Introduction to Differentiable Manifolds	
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Exercise series 3, with solutions	**2021–10–19**

Note: Here we will use the notation T_pf rather than D_pf to denote the differential of a map f between manifolds. The symbol D_pf will only be used to denote the usual differential, defined when f is a map between open subsets of Euclidean spaces.

Exercise 3.1. Prove that for any open cover $\mathcal{U} = \{U_j\}_{j\in J}$ of a $\mathcal{C}^k$ manifold M there exists a partition of unity $(\xi_j)_{j\in J}$ such that $\operatorname{supp}(\xi_j) \subseteq U_j$ for all j.

Solution. This theorem is a second version of the theorem of existence of partitions of unity. By the first version of the theorem, there exists a partition of unity $(\eta_i)_{i\in I}$ where each η_i has its support $\operatorname{supp}(\eta_i)$ contained in U_j for some $j = f(i) \in J$. (We also know that $\operatorname{supp}(\eta_i)$ compact, but that is not useful for the proof.) Note that the functions η_i may be many more than the sets U_j. For example, if M is noncompact, since the compact supports $\operatorname{supp}(\eta_i)$ cover M, we see that I is infinite even if J is finite.

However, we can "group" the functions η_i as follows. For each $j \in J$ let $I_j = \{i \in I : f(i) = j\}$. The sets $(I_j)_{j\in J}$ form a partition of I, i.e., each $i \in I$ is contained in exactly one of the sets I_j (namely, when $j = f(i)$).

Define for each $j \in J$ the function $\xi_j = \sum_{i\in I_j} \eta_i$. This function ξ_j is $\mathcal{C}^k$ because locally it is a finite sum of $\mathcal{C}^k$ functions. The functions η_j are $\mathcal{C}^k$ and form a partition of unity because

$$\sum_{j\in J} \xi_j = \sum_{j\in J} \sum_{i\in I_j} \eta_i = \sum_{i\in I} \eta_i = 1$$

To finish, we must check that each ξ_j is supported on U_j. And indeed,

$$\operatorname{supp}(\xi_j) = \overline{\bigcup_{i\in I_j} \{x \in \mathbb{R} : \eta_i(x) > 0\}} \subseteq \bigcup_{i\in I_j} \overline{\{x \in \mathbb{R} : \eta_i(x) > 0\}} = \bigcup_{i\in I_j} \operatorname{supp}(\eta_i).$$

Here we are using again the local finiteness of the family $(\operatorname{supp}(\eta_i))_{i\in I}$. In general, if $(A_i)_i$ is a family of subsets of a topological space, then we have just an inclusion $\overline{\bigcup A_i} \supseteq \bigcup_i \overline{A_i}$. The equality holds, for instance, if the family $(A_i)_i$ is locally finite, which is the case here. ☐

Exercise 3.2. A continuous map $f : X \to Y$ is called *proper* if $f^{-1}(K)$ is compact for every compact set $K \subseteq Y$. Show that for every $\mathcal{C}^k$ manifold M there exists a $\mathcal{C}^k$ map $f : M \to [0, +\infty)$ that is proper.

Hint: Note that f must be unbounded unless M is compact. Use a function of the form $f = \sum_{i\in\mathbb{N}} c_i f_i$, where $(f_i)_{i\in\mathbb{N}}$ is a partition of unity and the c_i's are real numbers.

Solution. Let $(U_i)_{i\in\mathbb{N}}$ be a countable topological basis for M such that $\overline{U_i}$ is compact for each i. Let (f_i) be a $\mathcal{C}^k$ partition of unity on M such that $\operatorname{supp}(f_i) \subseteq U_i$ for each i. Define the $\mathcal{C}^k$ function $f : M \to \mathbb{R}$ by the formula $f(x) = \sum_{i\in\mathbb{N}} c_i\, f_i(x)$, where $c_i \geq 0$ are numbers satisfying $\lim_{i\to\infty} c_i = +\infty$. (For instance, we may put $c_i = i$.)

We can view $f(x)$ as a weighted average of the numbers c_i, using as weights the coefficients $f_i(x) \geq 0$, which satisfy $\sum_i f_i(x) = 1$. In particular, note that if $I_x \subseteq \mathbb{N}$ is the set of indices such that U_i contains the point x, then any upper or lower bound for the numbers c_i with $i \in I_x$ is also an upper or lower bound for $f(x)$. It follows that if $f(x) < c$, then x is contained in the union of the first few U_i's which satisfy $c_i < c$.

To see that f is proper, let $K \subseteq \mathbb{R}$ be a compact set. Take any number $c \geq 0$ such that $K \subseteq (-c, c)$, and let $i_c \in \mathbb{N}$ such that $c_i \geq c$ for $i \geq i_c$. The preimage $f^{-1}(K)$ consists of points x satisfying $f(x) < c$, and is therefore contained in the compact set $\bigcup_{i<i_c} \overline{U_i}$. Since the set $f^{-1}(K)$ is closed, we conclude that it is compact. ☐

Exercise 3.3. Let M be a $\mathcal{C}^k$ manifold and let U be an open neighborhood of the set $M \times \{0\}$ in the space $M \times [0, +\infty)$. Show that there exists a $\mathcal{C}^k$ function $f : M \to (0, +\infty)$ whose graph is contained in U.

Solution. Every point $\{x\} \times \{0\}$ of the set $M \times \{0\}$ has a neighborhood $V \times [0, \varepsilon)$ contained in U, where $V \subseteq M$ is an open neighborhood of x and $\varepsilon > 0$. Thus there is a covering of M by open sets V_i and numbers $\varepsilon_i > 0$ such that $V_i \times [0, \varepsilon_i) \subseteq U$ for all i. Let $(f_i)_i$ be a partition of unity with $\text{supp}(f_i) \subseteq U_i$. Then we can take the $\mathcal{C}^k$ function $f = \sum_i \frac{\varepsilon_i}{2} f_i$. □

Exercise 3.4. Let M be a $\mathcal{C}^k$-differentiable n-manifold. Show that:

(1)

$$(p, \phi, v) \sim (q, \psi, w) \quad \iff \quad q = p \quad \text{and} \quad w = \mathrm{D}_{\phi(p)}(\psi\,\phi^{-1})(v)$$

is an equivalence relation between coordinatized tangent vectors.

Solution. Let us check that $\sim$ is an equivalence relation:

- reflexivity: Let (p, ϕ, v) be a coordinatized tangent vector, with $\phi : W \to U$. To see that $(p, \phi, v) \sim (p, \phi, v)$, we compute

$$D_{\phi(p)}(\phi \circ \phi^{-1})(v) = D_{\phi(p)}(\text{id}_U)(v) = \text{id}_{\mathbb{R}^n}(v) = v$$

- symmetry : Suppose that $(p, \phi, v) \sim (q, \psi, w)$.
This means that $p = q$ and $w = D_{\psi(p)}(\psi \circ \phi^{-1})(v)$. Therefore

$$\begin{aligned} D_{\psi(q)}(\phi\,\psi^{-1})(w) &= D_{\psi(q)}(\phi\,\psi^{-1})(D_{\phi(p)}(\psi\,\phi^{-1})(v)) \\ &= D_{\phi(p)}((\phi\,\psi^{-1}) \circ (\psi\,\phi^{-1}))(v) \\ &= D_{\phi(p)}(\text{id}_U)(v) = v \end{aligned}$$

which implies that $(q, \psi, w) \sim (p, \phi, v)$.

- transitivity:
Suppose that $(p, \phi, v) \sim (q, \psi, w)$ and $(q, \psi, w) \sim (r, \eta, z)$.
We know from the first relation that $p = q$ and $D_{\phi(p)}(\psi \circ \phi^{-1})(v) = w$;
from the second relation we know that $q = r$ and $D_{\psi(q)}(\eta \circ \psi^{-1})(w) = z$.
Then we observe that $p = r$ and by the chain rule,

$$\begin{aligned} D_{\phi(p)}(\eta\,\phi^{-1})(v) &= D_{\phi(p)}((\eta\,\psi^{-1}) \circ (\psi\,\phi^{-1}))(v) \\ &= D_{\psi(p)}(\eta\,\psi^{-1})((D_{\phi(p)}(\psi\,\phi^{-1})(v)) \\ &= D_{\psi(p)}(\eta \circ \psi^{-1})(w) = z \end{aligned}$$

We conclude that $(p, \phi, v) \sim (r, \eta, z)$.

□

(2) Fixed a point $p \in M$ and a $\mathcal{C}^k$ chart φ defined on p, the function $\mathbb{R}^n \to T_pM$ sending $v \mapsto [p, \varphi, v]$ is a bijection.

Solution. The function defined in this point has been defined in the course and it was denoted by ι (we will keep this notation here).

- Injectivity:
Let $v, v' \in \mathbb{R}^n$ such that $\iota(v) = \iota(v')$. Thus $(p, \phi, v) \sim (p, \phi, v')$. It follows that $v' = D_{\phi(p)}(\phi \circ \phi^{-1})(v) = D_{\phi(p)}(\text{id}_U)(v) = \text{id}_{\mathbb{R}^n}(v) = v$ where U is the image of ϕ.

- Surjectivity:
Let $[p, \psi, w] \in T_pM$, we want to find a $v \in \mathbb{R}^n$ such that $\iota(v) = [p, \psi, w]$. We have to ensure $[p, \phi, v] = [p, \psi, w]$, which means that $(p, \phi, v) \sim (p, \psi, w)$. It suffices to put $v = D_{\psi(p)}(\phi \circ \psi^{-1})(w)$.

□

(3) Vector addition and scalar multiplication are well defined and make T_pM a real vector space of dimension n.

Solution. We have to show that the sum $[p,\phi,v]+[p,\phi,w]=[p,\phi,v+w]$ is well defined, i.e., it does not depend on the representatives we take of the equivalence classes $[p,\phi,v]$ and $[p,\phi,w]$. Initially we have the representatives (p,ϕ,v) and (p,ϕ,w), and we take a second set of representatives using a different chart ψ, i.e.,

$$(p,\phi,v)\sim(p,\psi,\overline{v}) \quad \text{and} \quad (p,\phi,w)\sim(p,\psi,\overline{w}).$$

and we have to show that $(p,\phi,v+w)\sim(p,\psi,\overline{v}+\overline{w})$. We compute

$$\overline{v}+\overline{w}=D_{\phi(p)}(\psi\circ\phi^{-1})(v)+D_{\phi(p)}(\psi\circ\phi^{-1})(w)=D_{\phi(p)}(\psi\circ\phi^{-1})(v+w).$$

We can use the same idea to show that scalar multiplication $\lambda\,[p,\phi,v]=[p,\phi,\lambda\,v]$ is well defined.

□

(4) The differential of a $\mathcal{C}^k$ map $f:M\to N$ at a point $p\in M$ is a well-defined linear map $T_pf:T_pM\to T_pN$.

Solution. We want to show that the map T_pf is well defined.

We take two representatives of the class $[p,\phi,v]$: $(p,\phi,v)\sim(p,\overline{\phi},\overline{v})$, then we apply D_pf on both of them and we get:

$$T_pf(p,\phi,v)=(f(p),\psi,\underbrace{D_{f(p)}(\psi\circ f\circ\phi^{-1})(v)}_{a})$$

$$T_pf(p,\overline{\phi},\overline{v})=(f(p),\overline{\psi},\underbrace{D_{\overline{f}(p)}(\overline{\psi}\circ f\circ\overline{\phi}^{-1})(\overline{v})}_{b})$$

We will show that the resulting expressions represent the same tangent vector by checking that $D_{\psi\circ f(p)}(\overline{\psi}\circ\psi^{-1})(a)=b$.

We compute

$$\begin{aligned}D_{\psi\circ f(p)}(\overline{\psi}\circ\psi^{-1})(D_{f(p)}(\psi\circ f\circ\phi^{-1})(v)) &= D_{\phi(p)}(\overline{\psi}\circ f\circ\phi^{-1}(v))\\ &= D_{\phi(p)}(\overline{\psi}\circ f\circ\phi^{-1}(D_{\overline{\phi}(p)}(\phi\circ\overline{\phi}^{-1})(\overline{v})))\\ &= D_{\overline{\phi}(p)}(\overline{\psi}\circ f\circ\overline{\phi}^{-1})(\overline{v}).\end{aligned}$$

This concludes the proof.

Now we want to show that T_pf is linear.

$$\begin{aligned}T_pf([p,\phi,v]+[p,\phi,w]) &= D_pf([p,\phi,v+w])\\ &= [f(p),\psi,D_{\phi(p)}\psi\circ f\circ\phi^{-1}(v+w)]\\ &= [p,\phi,D_{\phi(p)}\psi\circ f\circ\phi^{-1}(v)+D_{\phi(p)}\psi\circ f\circ\phi^{-1}(w)]\\ &= [p,\phi,D_{\phi(p)}\psi\circ f\circ\phi^{-1}(v)]+[p,\phi,D_{\phi(p)}\psi\circ f\circ\phi^{-1}(w)]\\ &= T_pf([p,\phi,v])+T_pf([p,\phi,w])\end{aligned}$$

□

(5) *Chain rule*: for $\mathcal{C}^k$ maps $f:M\to N$, $g:N\to L$ and a point $p\in M$,

$$T_p(g\circ f)=T_{f(p)}g\circ T_pf.$$

In particular, if f is a diffeo, then T_pf has inverse $(T_pf)^{-1}=T_{f(p)}(f^{-1})$.

Solution. Here we consider the following charts : ϕ for the point p, ψ for the point $f(p)$ and η for $g(f(p))$. We check that

$$T_p(g\circ f)([p,\phi,v])=[g\circ f(p),\psi,D_{\phi(p)}(\psi\circ(g\circ f)\circ\phi^{-1})(v)]$$

coincides with

$$\begin{aligned}T_{f(p)}g \circ T_pf([p,\phi,v]) &= T_{f(p)}g([f(p),\psi,D_{\phi(p)}(\psi\circ f\circ\phi^{-1})(v)])\\ &= [g(f(p)),\eta,D_{\psi\circ f(p)}(\eta\circ g\circ h^{-1})(D_{\phi(p)}(\psi\circ f\circ\phi^{-1})(v))]\\ &= [g\circ f(p),\eta,D_{\phi(p)}(\eta\circ(g\circ f)\circ\phi^{-1}(v))].\end{aligned}$$

Before continuing, let us prove that for any manifold M and any $p \in M$ we have $T_p(Id_M) = \mathrm{id}_{T_pM}$. To see this we take a chart $\phi : W \to U$ that is defined at p and compute

$$T_p(\mathrm{id}_M)([p,\phi,v]) = [p,\phi,D_{\phi(p)}(\phi\circ\mathrm{id}_M\circ\phi^{-1})(v)] = [p,\phi,D_{\phi(p)}(\mathrm{id}_U)v] = [p,\phi,\mathrm{id}_{\mathbb{R}^n}v] = [p,\phi,v].$$

Now let $f : M \to N$ be a diffeomorphism, and let $p \in M$ and $q = f(p) \in N$. Then, applying the chain rule to the equation $f^{-1}\circ f = \mathrm{id}_M$ we get

$$T_qf^{-1}\circ T_pf = T_p\mathrm{id}_M = \mathrm{id}_{T_pM}.$$

Similarly, from $f\circ f^{-1} = \mathrm{id}_N$ we obtain

$$T_pf\circ T_qf^{-1} = T_q\mathrm{id}_N = \mathrm{id}_{T_qN}.$$

We conclude that T_pf is invertible, with inverse T_qf^{-1}. □

(6) *Change of coordinates*: Let $X \in T_pM$ be a tangent vector and let φ, $\widetilde{\varphi}$ be $\mathcal{C}^k$ charts of M defined at a p. Let $(X^i)_i$ be coordinate tuple of X with respect to the basis $\left(\frac{\partial}{\partial\varphi^i}\big|_p\right)_i$, and let $(\widetilde{X}^j)_j$ be the coordinate tuple of X with respect the basis $\left(\frac{\partial}{\partial\widetilde{\varphi}^j}\big|_p\right)_j$, so that

$$X = \sum_i X^i\frac{\partial}{\partial\varphi^i}\Big|_p = \sum_j \widetilde{X}^j\frac{\partial}{\partial\widetilde{\varphi}^j}\Big|_p.$$

Show that

$$\widetilde{X}^j = \sum_i X^i\frac{\partial\widetilde{\varphi}^j}{\partial\varphi^i}\Big|_{\varphi(p)},$$

where $\frac{\partial\widetilde{\varphi}^j}{\partial\varphi^i}\Big|_{\varphi(p)}$ is the partial derivative that appears in the position (j,i) of the Jacobian matrix $J_{\varphi(p)}(\widetilde{\varphi}\circ\varphi^{-1}) : \mathbb{R}^n \to \mathbb{R}^n$.

Solution. Using the equation $\frac{\partial}{\partial\varphi^i} = \sum_j \frac{\partial\widetilde{\varphi}^j}{\partial\varphi^i}\frac{\partial}{\partial\widetilde{\varphi}^j}$, we get

$$X = \sum_i X^i\frac{\partial}{\partial\varphi^i} = \sum_i X^i\sum_j\frac{\partial\widetilde{\varphi}^j}{\partial\varphi^i}\frac{\partial}{\partial\widetilde{\varphi}^j} = \sum_j\sum_i X^i\frac{\partial\widetilde{\varphi}^j}{\partial\varphi^i}\frac{\partial}{\partial\widetilde{\varphi}^j}$$

Since on the other hand we have

$$X = \sum_j \widetilde{X}^j\frac{\partial}{\partial\widetilde{\varphi}^j}$$

and the vectors $\frac{\partial}{\partial\widetilde{\varphi}^j}$ are linearly independent, we conclude that

$$\widetilde{X}^j = \sum_i X^i\frac{\partial\widetilde{\varphi}^j}{\partial\varphi^i}$$

Donc la morale c'est que nous pouvons exprimer les coefficiens des vecteurs tangents d'une base par rapport à une autre base en utilisant les coefficients (j,i) de la matrice de la transformation linéaire $D_{\varphi(p)}(\widetilde{\varphi}\circ\varphi^{-1})$. □

Exercise 3.5 (Velocity vectors of curves)**.** Let M be a $\mathcal{C}^k$ differentiable manifold. The *velocity vector* of a differentiable curve $\gamma : I \subseteq \mathbb{R} \to M$ at an instant $t \in I$ is the vector $\gamma'(t) := T_t\gamma(1|_t) \in T_{\gamma(t)}M$. (Here the $1|_t$ represents the element $[t,\mathrm{id}_I,1]$ of $T_tI \simeq \mathbb{R}$.)

Show that for any tangent vector $X \in TM$ there exists a $\mathcal{C}^k$ curve $\gamma : (-\varepsilon,\varepsilon) \to M$ such that $\gamma'(0) = X$.

Solution. By definition we have $\gamma'(t) := T_t\gamma[t, \mathrm{id}_U, 1] \in T_{\gamma(0)}M$

Note that the equation $\gamma'(0) = X$ implies that $\gamma(0) = p$.

Write $X = [p, \varphi, v]$, where $p \in M$ and φ is a $\mathcal{C}^k$ chart of M that is defined at p, and $v \in \mathbb{R}^n$. If $\varepsilon > 0$ is small enough, we may define on the interval $I = (-\varepsilon, \varepsilon)$ a function $\gamma : I \to M$ by the formula

$$\gamma(t) = \varphi^{-1}(\varphi(p) + tv).$$

In other words, γ is the function whose local expression $g = \gamma|^{\varphi}_{\mathrm{id}_I}$ with respect to the charts id_I of I and φ of M is $g(t) = \varphi(p) + tv$. Note that $g'(t) = v$ for all $t \in I$.

Let us verify that $\gamma'(0) = X$. For all $t \in I$ we have

$$\begin{aligned} \gamma'(t) &= T_t\gamma[t, \mathrm{id}_I, 1] \quad \text{by definition of } \gamma'(t) \\ &= [\gamma(t), \varphi, \mathrm{D}_t\, \gamma|^{\mathrm{id}_{\mathbb{R}}}_{\varphi}(1)] \quad \text{by definition of } T_t\gamma \\ &= [\gamma(t), \varphi, g'(t)] \end{aligned}$$

In particular, $\gamma'(0) = [p, \varphi, v] = X$, as intended. □

Exercise 3.6 (Spherical coordinates on $\mathbb{R}^3$)**.** Consider the following map defined for $(r, \varphi, \theta) \in W := \mathbb{R}^+ \times (0, 2\pi) \times (0, \pi)$:

$$\Psi(r, \varphi, \theta) = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta) \in \mathbb{R}^3.$$

Check that Ψ is a diffeomorphism[1] onto its image $\Psi(W) =: U$. We can therefore consider Ψ^{-1} as a smooth chart on $\mathbb{R}^3$ and it is common to call the component functions of Ψ^{-1} the **spherical coordinates** (r, φ, θ).

Express the coordinate vectors of this chart

$$\frac{\partial}{\partial r}\Big|_p, \frac{\partial}{\partial \varphi}\Big|_p, \frac{\partial}{\partial \theta}\Big|_p$$

at some point $p \in U$ in terms of the standard coordinate vectors $\frac{\partial}{\partial x}\big|_p$, $\frac{\partial}{\partial y}\big|_p$, $\frac{\partial}{\partial z}\big|_p$.

Solution. Consider the transition from spherical coordinates (r, φ, θ) to Cartesian coordinates (x, y, z), given by the map

$$\begin{aligned} \Psi : W &\to U \\ (r, \varphi, \theta) &\mapsto (x, y, z) \end{aligned}$$

where

$$\begin{cases} x = r\cos\varphi\,\sin\theta \\ y = r\sin\varphi\,\sin\theta \\ z = r\cos\theta \end{cases}$$

Let $p \in U$. The general formula for the change of coordinates is

$$\frac{\partial}{\partial \psi^i}\Big|_p = \sum_j \frac{\partial}{\partial \psi^i}\Big|_p \widetilde{\psi}^j \frac{\partial}{\partial \widetilde{\psi}^j}\Big|_p.$$

We apply this formula in the current setting where $\psi = \Psi^{-1}$ is the given chart on U (by abuse of notation we denote its coordinate functions by (r, φ, θ)) and $\widetilde{\psi} = \mathrm{id}_{\mathbb{R}^3}$

[1] Here "diffeomorphism" is meant in the standard sense of maps between open subsets of $\mathbb{R}^3$. The inverse function theorem can be useful here.

(we denote its coordinate functions by (x, y, z)). Then for $p \in U$ we have[2]

$$\begin{aligned}
\frac{\partial}{\partial r}\Big|_p &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x}\Big|_p + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}\Big|_p + \frac{\partial z}{\partial r}\frac{\partial}{\partial z}\Big|_p \\
&= \cos\varphi\sin\theta\frac{\partial}{\partial x}\Big|_p + \sin\varphi\sin\theta\frac{\partial}{\partial y}\Big|_p + \cos\theta\frac{\partial}{\partial z}\Big|_p \\
&= \frac{1}{(x^2+y^2+z^2)^{1/2}}\Big(x\frac{\partial}{\partial x}\Big|_p + y\frac{\partial}{\partial y}\Big|_p + z\frac{\partial}{\partial z}\Big|_p\Big) \\
\frac{\partial}{\partial \varphi}\Big|_p &= \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x}\Big|_p + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y}\Big|_p + \frac{\partial z}{\partial \varphi}\frac{\partial}{\partial z}\Big|_p \\
&= -r\sin\varphi\sin\theta\frac{\partial}{\partial x}\Big|_p + r\cos\varphi\sin\theta\frac{\partial}{\partial y}\Big|_p \\
&= -y\frac{\partial}{\partial x}\Big|_p + x\frac{\partial}{\partial y}\Big|_p \\
\frac{\partial}{\partial \theta}\Big|_p &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x}\Big|_p + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y}\Big|_p + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z}\Big|_p \\
&= r\cos\varphi\cos\theta\frac{\partial}{\partial x}\Big|_p + r\sin\varphi\cos\theta\frac{\partial}{\partial y}\Big|_p - r\sin\theta\frac{\partial}{\partial z}\Big|_p \\
&= \frac{xz}{(x^2+y^2)^{\frac{1}{2}}}\frac{\partial}{\partial x}\Big|_p + \frac{yz}{(x^2+y^2)^{\frac{1}{2}}}\frac{\partial}{\partial y}\Big|_p - (x^2+y^2)^{\frac{1}{2}}\frac{\partial}{\partial z}\Big|_p
\end{aligned}$$

□

Exercise 3.7 (The tangent plane of the sphere)**.** Consider the inclusion $\iota : S^2 \to \mathbb{R}^3$, where we endow both spaces with the standard smooth structure. Let $p \in S^2$. What is the image of $D_p\iota : T_pS^2 \to T_p\mathbb{R}^3$? (Identify $T_p\mathbb{R}^3$ with $\mathbb{R}^3$ in the standard way. So the result should be the equation for a plane in $\mathbb{R}^3$.)

Hint: Use Exercise 6 on spherical coordinates.

Solution. Let $p \in \mathbb{S}^2$ and let us use spherical coordinates to parametrize a neighborhood of p. Define the map $\Psi(\varphi, \theta) = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$ from $W = (0, 2\pi) \times (0, \pi)$ to $U = \Psi(W)$. For simplicity we assume that $p \in U$ (otherwise we just take $\tau \circ \Psi$ instead of Ψ, where τ is a rotation of $\mathbb{R}^3$).

Then Ψ^{-1} is a smooth local chart for the sphere and, denoting its coordinate functions by (φ, θ), a basis of the tangent space $T_p\mathbb{S}^2$ is given by $\frac{\partial}{\partial\varphi}\big|_p, \frac{\partial}{\partial\theta}\big|_p$.

Now $T_p\iota : T_p\mathbb{S}^2 \to T_p\mathbb{R}^3$ sends $\frac{\partial}{\partial\varphi}\big|_p, \frac{\partial}{\partial\theta}\big|_p$ to the corresponding coordinate vectors[3] $\frac{\partial}{\partial\varphi}\big|_p, \frac{\partial}{\partial\theta}\big|_p$ of the spherical coordinates on $\mathbb{R}^3$. We have already seen the expression in cartesian coordinates for the latter (here $p = (x, y, z)^T$):

$$\begin{aligned}
\frac{\partial}{\partial \varphi}\Big|_p &= -y\frac{\partial}{\partial x}\Big|_p + x\frac{\partial}{\partial y}\Big|_p \\
\frac{\partial}{\partial \theta}\Big|_p &= \frac{xz}{(x^2+y^2)^{\frac{1}{2}}}\frac{\partial}{\partial x}\Big|_p + \frac{yz}{(x^2+y^2)^{\frac{1}{2}}}\frac{\partial}{\partial y}\Big|_p - (x^2+y^2)^{1/2}\frac{\partial}{\partial z}\Big|_p.
\end{aligned}$$

[2]There is a bit of abuse of notation going on; e.g. $\frac{\partial x}{\partial r}$ really means $\frac{\partial}{\partial r}|_p(x)$, i.e. the coordinate vector $\frac{\partial}{\partial r}$ applied to the function $x : \mathbb{R}^3 \to \mathbb{R}$ and this by definition is $\frac{\partial(r\cos\varphi\sin\theta)}{\partial r}|_{\psi(p)}$. The potentially confusing thing here is that r denotes at the same time the first component of the chart ψ (in the lecture this was φ^i) and the coordinate on the image of the chart in $\mathbb{R}^3$ (in the lecture this was x^i). But this sloppiness is common and actually helps with computations as you see above.

[3]Note that while these tangent vectors have the same symbol, they really are different objects (the latter acts on functions defined on (an open subset of) $\mathbb{R}^3$, the former on functions defined on (an open subset of) S^2: To make the distinction clear, let us denote the spherical coordinate chart on $\mathbb{R}^3$ by $(\overline{r}, \overline{\varphi}, \overline{\theta})$. Then the statement in the text reads $\iota_*\frac{\partial}{\partial\varphi}\big|_p = \frac{\partial}{\partial\overline{\varphi}}\big|_p$, $\iota_*\frac{\partial}{\partial\theta}\big|_p = \frac{\partial}{\partial\overline{\theta}}\big|_p$. But with respect to the chart (φ, θ) on $\mathbb{S}^2$ and $(\overline{r}, \overline{\varphi}, \overline{\theta})$ on $\mathbb{R}^3$, the coordinate representation $\hat{\iota}$ of ι is just $\hat{\iota}(\varphi, \theta) = (1, \varphi, \theta)$ and so the result follows by looking at the Jacobian of $\hat{\iota}$.

Under the "standard identification" of $T_p\mathbb{R}^3$ with $\mathbb{R}^3$ discussed in the lecture, the first coordinate vector $\frac{\partial}{\partial x}\Big|_p$ corresponds to $e_0 = (1, 0, 0)^T$, etc.

So under this identification $\frac{\partial}{\partial \varphi}\big|_p$, $\frac{\partial}{\partial \theta}\big|_p$ span the plane

$$\operatorname{span}\big\{(-y, x, 0)^T, (xz, yz, -(x^2 + y^2))^T)\big\}.$$

Clearly the vector $p = (x, y, z)^T$ is perpendicular to this plane and hence an equation for the latter is

$$T_p\iota(T_p\mathbb{S}^2) = \{v \in \mathbb{R}^3 : p \cdot v = 0\}.$$

□