Introduction to Differentiable Manifolds
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Exercise series 4 **2021–10–12**

Exercise 4.1 (A little heads-up regarding coordinate vectors)**.** Let φ and ψ be smooth charts on a smooth manifold M defined on the same domain U. If the first coordinate functions φ^0 and ψ^0 agree ($\varphi^0 \equiv \psi^0$ on U), this does *not* imply $\frac{\partial}{\partial \varphi^0}|_p = \frac{\partial}{\partial \psi^0}|_p$ for $p \in U$.

Work out a simple example of this fact e.g. on $M = \mathbb{R}^2$ by considering on the one hand the Cartesian coordinates (x, y) and on the other hand the chart (u, v) given by $u = x$, $v = x + y$.
This shows that $\frac{\partial}{\partial \varphi^i}|_p$ *depends on the whole system* $(\varphi^0, \dots, \varphi^{n-1})$, *not only on* φ^i.

Exercise 4.2 (The tangent space of a vector space)**.** Let V be an n-dimensional vector space.

(a) Let $\mathcal{A}$ be the set of linear isomorphisms $\varphi : V \to \mathbb{R}^n$. Show that there is a topology on V such that all $\varphi \in \mathcal{A}$ are homeomorphisms. Show that $\mathcal{A}$ is a smooth atlas on V. *In other words, any vector space has a natural smooth structure.*

(b) Fix $a \in V$. To every $v \in V$ we associate the curve passing through a

$$\gamma_v : \mathbb{R} \to V : t \mapsto a + tv$$

Show that the map $\Phi_a : V \to T_a V : v \mapsto \gamma_v'(0)$ is an isomorphism of vector spaces. *Hence we can identify a vector space with its tangent space in a canonical way.*

(c) Let $f : V \to W$ be a *linear* map between vector spaces V, W. Consider the differential $D_a f : T_a V \to T_{F(a)} W$ at any point $a \in V$. Identifying $T_a V \cong V$ and $T_{f(a)} W \cong W$ via the isomorphisms Φ_a, $\Phi_{f(a)}$, show that $D_a f$ is identified with f. That is, show that the following diagram commutes:

$$\begin{array}{ccc} T_a V & \xrightarrow{D_a f} & T_{f(a)} W \\ \uparrow{\scriptstyle \Phi_a} & & \uparrow{\scriptstyle \Phi_{f(a)}} \\ V & \xrightarrow[f]{} & W \end{array}$$

Exercise 4.3 (Differential of the determinant function)**.** Consider the determinant function $\det : M_n(\mathbb{R}) \to \mathbb{R}$, where $M_n(\mathbb{R}) \simeq \mathbb{R}^{n \times n}$ is the vector space of real $n \times n$, with its natural smooth structure. We want to compute its differential transformation $D_A \det$ at any matrix $A \in \mathrm{GL}_n(\mathbb{R})$ (i.e. at any invertible matrix),

$$D_A \det : T_A M_n(\mathbb{R}) \to T_{\det(A)} \mathbb{R}$$

(Note that we may identify $T_A M_n(\mathbb{R})$ with $M_n(\mathbb{R})$ and $T_{\det(A)}\mathbb{R}$ with $\mathbb{R}$.)

(a) Verify that det is a smooth function.
Hint: Write the determinant as a sum over all n-permutations.

(b) Show that the differential of det at the identity matrix $I \in M_n(\mathbb{R})$ is

$$D_I \det(B) = \mathrm{tr}(B).$$

where tr denotes the trace.

(c) Show that for arbitrary $A \in \mathrm{GL}_n(\mathbb{R}), B \in M_n(\mathbb{R})$.

$$D_A \det(B) = (\det A) \,\mathrm{tr}(A^{-1}B)$$

Hint: Write $\det(A + tB) = (\det A)(\det(I + tA^{-1}B))$.

(d) Show that $D_A \det$ is the null linear transformation if $A = 0$ and $n \geq 2$.

Exercise 4.4 (Diffeomorphic manifolds have the same dimension)**.** Let M and N be nonempty diffeomorphic manifolds. Show that $\dim M = \dim N$.

Exercise 4.5 (Tangent vectors as derivations)**.** Let M be a $\mathcal{C}^k$ manifold, $k \geq 1$, and let $p \in M$. Show that the map $\nu_p : X \in T_pM \mapsto D_X \in Der_pM$ defined by

$$D_X : f \in C^k(M, \mathbb{R}) \mapsto D_pf(X) \in T_{f(p)}\mathbb{R} \cong \mathbb{R}$$

is linear and injective.

Hint: To prove injectivity, take a chart ϕ that is defined at p. Any vector $X \in T_pM$ can be written as $X = \sum_i X^i \frac{\partial}{\partial \phi^i}|_p$. Show that $D_X(\phi^j) = X^j$.

Exercise 4.6 (Nonvectorial derivations* – optional)**.** Let M be a $\mathcal{C}^k$-differentiable n-manifold and let Der_pM be the vector space of derivations at some point $p \in M$.

(a) If M is a smooth manifold, show that $Der_p(M)$ has dimension n.
Hint: Prove Hadamard's lemma: any $f \in \mathcal{C}^{1+k}(M)_p$ can be locally written as $f(p) + \sum_i \varphi^i f_i$, with $f_i \in \mathcal{C}^k(M)$, φ a chart satisfying $\varphi(p) = 0$.

(b) Let $I \subseteq C^k(M, \mathbb{R})$ be the ideal of functions that vanish at p. Show that a linear map $X : \mathcal{C}^k(M) \to \mathbb{R}$ satisfies the Leibniz identity iff it vanishes on I^2 and on $\mathbb{R}$. Conclude that $Der_p(M) \equiv (I/I^2)^*$.

(c) (Newns–Walker, 1956) If $k < \infty$, show that I/I^2 is infinite dimensional if $k < \infty$. Conclude that Der_pM is infinite dimensional.
Hint: (From Laird E Taylor (1972), "The tangent space of a C^k manifold") For the case $M = \mathbb{R}$, $p = 0$, show that the functions $f_\sigma(t) = |t|^\sigma$ with $k < \sigma < k + 1$, taken modulo I^2, are linearly independent. To distinguish these functions, define the vanishing order $\mathrm{ord}(f)$ of a function $f \in I$ as the maximum $\alpha \geq 0$ such that $\lim_{t \to 0} \frac{f(t)}{|t|^\alpha} = 0$ and use Taylor's theorem to show that $\mathrm{ord}(f) \notin (k, k+1)$ if $f \in I^2$.