1-e^2}\,\rho\,\delta\beta}{\sqrt{1-e^2\sin^2 v}}$$

Volume of the ellipsoid : $V = \frac{4}{3}\pi\,a^2c\,\beta^3 = \frac{4}{3}\pi\,a^3\beta^3\sqrt{1-e^2}$

$$\delta M = \delta(\rho M) = 4\pi\,a^3\beta^2\rho\sqrt{1-e^2}\,\delta\beta$$

introduce $\delta\beta$ in $\Sigma = \dfrac{a\sqrt{1-e^2}\,\rho\,\delta\beta}{\sqrt{1-e^2\sin^2 v}}$

and set $\beta = 1$

Demonstration

$$\Sigma(\nu) = \frac{a\sqrt{1-e^2}\,\rho\,\delta M}{4\pi a^3 \beta^2 \rho \sqrt{1-e^2}\,\sqrt{1-e^2\Omega^2\nu}}$$

$$\boxed{\Sigma(\nu) = \frac{\delta M}{4\pi a^2 \sqrt{1-e^2\Omega^2\nu}}}$$

$\#$

Newton’s Theorems

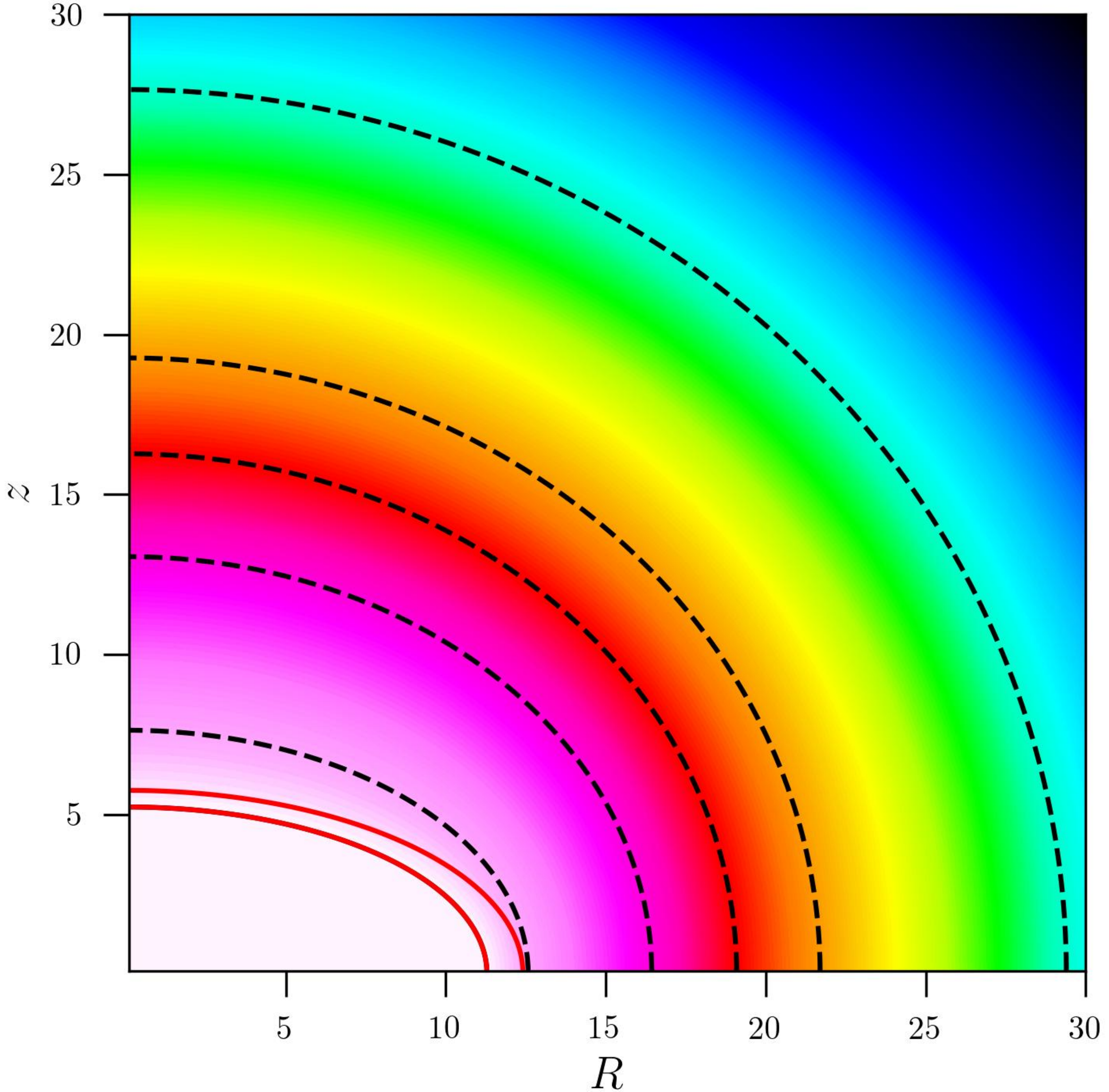
Newton’s third theorem:

- A mass that is inside a homoeoid experiences no net gravitational force from the homoeoid.

Homoeoid theorem:

- The exterior iso-potential surfaces of a thin homoeoid are the spheroids that are confocal with the shell itself. Inside the shell, the potential is constant.

potential of homoeoids

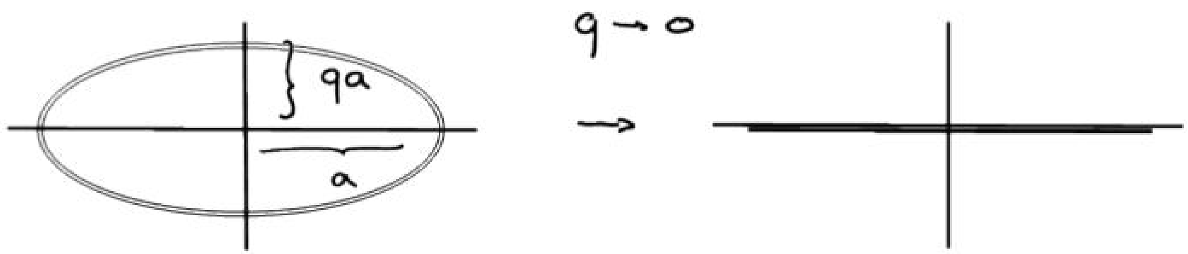


The potential of infinite thin (razor-thin) disks from homoeoids

The potential of zero thickness (razor-thin) disks from homoeoids

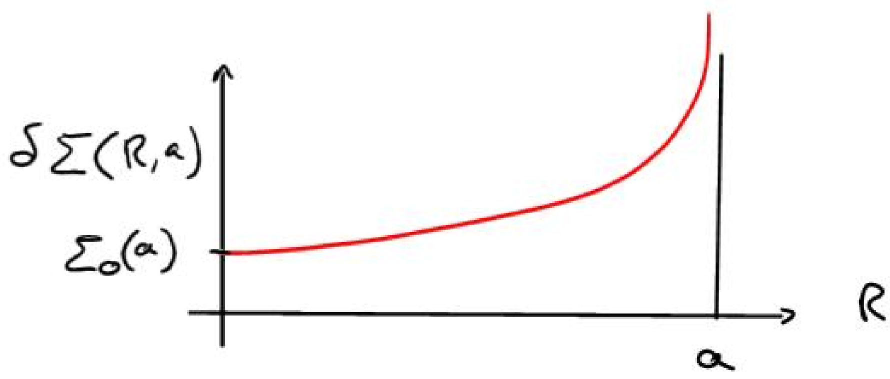
Idea Reproduce any surface density $\Sigma(R)$ by summing a set of infinitely flattened homoeoids

Infinitely flattened homoeoids



The surface density remains the same (indep. of q)

$$\delta\Sigma(a,R) = \frac{\Sigma_0(a)}{\sqrt{a^2 - R^2}}\,\delta a$$



Summing infinitely flattened homoeoids

$$\Sigma(R) = \sum_{a \geq R} \delta\Sigma(a,R) = \sum_{a \geq R} \frac{\Sigma_0(a)}{\sqrt{a^2 - R^2}} \, \delta a$$

$$= \int_R^\infty \frac{\Sigma_0(a)}{\sqrt{a^2 - R^2}} \, da$$

Abel integral

Solution:

$$\Sigma_0(a) = -\frac{2}{\pi} \frac{d}{da} \left(\int_a^\infty dR \, \frac{R\,\Sigma(R)}{\sqrt{R^2 - a^2}} \right)$$

For a given $\Sigma(R)$ we can compute $\Sigma_0(a)$ (the weights)

such that $\Sigma(R) = \int_R^\infty \delta\Sigma(a,R)$

Potential of infinitely flattened homoeoids

$$\phi(u) = -\frac{G\delta M}{ae}\arcsin\left(\frac{1}{\cosh(u)}\right) \qquad u \geq u_0$$

The potential is continuous across the plane $z=0$
we can compute it just above the plane
i.e, outside the shell

with $\delta M = 2\pi a\,\Sigma_0(a)\,\delta a$ and for $u \geq u_0$

and noting that for $q \to 0$ $e \to 1$

$$\delta\phi_a(R,z) = -\frac{G\,2\pi a\,\Sigma_0(a)\,\delta a}{ae}\arcsin\left(\frac{1}{\cosh(u)}\right)$$

$$= -2\pi G\,\Sigma_0(a)\,\delta a\,\arcsin\left(\frac{1}{\cosh(u)}\right)$$

Expression for u

from $\begin{cases} R = \Delta\cosh u\,\sin v \\ z = \Delta\sinh u\,\cos v \end{cases}$ and $\cos^2 v + \sin^2 v = 1$

$$\cosh^2 u = \frac{1}{4a^2}\Big[\underbrace{\sqrt{z^2+(a+R)^2}}_{\sqrt{+}} + \underbrace{\sqrt{z^2\;(a-R)^2}}_{\sqrt{-}}\Big]^2$$

Thus

$$\delta\phi_a(R, z) = -2\pi G\,\Sigma_0(a)\,\arcsin\left(\frac{2a}{\sqrt{+} + \sqrt{-}}\right)\delta a$$

Summing the contribution of all homoeoids

$$\phi(R, z) = \int_0^\infty \delta\phi_a(R, z) = -2\pi G \int_0^\infty \Sigma_0(a)\,\arcsin\left(\frac{2a}{\sqrt{+} + \sqrt{-}}\right) da$$

but $\Sigma_0(a) = -\frac{2}{\pi}\frac{d}{da}\left(\int_a^\infty dR' \,\frac{R'\Sigma(R')}{\sqrt{R'^2 - a^2}}\right)$

$$\phi(R, z) = 4G \int_0^\infty da\,\arcsin\left(\frac{2a}{\sqrt{+} + \sqrt{-}}\right)\frac{d}{da}\left(\int_a^\infty dR'\,\frac{R'\Sigma(R')}{\sqrt{R'^2 - a^2}}\right)$$

dep. only on "a" : can be tabulated

Integrating by part gives

$$\phi(R,z) = -2\sqrt{2}\,G \int_0^\infty da\, \frac{\left[(a+R)/\sqrt{+}\right] - \left[(a-R)/\sqrt{-}\right]}{\sqrt{R^2 - z^2 - a^2 + \sqrt{+}\cdot\sqrt{-}}} \int_a^\infty dR' \frac{R'\Sigma(R')}{\sqrt{R'^2 - a^2}}$$

Circular velocity

$$V_c^2(R) = R \left.\frac{\partial \phi}{\partial R}\right|_{z=0}$$

$$\frac{d}{dR} \arcsin\left(\frac{2a}{|a+R| + |a-R|}\right)$$

- $R < a \rightarrow a+R+a-R = 2a \Rightarrow \arcsin(1) \rightarrow \frac{d}{dR} = 0$
- $R > a \rightarrow a+R-a+R = 2R \Rightarrow \arcsin(\frac{a}{R}) \Rightarrow \frac{d}{dR} = -\frac{a/R^2}{\sqrt{1-(\frac{a}{R})^2}}$

$$V_c^2(R) = -4G \int_0^R da\, \frac{a}{\sqrt{R^2 - a^2}}\, \frac{d}{da}\left(\int_a^\infty dR' \frac{R'\Sigma(R')}{\sqrt{R'^2 - a^2}}\right)$$

Exponential disk

$$\Sigma(R) = \Sigma_0 \, e^{-R/R_d}$$

The integral in the razor-thin potential equation is then:

$$\int_a^\infty R' \frac{R'\Sigma_0 e^{-R'/R_d}}{\sqrt{R'^2 - a^2}} = \Sigma_0 a K_1(a/R_d)$$

The potential:

$$\Phi(R,z) = -2\sqrt{2}\,G \int_0^\infty a \frac{\frac{a+R}{\sqrt{z^2+(a+R)^2}} - \frac{a-R}{\sqrt{z^2+(a-R)^2}}}{\sqrt{R^2 - z^2 - a^2 + \sqrt{z^2+(a+R)^2}\sqrt{z^2+(a-R)^2}}} \times \Sigma_0 a K_1(a/R_d)$$

The circular velocity:

$$v_c^2 = 4\pi G \Sigma_0 R_d y^2 \left[I_0(y)K_0(y) - I_1(y)K_1(y)\right]$$

$$y = \frac{R}{2\,R_\mathrm{d}}$$

Bessel functions

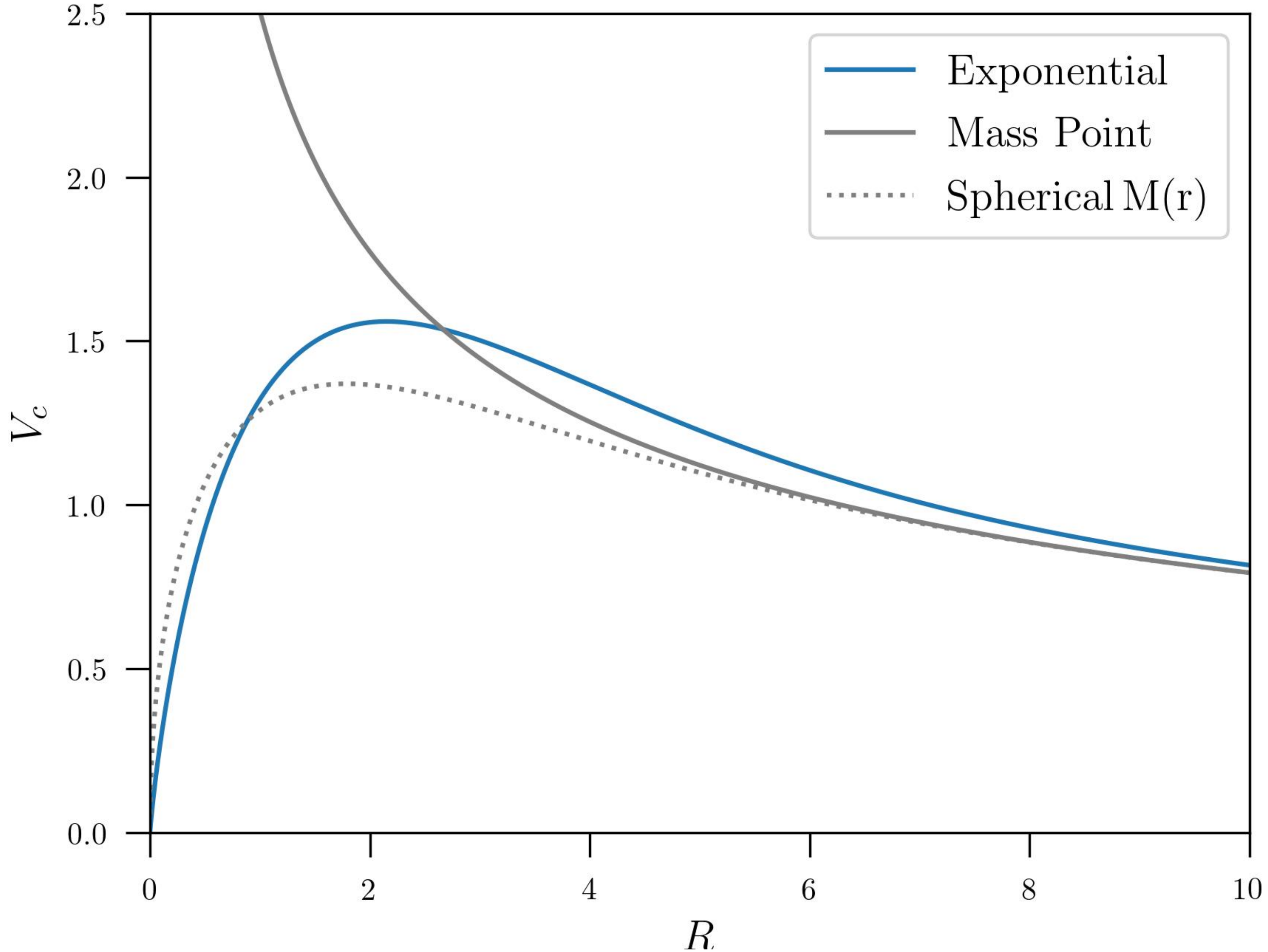
$$I_\nu(z) = i^{-\nu} J_\nu(iz)$$

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\nu+k)!}\left(\frac{1}{2}z\right)^{\nu+2k}$$

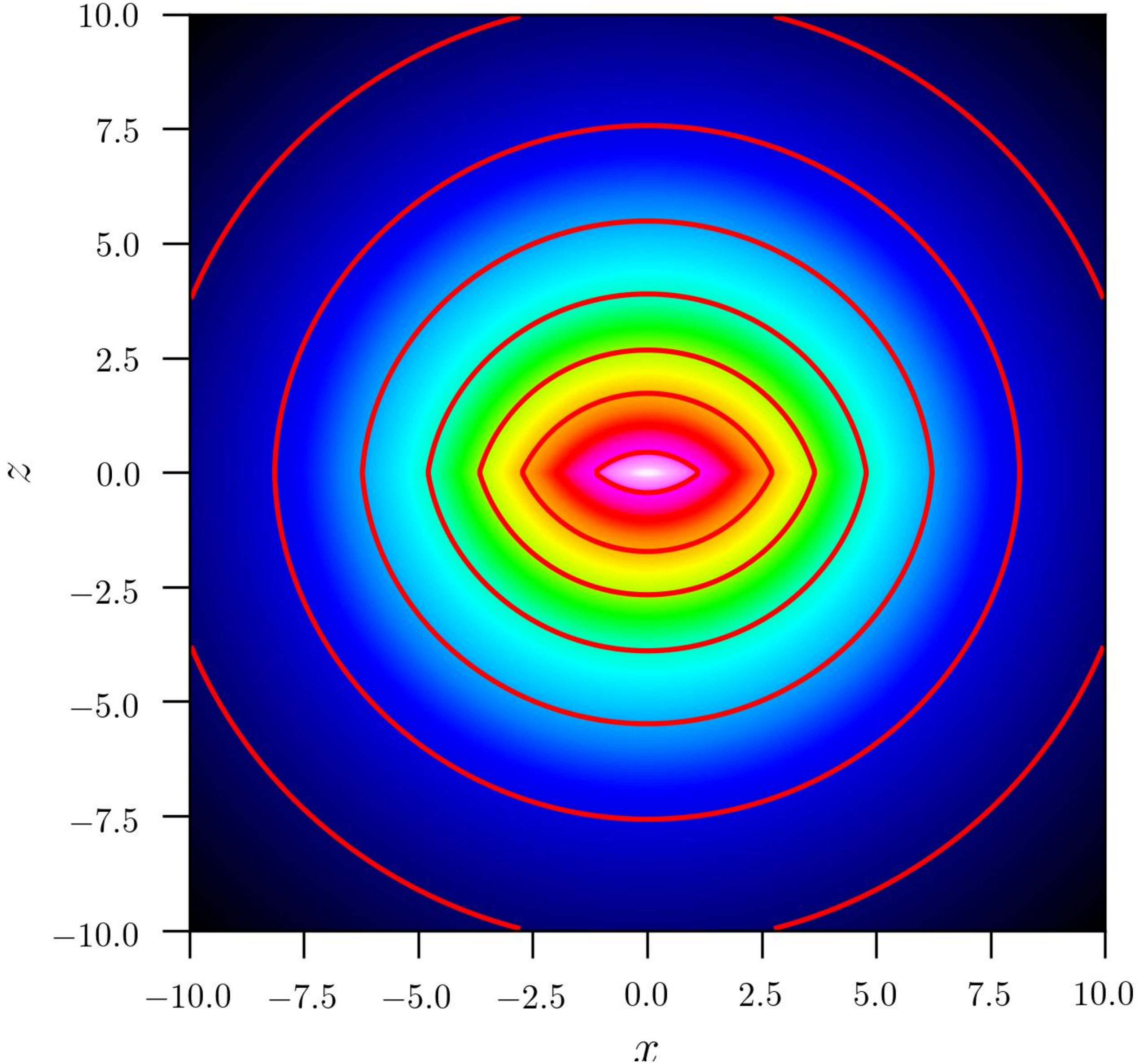
Exponential disk

Circular velocity rotation curve



Exponential disk

Potential



Mestel disk

$$\Sigma(R) = \begin{cases} \frac{v_0^2}{2\pi G R} & (R < R_{\max}) \\ 0 & (R \geq R_{\max}) \end{cases}$$

"2D" version of the Isothermal sphere

$$\text{for } \mathrm{R}_{\max} \to \infty$$

$$v_c^2 = \frac{2v_0^2}{\pi} \int_0^R \frac{a}{\sqrt{R^2 - a^2}} = v_0^2 = \mathrm{cte}$$

Computing the cumulative mass:

$$M(R) = 2\pi \int_0^R R' \, R' \, \Sigma(R') = \frac{v_0^2 R}{G}$$

we get:

$$v_0^2 = v_c^2(R) = \frac{GM(R)}{R}$$

This is very specific to the Mestel disk…
In general the external mass matter.

The End