Potential Theory

end of the 2nd part

Stellar orbits

1st part

Outlines

Ideal but useful models
- the infinite wire, the infinite slab
- infinite slab with oscillatory surface density, tightly wound spiral

Orbits
- some generalities

Lagrangian and Hamiltonian mechanics
- Euler-Lagrange equations
- Hamilton's equations

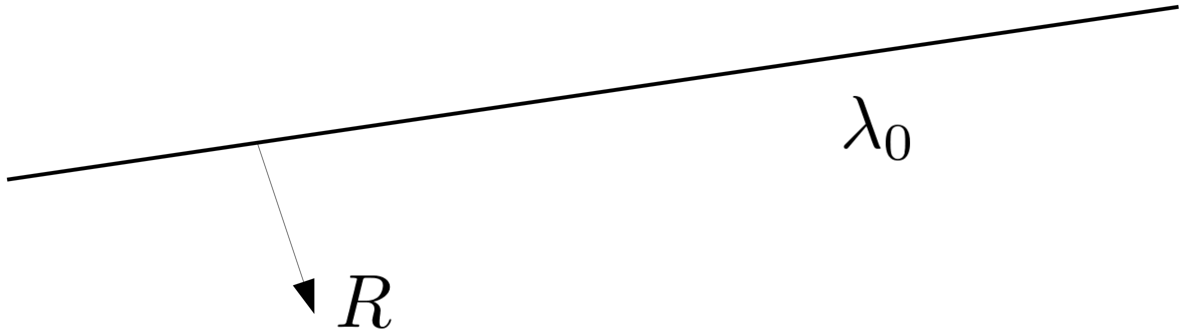
Orbits in spherical potentials
- angular momentum conservation
- equations of motion
- radial orbits
- non radial orbits

Examples
- Keplerian orbits
- Orbits in an homogeneous sphere
- Orbits in isochrone potentials

Potential Theory

Ideal but useful models

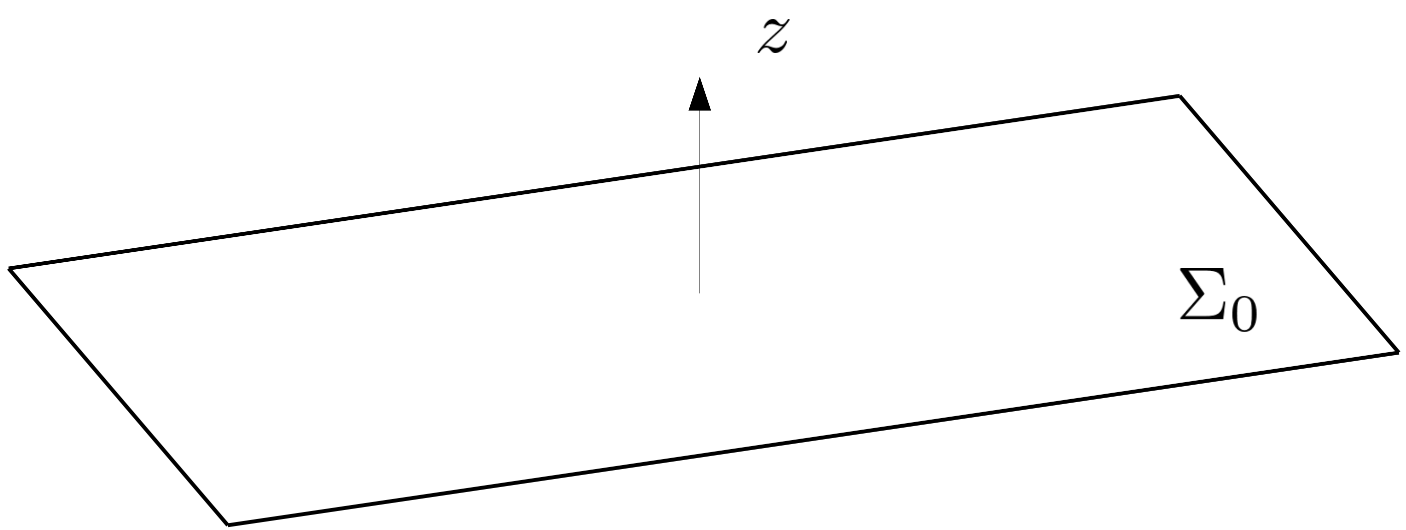
Potential of an infinite wire of constant linear density



$$\boxed{\Phi(R) = 2\,G\,\lambda_0\,\ln(R) + C}$$

EXERCICE

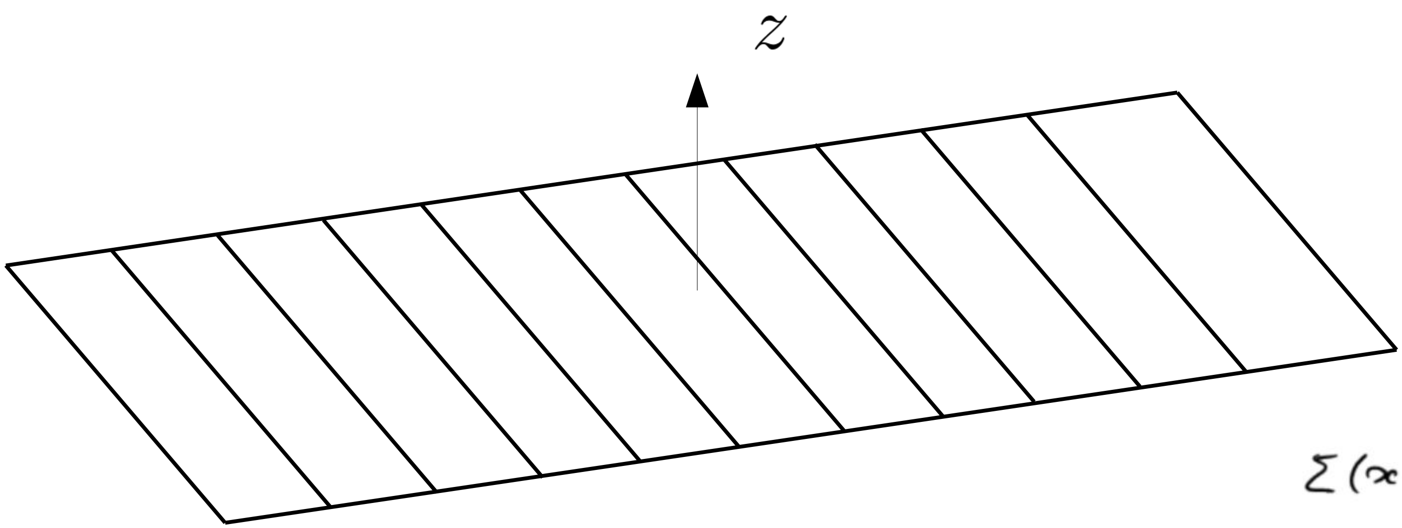
Potential of an infinite slab of constant surface density



$$\boxed{\Phi(z) = 2\pi G\,\Sigma_0\,|z| + C}$$

EXERCICE

Potential of an infinite slab with an oscillatory surface density



$$|\vec{k}| = \frac{2\pi}{\lambda}$$

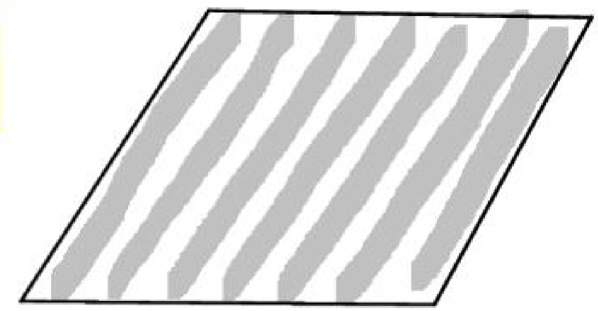
$$\Sigma(x, y) = \mathrm{Re}\left(\Sigma_1 \, e^{i\left(\vec{k}\cdot\vec{x}\right)}\right)$$

! will be negative !

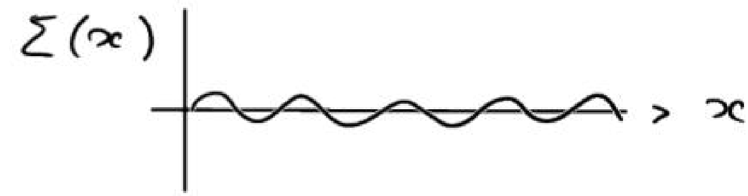
$$\boxed{\Phi(x, y, z) = -\frac{2\pi G\, \Sigma_1}{|\vec{k}|} \mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right) - |\vec{k}\, z|}\right)}$$

Potential of an infinite slab with an oscillatory surface density

$$\Sigma(x,y) = \mathrm{Re}\left(\Sigma_0\, e^{i(\vec{k}\vec{x})}\right)$$



Without loss of generality we can restrict to:



$$\Sigma(x) = \Sigma_0\, e^{ikx}$$

Poisson equation $\quad \vec{\nabla}^2\phi(x,z) = 4\pi G\, \Sigma(x)\, \delta(z)$

Assume a corresponding potential of the type

$$\phi(x,z) = \phi_0\, e^{ikx - |kz|}$$

Integrate Poisson over z (variation of Gauss th.)

$$\lim_{\xi \to 0} \int_{-\xi}^{\xi} \nabla^2 \phi \cdot dz = \lim_{\xi \to 0} \int_{-\xi}^{\xi} 4\pi G \rho \, dz = \lim_{\xi \to 0} \int_{-\xi}^{\xi} 4\pi G \Sigma_0 e^{ikx} \delta(z) \, dz$$

$$= 4\pi G \Sigma_0 e^{ikx}$$

Note $\frac{\partial^2 \phi}{\partial x^2}$ and $\frac{\partial^2 \phi}{\partial y^2}$ are continuous

$$\Rightarrow \lim_{\xi \to 0} \int_{-\xi}^{\xi} \frac{\partial^2 \phi}{\partial x^2} dz + \int_{-\xi}^{\xi} \frac{\partial^2 \phi}{\partial y^2} dz = 0$$

$$\lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \frac{\partial^2 \phi}{\partial z^2} = \lim_{\xi \to 0} \left. \frac{\partial \phi}{\partial z} \right|_{-\xi}^{\xi} = \lim_{\xi \to 0} \phi_0 |k| \operatorname{sgn}(z) \left. e^{ikx - |kz|} \right|_{-\xi}^{\xi}$$

$$= -2|k| \phi_0 e^{ikx}$$

$$\phi_0 = - \frac{2\pi G \Sigma_0}{|k|}$$

Thus for $\Sigma(x,y) = \Sigma_0 \, e^{i\vec{k}\vec{x}}$

$$\boxed{\phi(x,y,z) = -\frac{2\pi G \Sigma_0}{|\vec{k}|} \, e^{i\vec{k}\vec{x} - |\vec{k}z|}}$$

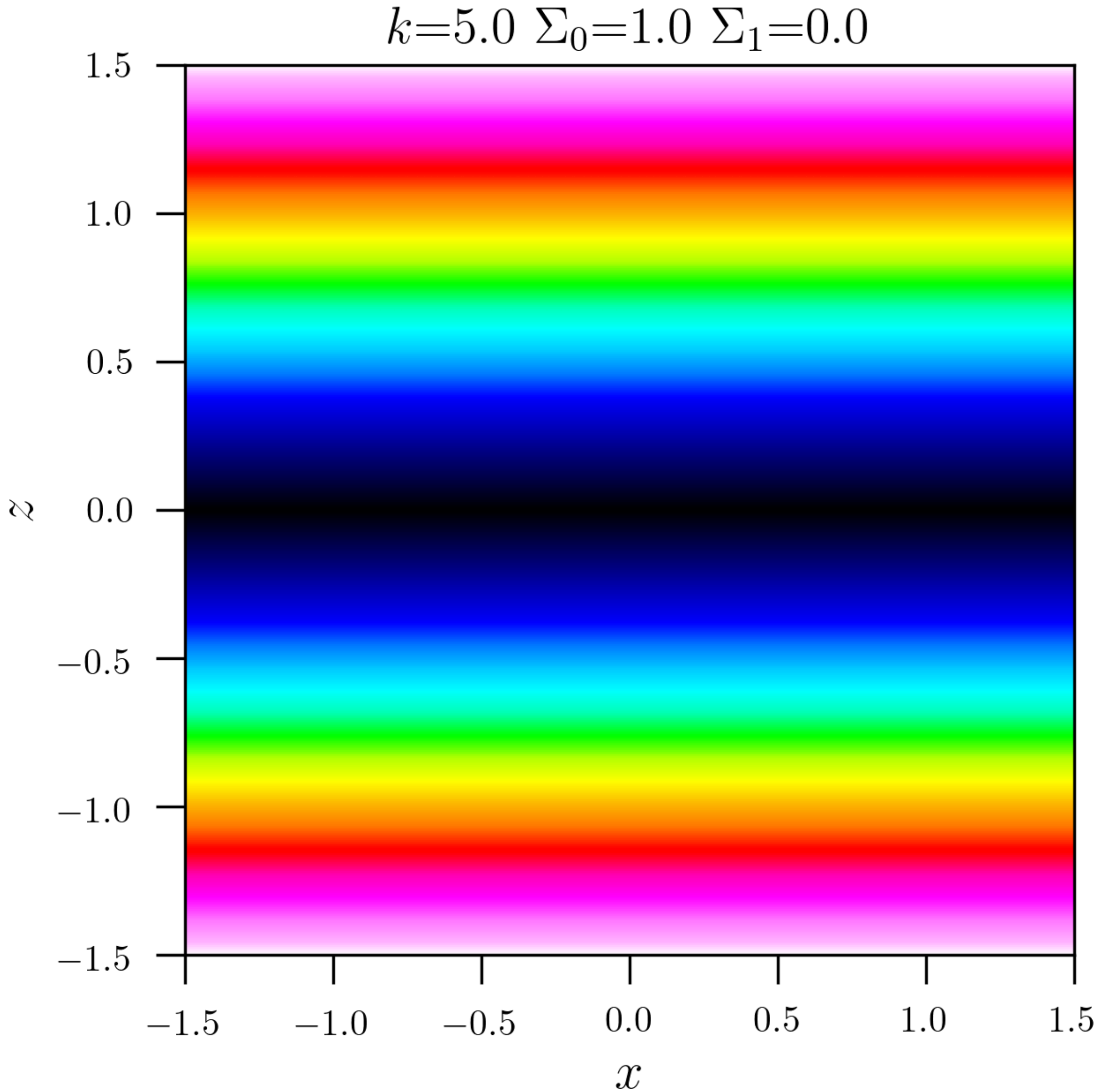
Note if the surface density evolves as a plane wave

$$\Sigma(x,y,t) = \Sigma_0 \, e^{i(\vec{k}\vec{x} - \omega t)}$$

$$\boxed{\phi(x,y,z,t) = -\frac{2\pi G \Sigma_0}{|\vec{k}|} \, e^{i(\vec{k}\vec{x} - \omega t) - |\vec{k}z|}}$$

Potential of an Infinite slab

$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$



Potential of an Infinite slab

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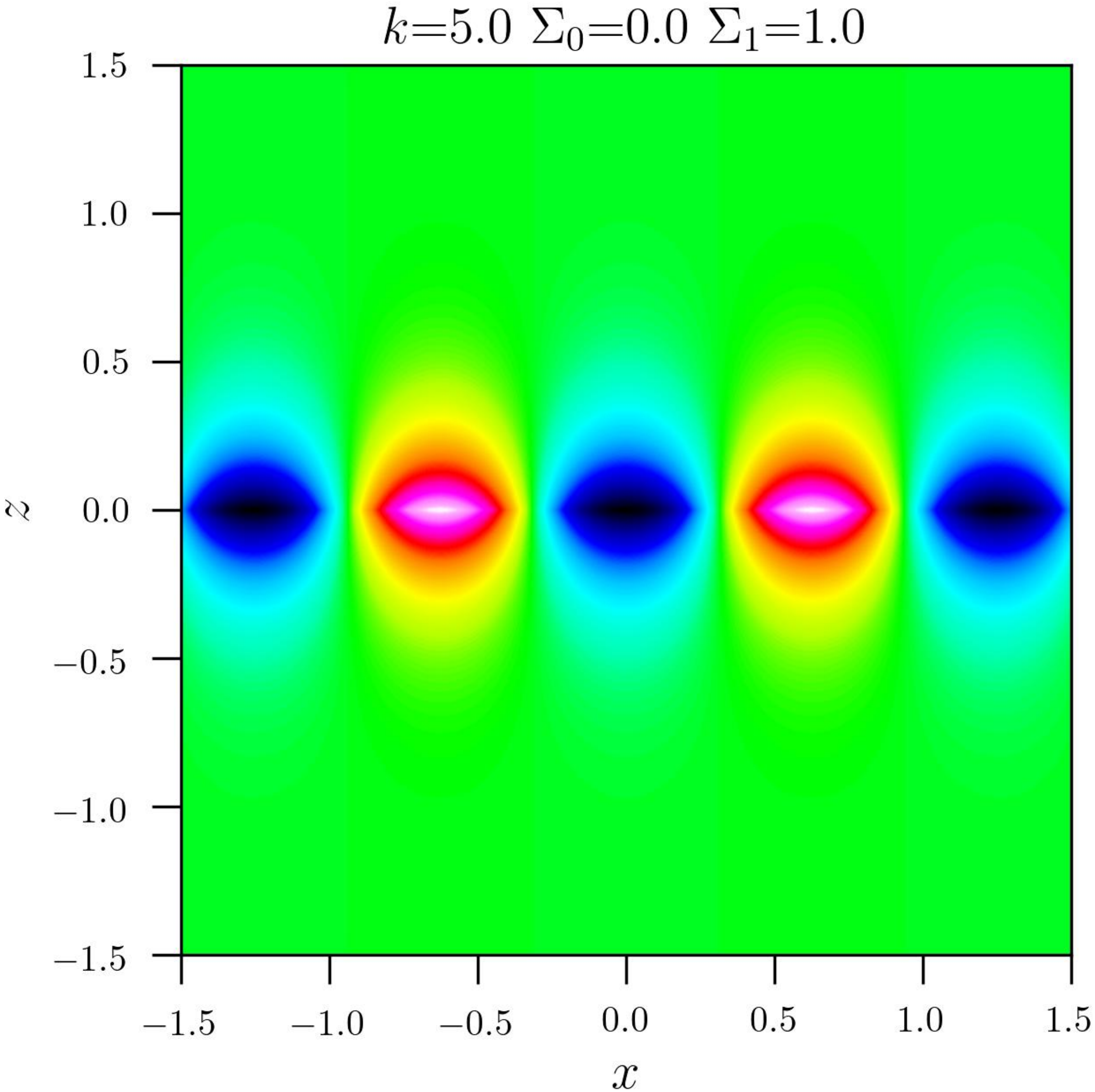

Potential of an Infinite slab

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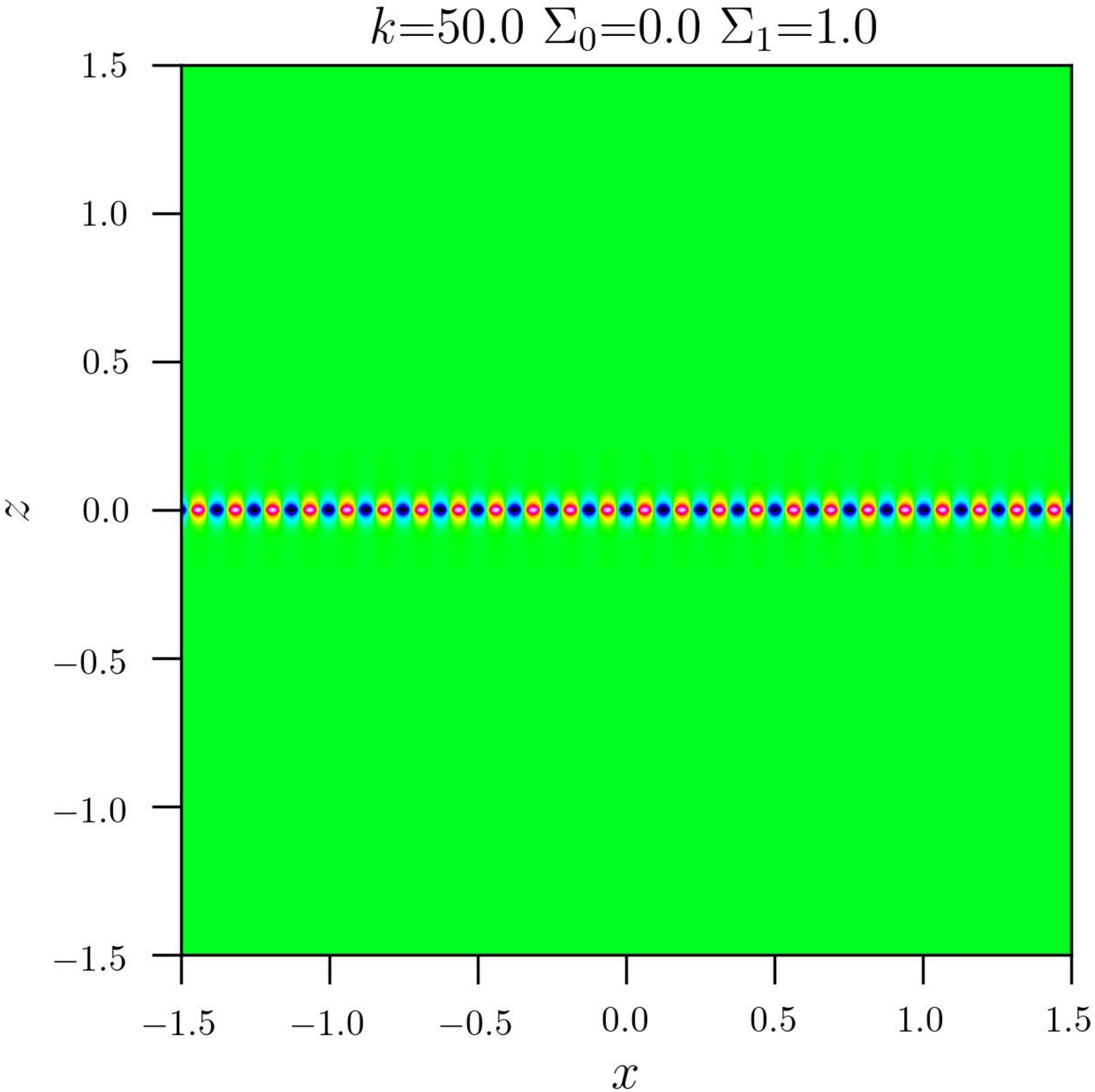

Potential of an Infinite slab

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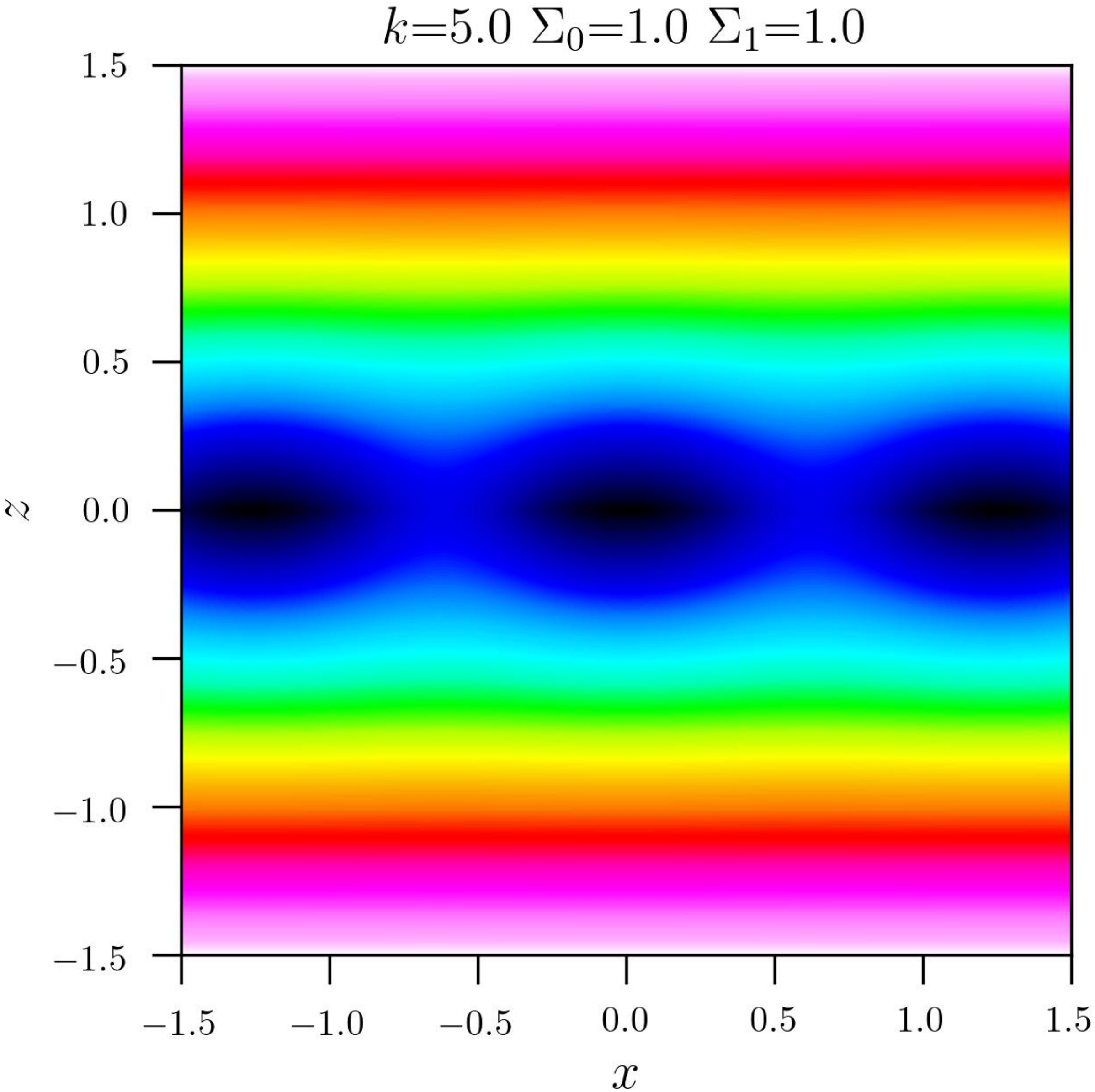
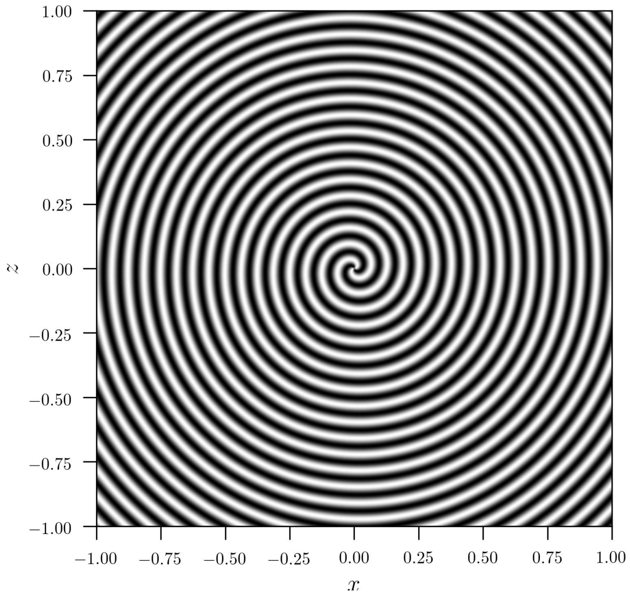


Potential of an infinite slab with a tightly wound spiral pattern

$$\Sigma(R,\phi) = \mathrm{Re}\left(H(R)\, e^{i(m\,\phi + f(R))}\right)$$

if $\left|\frac{\partial f}{\partial R}\cdot R\right| \ll 1$ WKB approximation

m=2

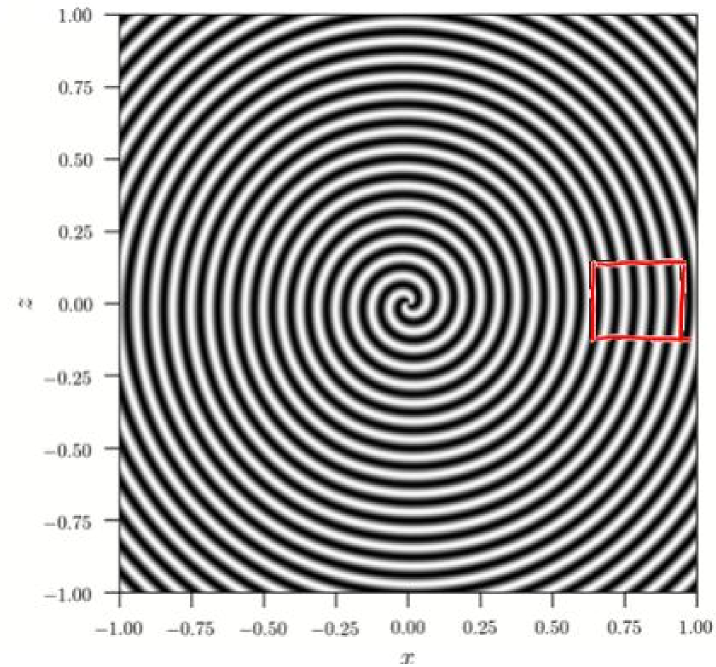


$$\boxed{\Phi(R,\phi) = -\frac{2\pi G\,\Sigma_0}{\left|\frac{\partial f}{\partial R}\right|}\,\mathrm{Re}\left(H(R)\, e^{i f(R) - \left|\frac{\partial f}{\partial R}\cdot z\right|}\right.}$$

Potential of an infinite slab with a tightly wound spiral pattern

m = 2

$$\boxed{\Sigma(R,\phi) = R_e\Big(\underbrace{H(R)}_{\text{slow variation}}\ \underbrace{e^{i(m\theta + g(R))}}_{\text{rapid variation}}\Big)}$$



Note $\boxed{m\theta + g(R) = \text{cte}}$

describe a spiral $g(R)$ = shape function

Idea : WKB approximation

far from the center, Σ is nearly $\sim e^{i(kx)}$

Indeed Developping $g(R)$ around R_0 gives

$$g(R) \cong g(R_0) + \left.\frac{\partial g}{\partial R}\right|_{R_0}(R-R_0)$$

For $\theta = 0$

$$\Sigma(R,\theta) = \underbrace{\mathcal{M}(R_0)\, e^{i f(R_0)}}_{\text{no radial dependency}} \; \underbrace{e^{i \frac{\partial f}{\partial R}\big|_{R_0} (R-R_0)}}_{e^{i k x}} \qquad \begin{cases} k = \frac{\partial f}{\partial R}\big|_{R_0} \\ x = R - R_0 \end{cases}$$

We directely have the solution from the infinite slab

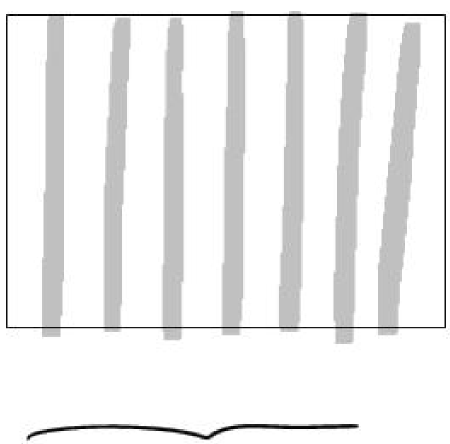
$$\phi(R,\theta) = -\frac{2\pi G}{\left|\frac{\partial f}{\partial R}\right|} \mathcal{M}(R_0)\, e^{i f(R_0)}\, e^{i \frac{\partial f}{\partial R}\big|_{R_0}(R-R_0)}\, e^{-\left|\frac{\partial f}{\partial R} z\right|}$$

Choosing $R_0 = R$

$$\boxed{\phi(R,\theta) = -\frac{2\pi G}{\left|\frac{\partial f}{\partial R}\right|} \mathcal{M}(R)\, e^{i f(R)}\, e^{-\left|\frac{\partial f}{\partial R} z\right|}}$$

Validity of the approximation

- we want a large number of "oscillations" over a small radius compared to R



$\sim R$

$$\boxed{\left|\frac{\partial S}{\partial R}\right| \cdot R \gg 1}$$

Stellar orbits

1st part

Generalities

Stellar orbits

Why studying stellar orbits ?

- understand the motion of stars in stellar systems and galaxies

→ understand the observed kinematics
→ constraints the mass model

We will assume :

- a smoothed gravitational field
- time independent potentials

Stellar orbits

Definitions

- trajectory — solution of the equation of motion

$$\ddot{\vec{x}} = -\vec{\nabla}\Phi(\vec{x})$$

defined on a finite interval:

$$\vec{x}(t), \vec{x}(t_0) = \vec{x_0}, t \in [t_0, t_1]$$

- orbit — a trajectory defined on an infinite time interval

$$\vec{x}(t), \vec{x}(t_0) = \vec{x_0}, t \in [-\infty, \infty[$$

- periodic orbit — a closed orbit

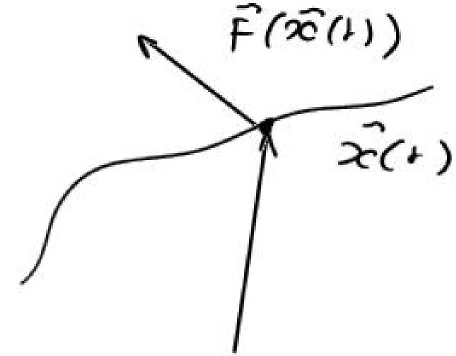
$$\forall\, t, \exists\, T, \vec{x}(t+T) = \vec{x}(t) \;\; \vec{v}(t+T) = \vec{v}(t)$$

- stationary point — a point such that:

$$\ddot{\vec{x}} = \dot{\vec{x}} = 0$$

Lagrangian and Hamiltonian mechanics

Lagrangian Mechanics



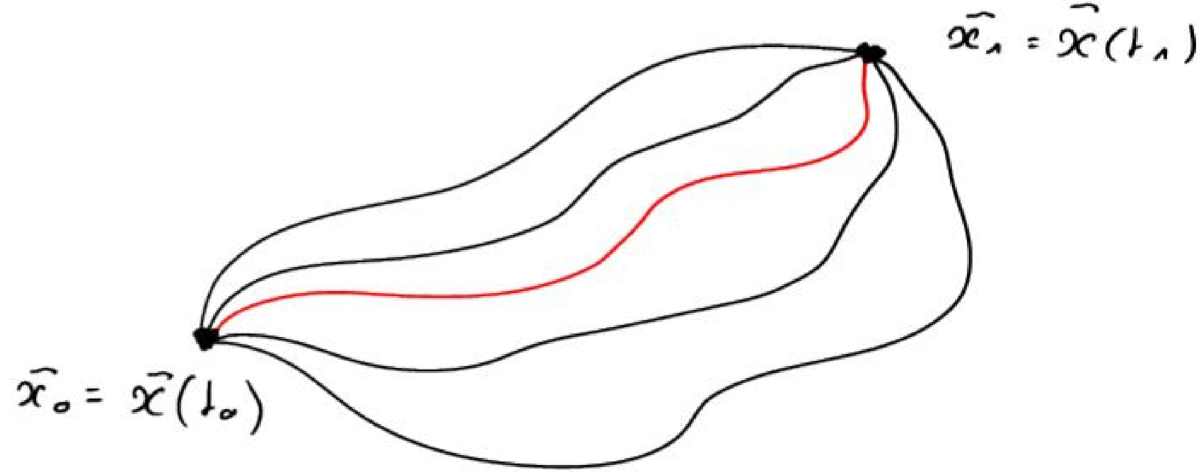
Assume a mass point moving in a force field $\vec{F}(\vec{x})$

Definition Lagrangian, a scalar function of $\vec{x}$, $\dot{\vec{x}}$, t

$$\boxed{\mathcal{L}(\vec{x}, \dot{\vec{x}}, t) = K - V = \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}, t)}$$

Principle of least action or Hamiltonian principle

The motion of the particle from $\vec{x}_0$ to $\vec{x}_1$ is along a curve $\vec{x}(t)$ such that $\vec{x}(t_0) = \vec{x}_0$, $\vec{x}(t_1) = \vec{x}_1$ that is an extremal of the action I.



$$\boxed{I = \int_{t_0}^{t_1} \mathcal{L}(\vec{x}, \dot{\vec{x}}, t)\, dt = \int_{t_0}^{t_1} K(t) - V(t)\, dt}$$

Euler - Lagrange equation

The trajectory is an extremal of I if and only if

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{x}} = 0}$$

With cathesian coordinates, we get:

$$\boxed{m\ddot{\vec{x}} = -\vec{\nabla} V(\vec{x})}$$

which is nothing else than the second Newton law.

However: $\mathcal{L}$ can be a fuction of arbitrary coordinates $(\vec{q}, \dot{\vec{q}})$ "generalized" coordinates $\mathcal{L}(\vec{q}, \dot{\vec{q}})$.

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = 0}$$

Lagrange's equations

We can easily write equations of motions in any coord. system.

Hamiltonian mechanics

Note : Lagrangian mechanics generate 2^{nd} order differential equations

$$m\ddot{\vec{x}} = -\vec{\nabla}V(\vec{x})$$

It is always possible to split a 2^{nd} order differential equation into two first order differential equations.

This is what is done in Hamiltonian mechanics

Definition

1. For $\vec{q}, \dot{\vec{q}}$, a set of generalized coordinates, the generalized momentum are :

$$\boxed{\vec{p} := \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}}$$

Note : inverting $\vec{p} = \vec{p}(\vec{q}, \dot{\vec{q}})$, it is possible to write $\dot{\vec{q}} = \dot{\vec{q}}(\vec{p}, \vec{q})$

2. Hamiltonian The scalar function

$$\boxed{H(\vec{q}, \vec{p}, t) := \vec{p} \cdot \dot{\vec{q}} - \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)}$$

Note : $\dot{\vec{q}}$ is replaced by $\vec{q}, \vec{p}$ through the definition of $\vec{p}$

Hamilton equations

Compute the total derivative of $H(\vec{q},\vec{p},t) = \vec{p}\cdot\dot{\vec{q}} - \mathcal{L}(\vec{q},\dot{\vec{q}},t)$

① right hand side (diff. with respect of $\vec{q},\vec{p}$)

$$\frac{\partial H}{\partial \vec{q}}\,d\vec{q} + \frac{\partial H}{\partial \vec{p}}\,d\vec{p} + \frac{\partial H}{\partial t}\,dt$$

② left hand side (diff with respect of $\vec{p}\,\vec{q}\,\dot{\vec{q}}$)

$$\vec{p}\cdot d\dot{\vec{q}} + \dot{\vec{q}}\,d\vec{p} - \frac{\partial \mathcal{L}}{\partial \vec{q}}\,d\vec{q} - \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\,d\dot{\vec{q}} - \frac{\partial \mathcal{L}}{\partial t}\,dt$$

def. of $\vec{p}$

$$= \cancel{\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\,d\dot{\vec{q}}} + \dot{\vec{q}}\,d\vec{p} - \frac{\partial \mathcal{L}}{\partial \vec{q}}\,d\vec{q} - \cancel{\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\,d\dot{\vec{q}}} - \frac{\partial \mathcal{L}}{\partial t}\,dt$$

$$= \dot{\vec{q}}\,d\vec{p} - \frac{\partial \mathcal{L}}{\partial \vec{q}}\,d\vec{q} - \frac{\partial \mathcal{L}}{\partial t}\,dt$$

Equating ① and ②

$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}} \qquad -\frac{\partial \mathcal{L}}{\partial \vec{q}} = \frac{\partial H}{\partial \vec{p}} \qquad \frac{\partial \mathcal{L}}{\partial t} = \frac{\partial H}{\partial t}$$

$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}} \qquad -\frac{\partial \mathcal{L}}{\partial \vec{q}} = \frac{\partial H}{\partial \vec{q}} \;(*) \qquad \frac{\partial \mathcal{L}}{\partial t} = \frac{\partial H}{\partial t}$$

Using Euler-Lagrange

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}}_{\vec{p}\ \text{def}\ \vec{p}}\Big) - \underbrace{\frac{\partial \mathcal{L}}{\partial \vec{q}}}_{\frac{\partial H}{\partial \vec{q}}\ (*)} = 0 \implies \frac{d}{dt}\vec{p} = -\frac{\partial H}{\partial \vec{q}}$$

In conclusion, we have transformed a set of 2^{nd} order differential equations into $2\times$ more 1^{st} order differential equations:

$$\boxed{\begin{aligned} \dot{\vec{q}} &= \frac{\partial H}{\partial \vec{p}} \\ \dot{\vec{p}} &= -\frac{\partial H}{\partial \vec{q}} \end{aligned}} \qquad \frac{\partial \mathcal{L}}{\partial t} = \frac{\partial H}{\partial t}$$

Hamilton's equations

Hamiltonian conservation

Lets compute the time derivative of $H(\vec{q},\vec{p},t)$

$$\frac{d}{dt} H(\vec{q},\vec{p},t) = \frac{\partial H}{\partial \vec{q}} \frac{d\vec{q}}{dt} + \frac{\partial H}{\partial \vec{p}} \frac{d\vec{p}}{dt} + \frac{\partial H}{\partial t}$$

$$- \dot{\vec{p}} \cdot \dot{\vec{q}} + \dot{\vec{q}}\, \dot{\vec{p}} = 0$$

If $\mathcal{L}$ is time independant, i.e. $\mathcal{L} = \mathcal{L}(\vec{q}, \dot{\vec{q}})$

($\equiv V(\vec{q})$ is time independant)

$\Longrightarrow$

$$\boxed{\text{By construction, } H(\vec{q},\vec{p}) \text{ is conserved along a trajectory}}$$

Definitions for an n-dimentional system

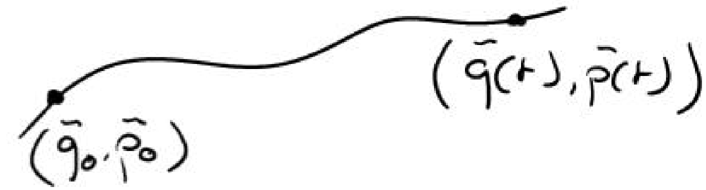
Configuration space $(q_1 \dots q_n)$ n-dimensions

Momentum space $(p_1 \dots p_n)$ n-dimensions

Phase space $(q_1 \dots q_n, p_1 \dots p_n)$ 2n-dimensions

$\equiv (w_1 \dots w_{2n})$

Note	As Hamilton's equations are 1^{st} order differential equations, a trajectory is uniquely defined by a point in the phase space



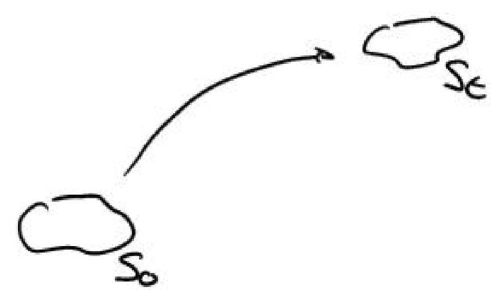
Time evolution operator

It is the possible to define a time evolution operator H_t that will bring $(\vec{q}_0, \vec{p}_0)$ to $(\vec{q}(t), \vec{p}(t))$

$$(\vec{q}(t), \vec{p}(t)) = H_t(\vec{q}_0, \vec{p}_0)$$

H_t will map any 2D surface S_0 in the phase space to an other 2D surface S_t in the phase space.

Poincaré invariant theorem



$$\boxed{\iint_{S_0} d\hat{q} \cdot d\hat{p} = \iint_{S_t} d\hat{q}\, d\hat{p}}$$

Poisson brackets

two operators A, B

$$[A, B] := \frac{\partial A}{\partial \vec{q}} \frac{\partial B}{\partial \vec{p}} - \frac{\partial A}{\partial \vec{p}} \frac{\partial B}{\partial \vec{q}} = \sum_i^n \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i}$$

Hamilton's equations

$$\boxed{\dot{w}_\alpha = [w_\alpha, H]} = \frac{\partial w_\alpha}{\partial \vec{q}} \frac{\partial H}{\partial \vec{p}} - \frac{\partial w_\alpha}{\partial \vec{p}} \frac{\partial H}{\partial \vec{q}}$$

$$\equiv \begin{cases} \dot{q}_\alpha = \dfrac{\partial H}{\partial p_\alpha} \\[2em] \dot{p}_\alpha = - \dfrac{\partial H}{\partial q_\alpha} \end{cases}$$

Orbits in Spherical Systems

Orbits in spherical potentials

Spherical coordinates

$$\vec{x} = r\,\hat{e}_r = \vec{r}$$

$$\phi(\vec{x}) = \phi(r) \qquad r = \sqrt{x^2 + y^2 + z^2}$$

Equation of motion

(Newton law)

$$\boxed{\frac{d^2}{dt^2}(\vec{x}) = \vec{g}(\vec{x})}$$

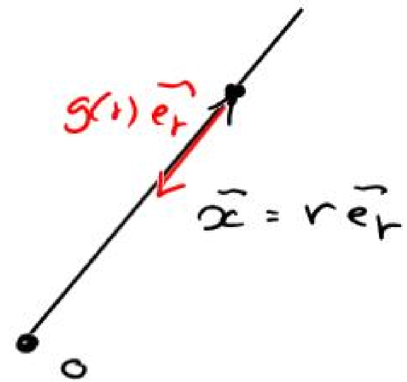
$$g(\vec{x}) = -\vec{\nabla}\phi(\vec{x}) = -\frac{\partial}{\partial r}\phi(r)\,\hat{e}_r - \frac{1}{r}\cancel{\frac{\partial}{\partial\theta}\phi(r)}\,\hat{e}_\theta - \frac{1}{r\sin\theta}\cancel{\frac{\partial}{\partial\varphi}\phi(r)}\,\hat{e}_\varphi$$

$$= g(r)\,\hat{e}_r$$

with $g(r) = -\frac{\partial}{\partial r}\phi(r)$

Angular momentum conservation

$$\vec{L} = \vec{x} \times \frac{d\vec{x}}{dt} \qquad \text{(specific angular momentum)}$$



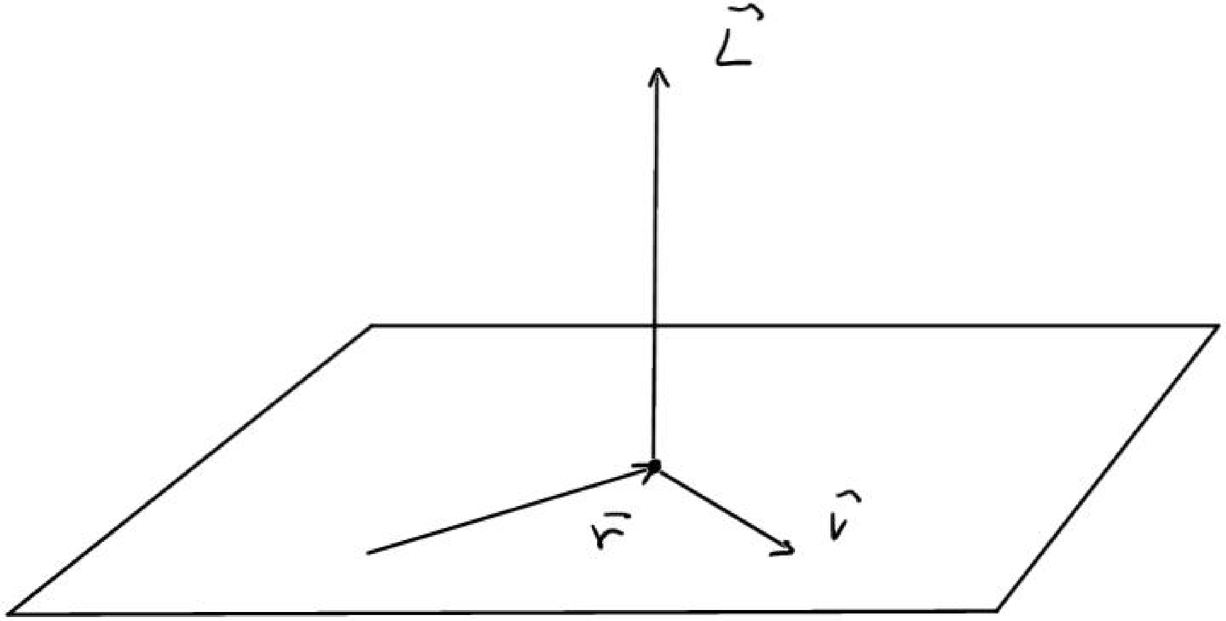
$$\frac{d}{dt}\left(\vec{L}\right) = \frac{d\vec{x}}{dt} \times \frac{d\vec{x}}{dt} + \vec{x} \times \frac{d^2\vec{x}}{dt^2}$$

$$= 0 + \underbrace{r\vec{e_r} \times g(r)\,\vec{e_r}}_{=\,0} = 0 \qquad (= \vec{N}, \text{ the tork})$$

> In a spherical system, the angular momentum of a particle is conserved ! $\vec{L} = \text{cte}$

(A spherical potential induces no tork $\vec{N} = \vec{x} \times \vec{F} = 0$)

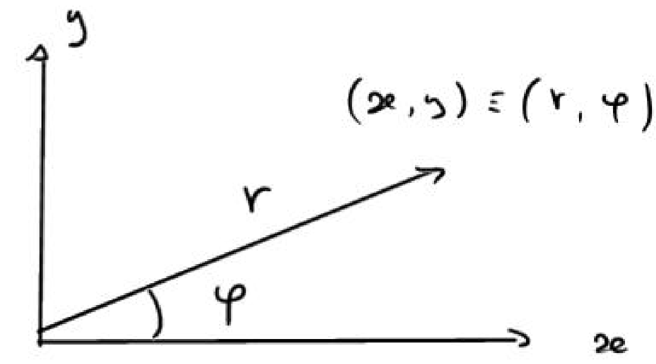
Corollary As $\vec{L}$ is conserved the orbit of a particle is limited to a plane (the orbital plane)



2D problem

Equations of motion in the orbital plane

Polar coordinates (in the orbital plane)



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \begin{cases} \dot{x} = \dot{r}\cos\varphi - r\sin\varphi\,\dot{\varphi} \\ \dot{y} = \dot{r}\sin\varphi + r\cos\varphi\,\dot{\varphi} \end{cases}$$

Lagrangian (specific) in polar coordinates

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \phi(\sqrt{x^2+y^2}) = \frac{1}{2}(\dot{r}^2 + (r\dot{\varphi})^2) - \phi(r)$$

Lagrange equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = 0}$$

$$\begin{cases} \ddot{r} - r\dot{\varphi}^2 + \dfrac{\partial \phi}{\partial r} = 0 \\ \dfrac{d}{dt}(r^2\dot{\varphi}) = 0 \end{cases} \Rightarrow \quad r^2\dot{\varphi} = |\vec{L}| = L$$

as $\vec{L} = \vec{x} \times \vec{v} = (\underbrace{r\vec{e}_r}_{\vec{x}} \times (\underbrace{\dot{r}\vec{e}_r + r\dot{\varphi}\vec{e}_\varphi}_{\vec{v}})) = r^2\dot{\varphi}\,\vec{e}_z$

Hamiltonian / Energy

$$\boxed{H(\vec{q},\vec{p},t) := \vec{p}\cdot\dot{\vec{q}} - \mathcal{L}(\vec{q},\dot{\vec{q}},t)}$$

$$\vec{q} = \begin{cases} r \\ \varphi \end{cases} \qquad \vec{p} = \begin{cases} \frac{\partial \mathcal{L}}{\partial \dot{r}} = \dot{r} \\ \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = r^2\dot{\varphi} \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{r} \\ \dot{\varphi} \end{cases}$$

$$H(r,\varphi,\dot{r},r^2\dot{\varphi}) = \dot{r}^2 + r^2\dot{\varphi}^2 - \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) + \phi(r)$$

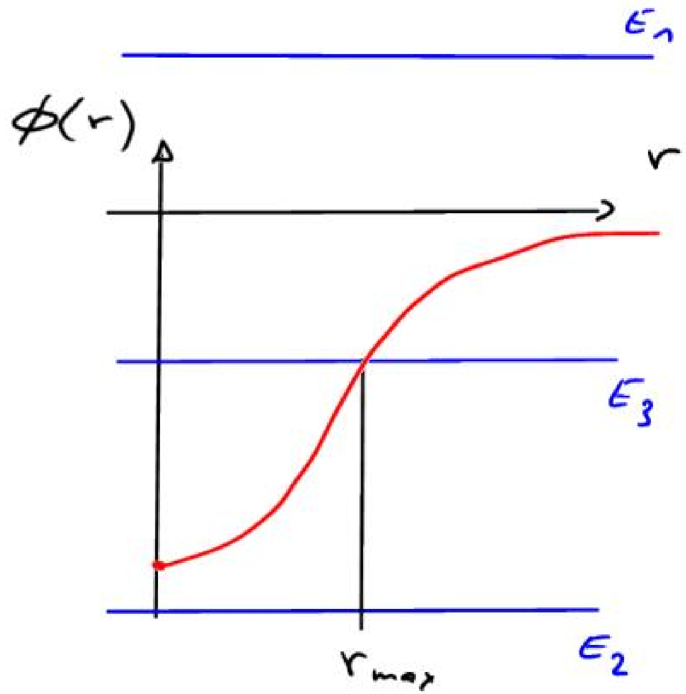
$$= \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) + \phi(r) = E$$

E (Energy) is conserved
as $\mathcal{L}$ is time independant

Radial orbits $\quad \dot{\varphi} = 0 \quad \Rightarrow \quad L = 0$

$$\begin{cases} \text{Equation of motion} : & \ddot{r} = -\dfrac{\partial \phi}{\partial r} \\ \text{Energy} : & E = \frac{1}{2}\dot{r}^2 + \phi(r) \end{cases}$$

3 cases



① $E > \phi(\infty) \quad \Rightarrow \quad \forall t,\ \dot{r}^2 > 0$

orbit not bounded

② $E < \phi(0) \quad \Rightarrow \quad$ impossible

③ $\phi(0) < E < \phi(\infty)$

$\exists\, r$ tq $\dot{r} = 0$ i.e $E = \phi(r)$

$r = r_{max}$

Non radial orbits

$r \neq 0 \quad \dot{\varphi} \neq 0 \quad L \neq 0$

EOM
$$\begin{cases} \ddot{r} - r\dot{\varphi}^2 + \dfrac{\partial \phi}{\partial r} = 0 & (1) \\ \dfrac{d}{dt}\left(r^2 \dot{\varphi}\right) = 0 \end{cases}$$

replace t by φ

$$\frac{d}{dt} = \frac{d}{d\varphi}\dot{\varphi} = \frac{L}{r^2}\frac{d}{d\varphi}$$

(1) becomes

$$\frac{L^2}{r^2}\frac{d}{d\varphi}\left(\frac{1}{r^2}\frac{dr}{d\varphi}\right) - \frac{L^2}{r^3} = -\frac{\partial \phi}{\partial r}$$

use $u = \dfrac{1}{r}$

$$\boxed{\frac{d^2 u}{d\varphi^2} + u = \frac{1}{L^2 u^2}\frac{\partial \phi}{\partial r}\left(\frac{1}{u}\right)}$$

No analytical general solution

Radial energy equation

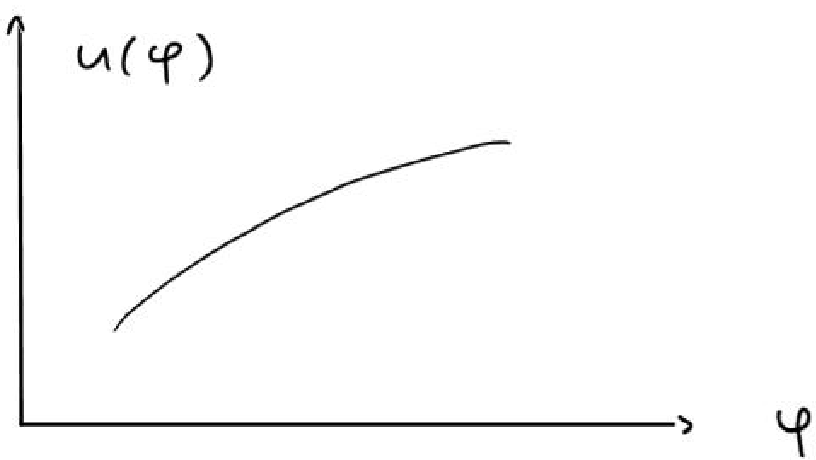
From the energy $\boxed{E = \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) + \phi(r)}$

1) multiply by $\frac{2}{L^2}$

2) use $u = \frac{1}{r}$ and $\frac{d}{dt} = \frac{L}{r^2}\frac{d}{d\varphi}$

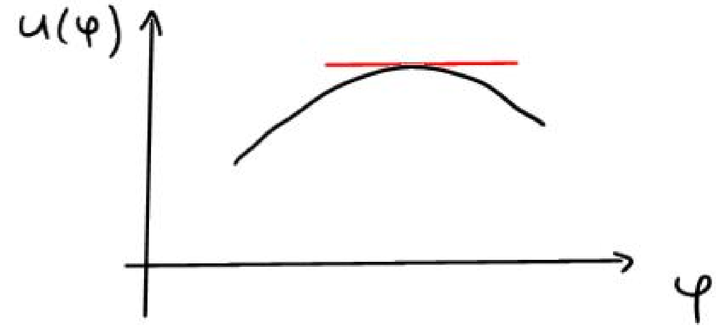
we get

$$\boxed{\left(\frac{du}{d\varphi}\right)^2 + u^2 + \frac{2\phi(\frac{1}{u})}{L^2} = \frac{2E}{L^2}}$$



Orbit properties

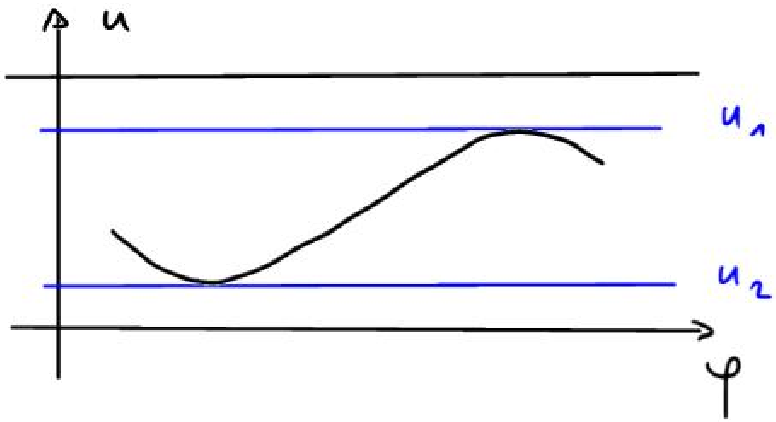
① bound orbits $\quad \exists \varphi$ tq $\dfrac{du}{d\varphi} = 0$

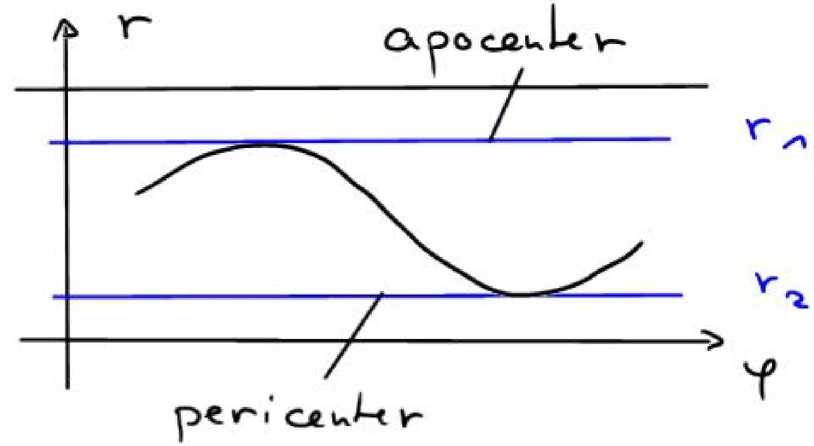


$$\boxed{u^2 = \frac{2\left[E - \phi(1/u)\right]}{L^2}}$$

we have one or two solutions for u

- $u_1 > u_2$
- $u_1 = u_2$

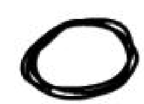




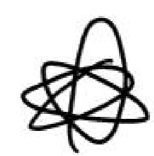
Notes

- if $u_1 = u_2$: periodic orbit

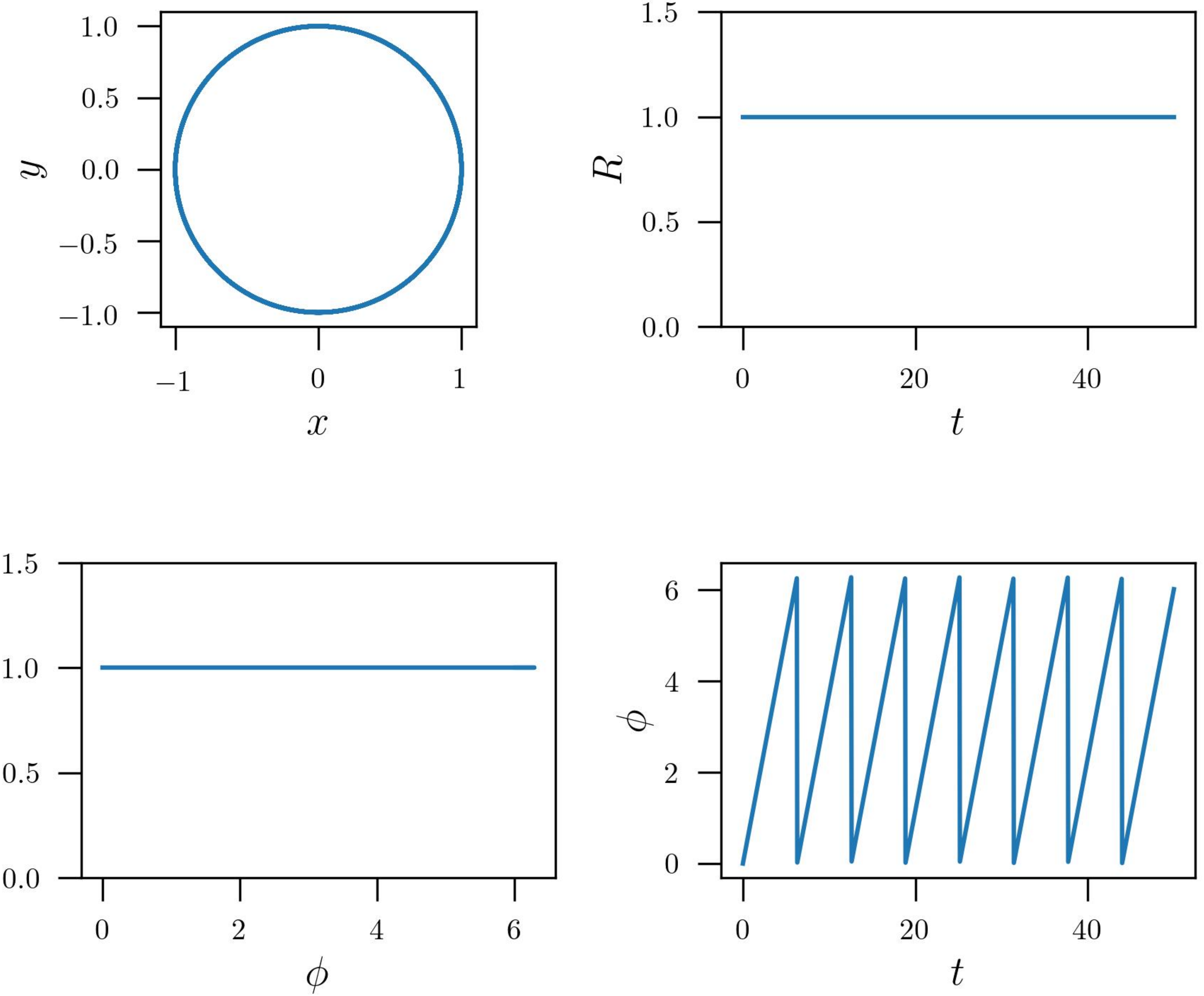
- if $u_1 \cong u_2$: orbit with a small eccentricity



- if $u_1 \gg u_2$: orbit eccentricity is nearly 1



Plummer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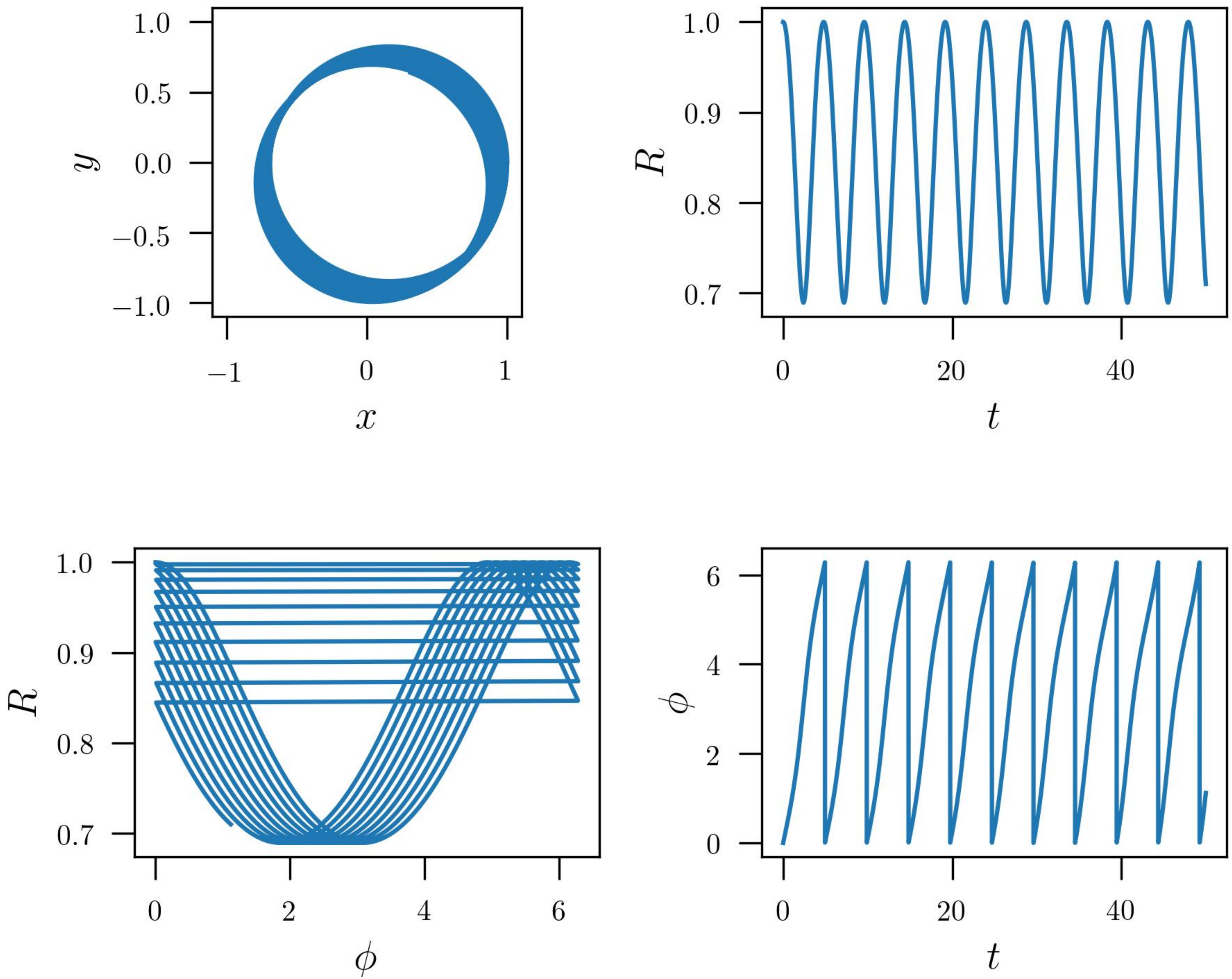
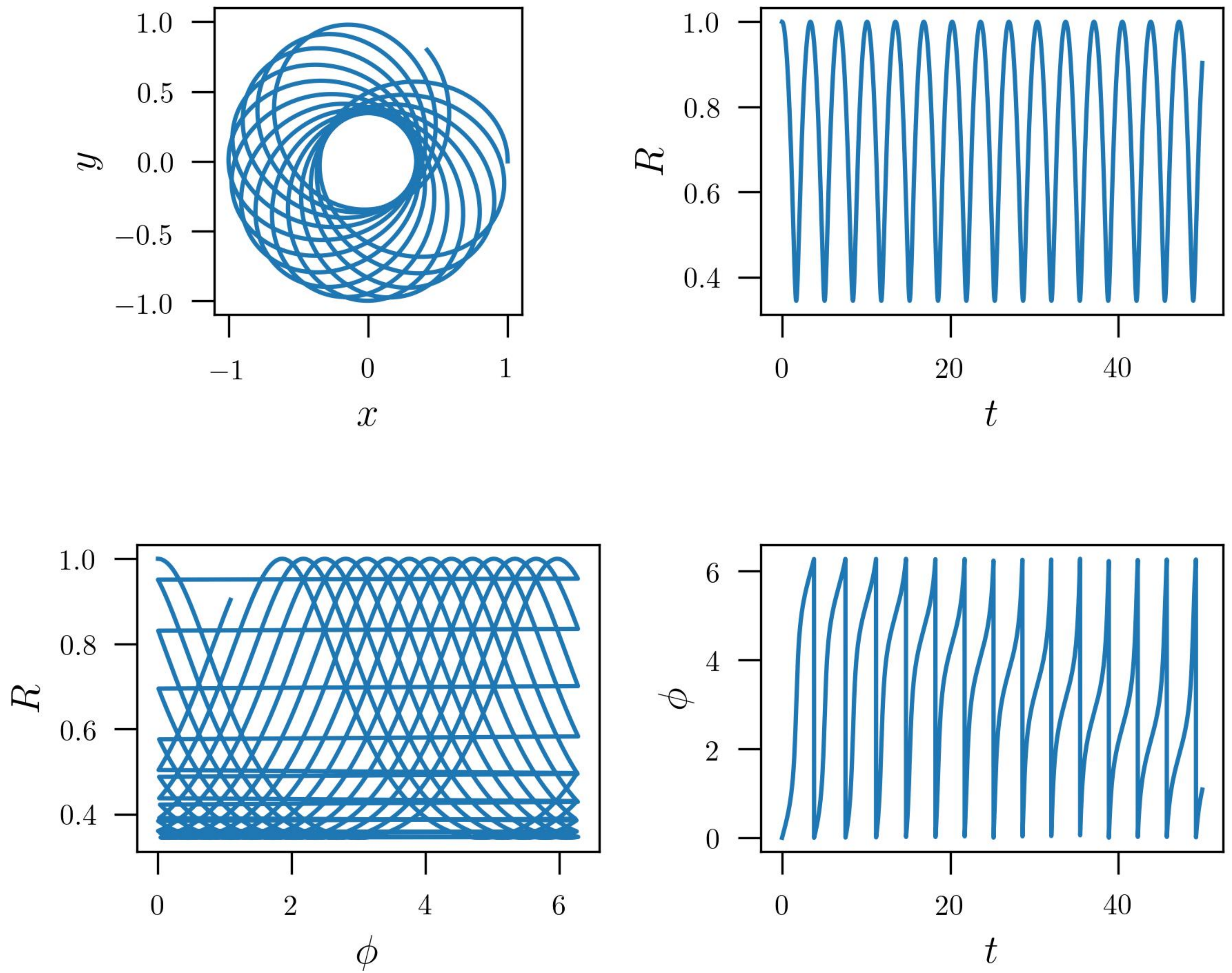
Plummer



Plummer



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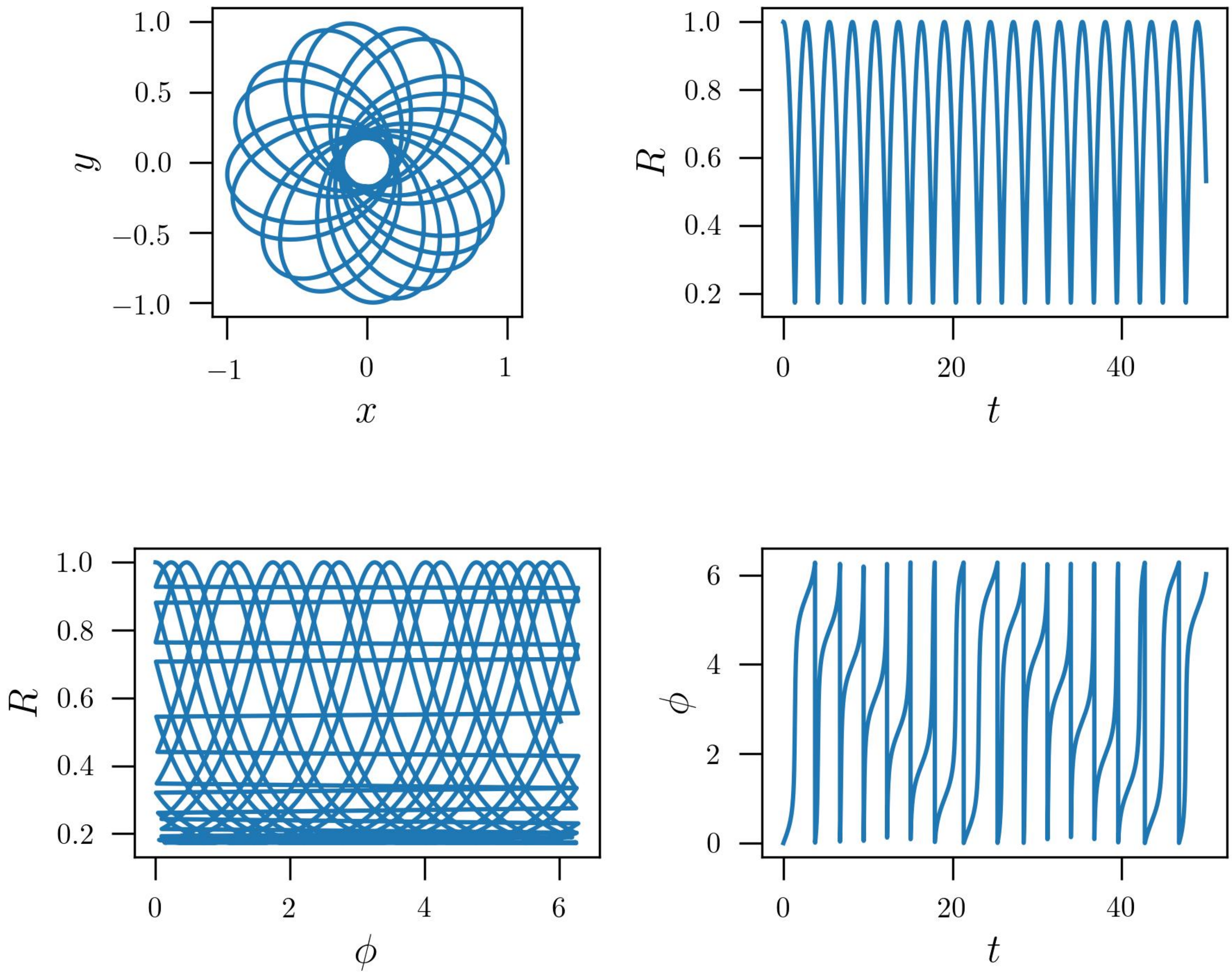
Plummer



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ϕ
6
4
2
0
ϕ
0
20
40
t

Radial period

Time to travel from the apocenter to the pericenter

$$T_r = 2\int_{t_1}^{t_2} dt = 2\int_{r_1}^{r_2} \frac{dt}{dr}\, dr \qquad \begin{cases} r(t_1) = r_1 \\ r(t_2) = r_2 \end{cases}$$

From

$$E = \frac{1}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) + \phi(r) = \frac{1}{2}\dot{r}^2 + \frac{L}{2r^2} + \phi(r)$$

$$\dot{r}^2 = 2(E - \phi(r)) - \frac{L^2}{r^2}$$

$$\frac{dr}{dt} = \sqrt{2(E-\phi(r)) - \frac{L^2}{r^2}}$$

$$\frac{dt}{dr} = \frac{1}{\sqrt{2(E-\phi(r)) - \frac{L^2}{r^2}}}$$

$$\boxed{T_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E-\phi(r)) - \frac{L^2}{r^2}}}}$$

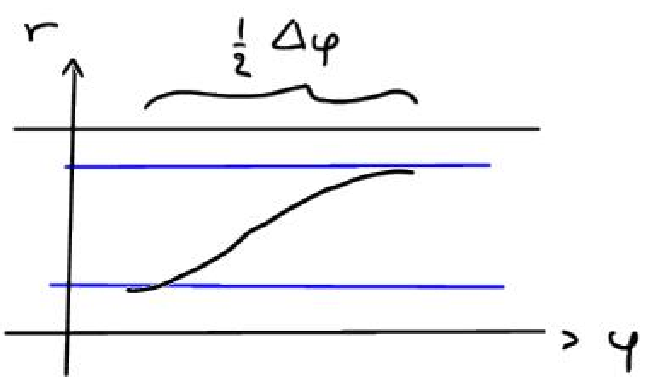
Azimuthal period

$$T_\varphi = \int_{t'_1}^{t'_2} dt \qquad \begin{cases} \varphi(t'_1) = 0 \\ \varphi(t'_2) = 2\pi \end{cases}$$

$\Delta\varphi$: increase of the azimuthal angle during T_r

$$\Delta\varphi = 2\int_{\varphi_1}^{\varphi_2} d\varphi = 2\int_{t_1}^{t_2} \frac{d\varphi}{dt}\, dt = 2\int_{r_1}^{r_2} \underbrace{\frac{d\varphi}{dt}}\, \frac{dt}{dr}\, dr$$

$$= 2\int_{r_1}^{r_2} \frac{L}{r^2}\, \frac{1}{\sqrt{2(E-\Phi(r)) - \frac{L^2}{r^2}}}\, dr$$



Azimuthal period: time to increase φ by 2π

$$T_\varphi = \frac{2\pi}{\langle \dot{\varphi} \rangle} \qquad \langle \dot{\varphi} \rangle = \frac{|\Delta\varphi|}{T_r}$$

mean azimuthal "velocity"

$$\boxed{T_\varphi = \frac{2\pi}{|\Delta\varphi|} T_r}$$

As in general $\frac{2\pi}{|\Delta\varphi|}$ is not a rational number

the orbit is not guarantee to be closed

Spherical Systems

Examples

Examples

① Kepler potential (potential of a mass point)

$$
\begin{cases}
\phi(r) = -\dfrac{GM}{r} \\[2mm]
\dfrac{\partial \phi}{\partial r}(r) = \dfrac{GM}{r^2} = GMu^2
\end{cases}
$$

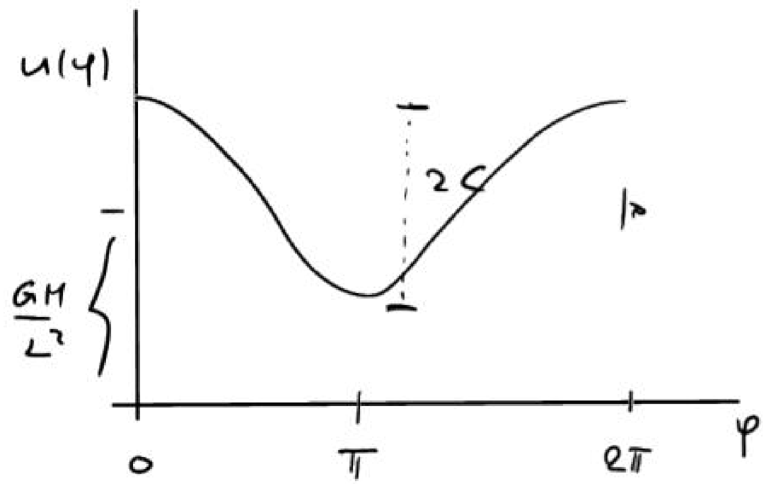
$$
\boxed{\frac{d^2u}{d\varphi^2} + u = \frac{1}{L^2u^2}\frac{\partial \phi}{\partial r}\left(\frac{1}{u}\right)} \Rightarrow \boxed{\frac{d^2u}{d\varphi^2} + u = \frac{GM}{L^2}}
$$

Harmonic equation,
with frequency 1

General solution

$$\boxed{\frac{d^2u}{d\varphi^2} + u = \frac{GM}{L^2}}$$

$$u(\varphi) = \underset{\text{free parameter}}{C} \cos(\varphi - \underset{\text{free parameter}}{\varphi_0}) + \frac{GM}{L^2}$$



period = 2π

In term of r

$$\boxed{r(\varphi) = \frac{1}{C \cos(\varphi - \varphi_0) + \frac{GM}{L^2}}}$$

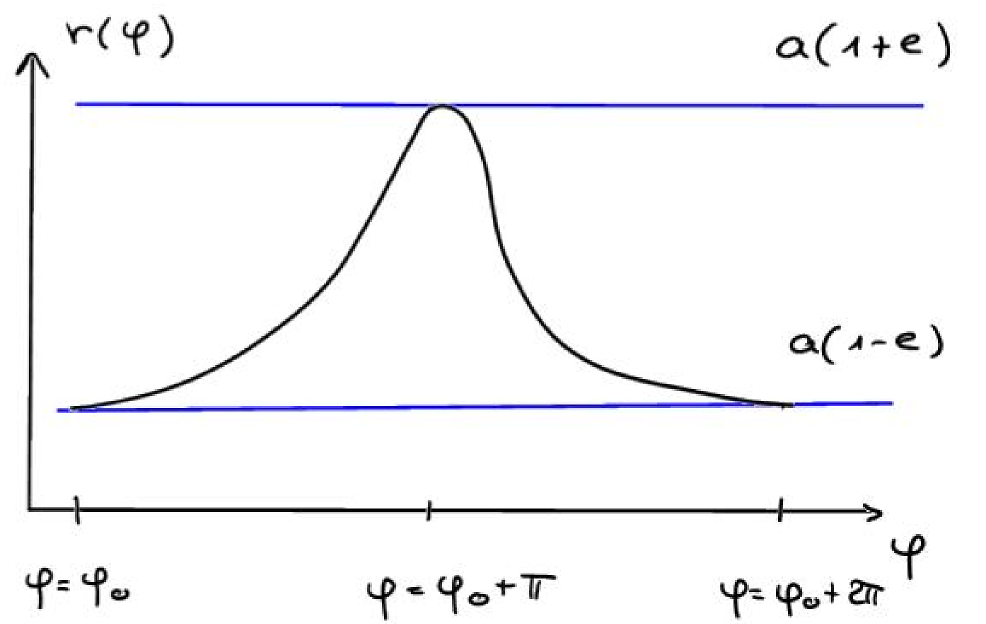
Introducing

$$\begin{cases} e = \dfrac{CL^2}{GM} & \text{eccentricity} \\[2ex] a = \dfrac{L^2}{GM(1-e^2)} & \text{semi-major axis} \end{cases}$$

evaluate u and $\dfrac{du}{dt}$ for $\varphi = \varphi_0$ $\left(u(\varphi) \overset{\varphi=\varphi_0}{=} C + \dfrac{GM}{L^2} \quad \dfrac{du}{dt}(\varphi) \overset{\varphi=\varphi_0}{=} 0 \right)$

+ using $\dfrac{d^2u}{d\varphi^2} + u = \dfrac{1}{L^2u^2} \dfrac{\partial \phi}{\partial r}\left(\dfrac{1}{u}\right)$

$$\boxed{\begin{cases} r(\varphi) = \dfrac{a(1-e^2)}{1 + e\cos(\varphi - \varphi_0)} \\[2ex] E = -\dfrac{GM}{a} \end{cases}}$$



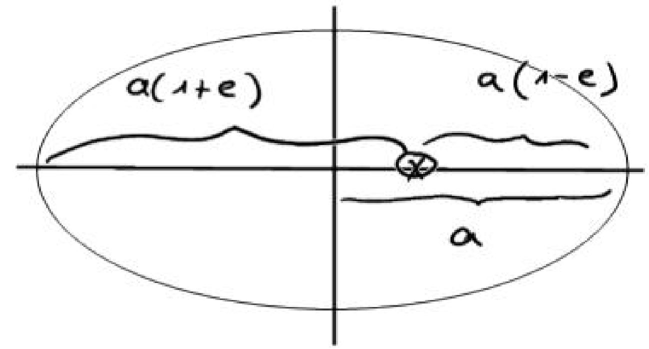
$$r(\varphi) = \boxed{\frac{a(1-e^2)}{1 + e\cos(\varphi - \varphi_0)}}$$

Cases

$e \geq 1$ unbound orbit as $1 + e\cos(\varphi-\varphi_0)$ can be $= 0$

$\Rightarrow r \to \infty$

$e < 1$ bound orbit (ellipse)



pericenter / apocenter

$$r_{min} = \frac{a(1-e^2)}{1+e} = a(1-e)$$

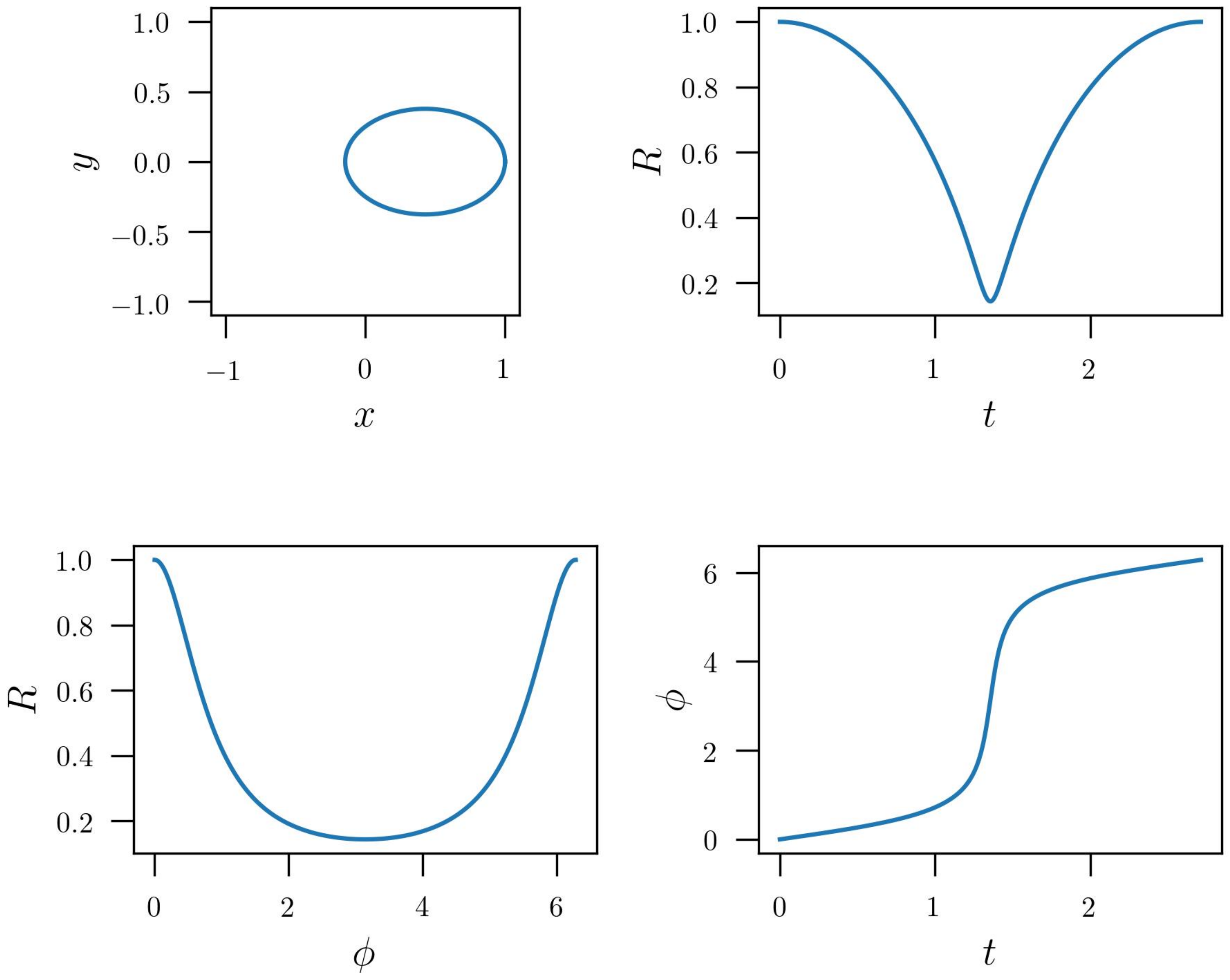
$$r_{max} = \frac{a(1-e^2)}{1-e} = a(1+e)$$

$e = 0$ $r_{min} = r_{max} = a$ (circular orbit)

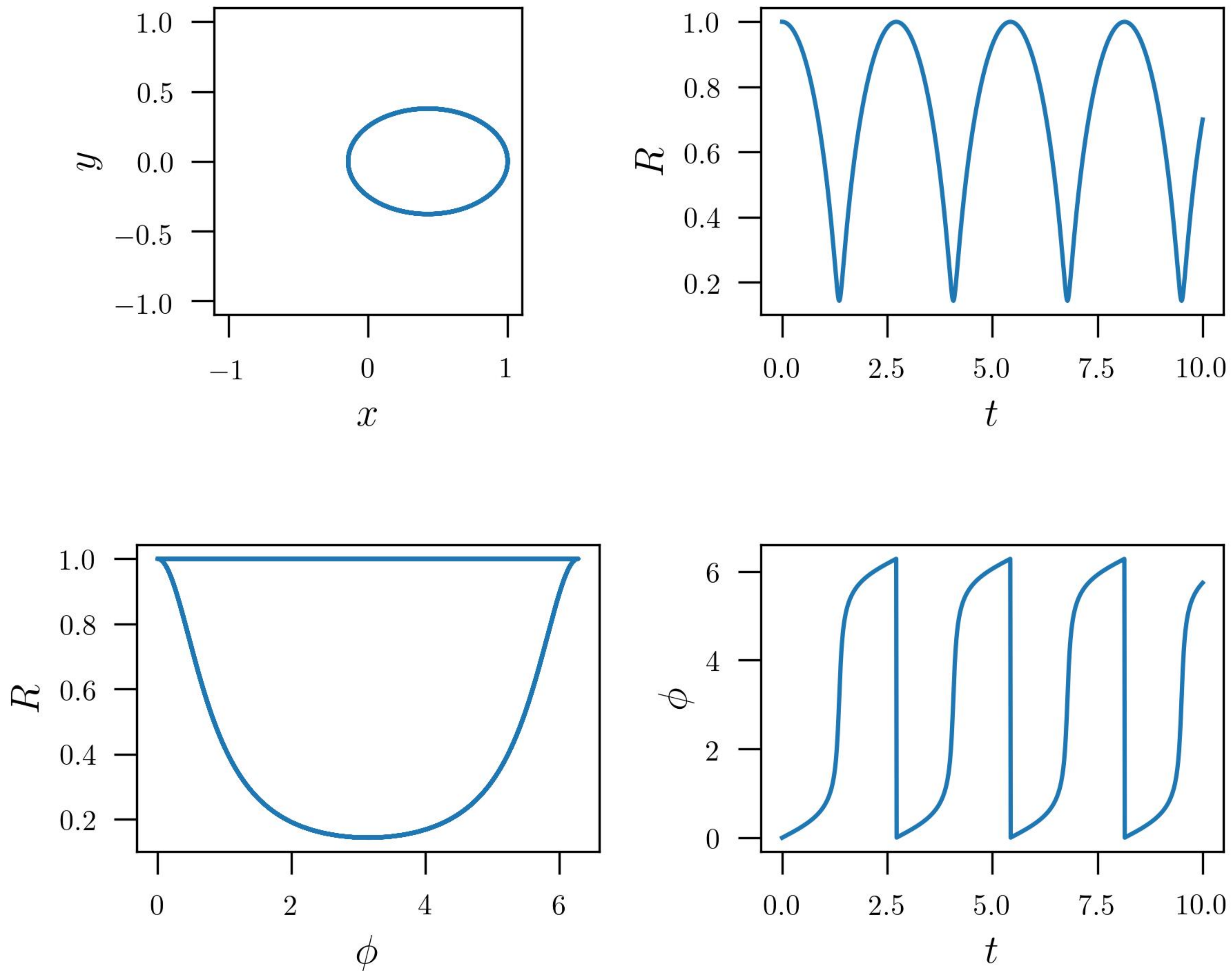
Periods

$$\begin{cases} T_r = 2\displaystyle\int_{r_1}^{r_2} \frac{dr}{\sqrt{2\left(E - \phi(r)\right) - \frac{L^2}{r^2}}} = 2\pi\sqrt{\frac{a^3}{GM}} \\ \\ T_r = T_\varphi \end{cases}$$

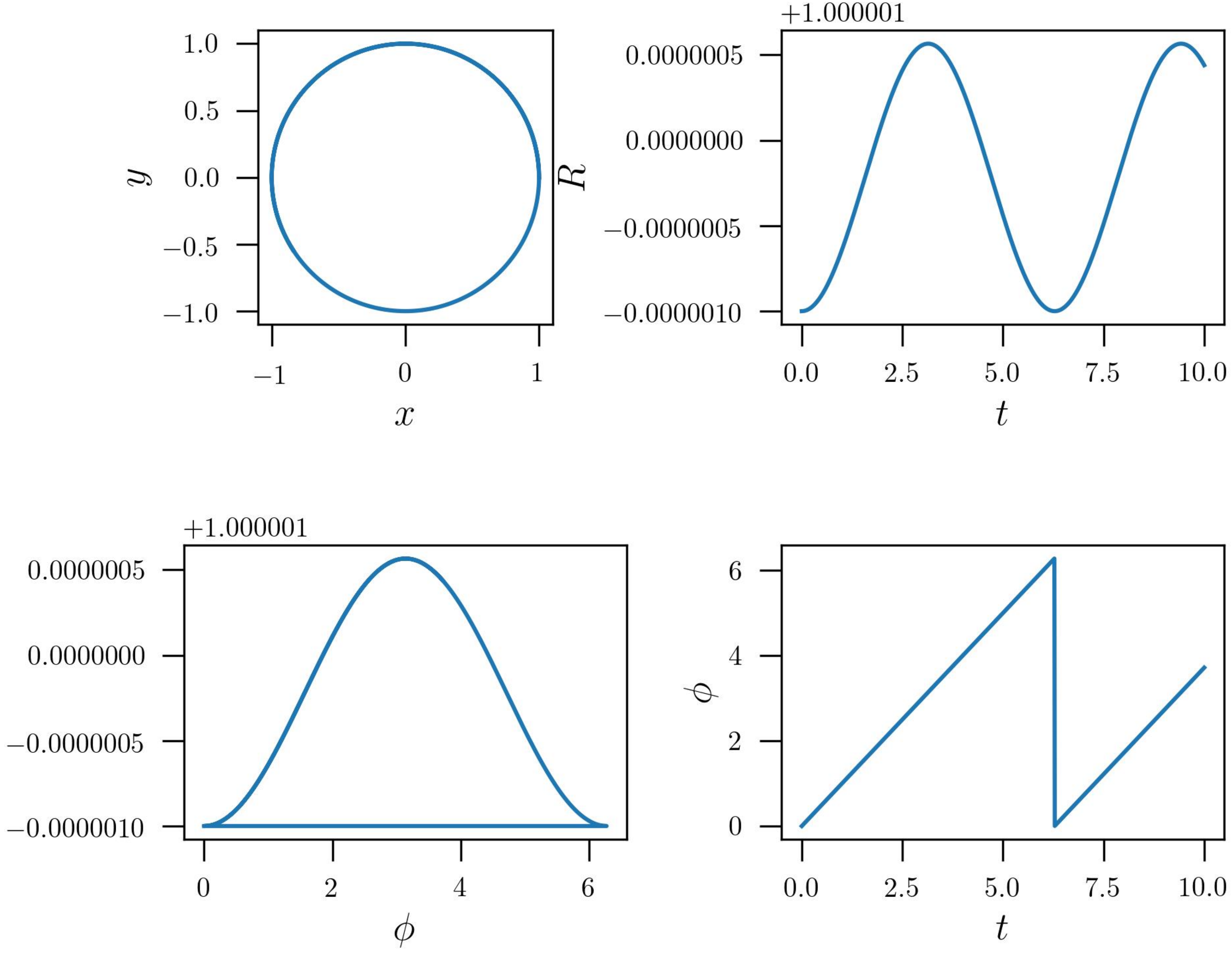
Keplerian orbits (point mass)



Keplerian orbits (point mass)



Keplerian orbits (point mass)



(2) Homogeneous sphere ρ_0, R_0 (Harmonic oscillations)

$$\phi(r) = \underbrace{-2\pi G \rho_0 R_0^2}_{\text{cte} \to 0} + \frac{2}{3}\pi G \rho_0 r^2$$

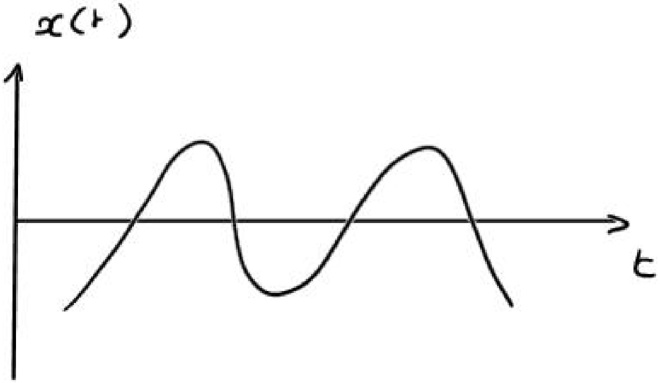
$$\phi(r) = \frac{1}{2}\Omega^2 r^2 \qquad \text{with } \Omega = \sqrt{\frac{4}{3}\pi G \rho_0}$$

Equations of motion (in carthesian coordinates)

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{2}\Omega^2(x^2 + y^2)$$

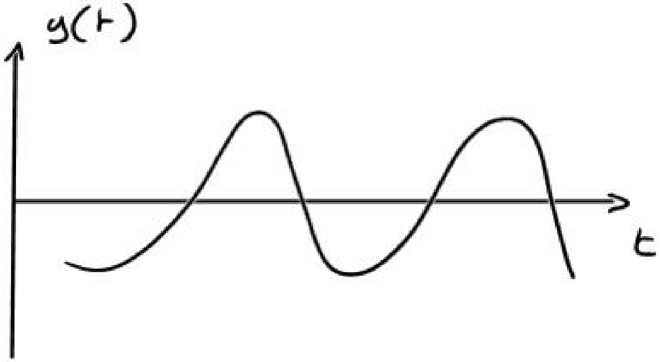
$$\begin{cases} \ddot{x} = -\Omega^2 x \\ \ddot{y} = -\Omega^2 y \end{cases} \qquad \begin{cases} x(t) = X \cos(\Omega t + \varepsilon_x) \\ y(t) = Y \cos(\Omega t + \varepsilon_y) \end{cases}$$

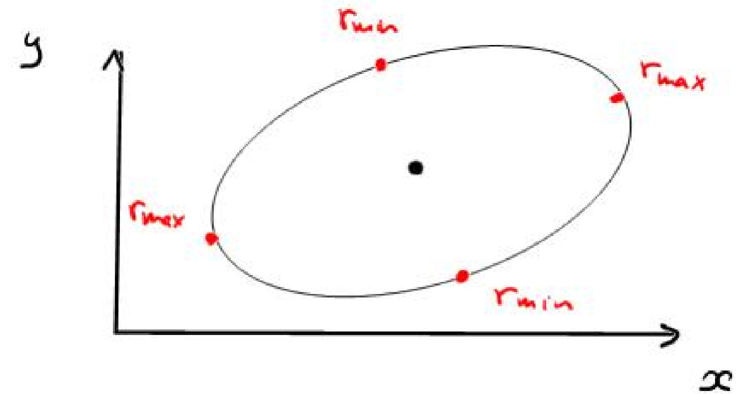
$X, Y, \varepsilon_x, \varepsilon_y$ constants fixed by the initial conditions



same period

$\Rightarrow$ closed orbits (ellipse)

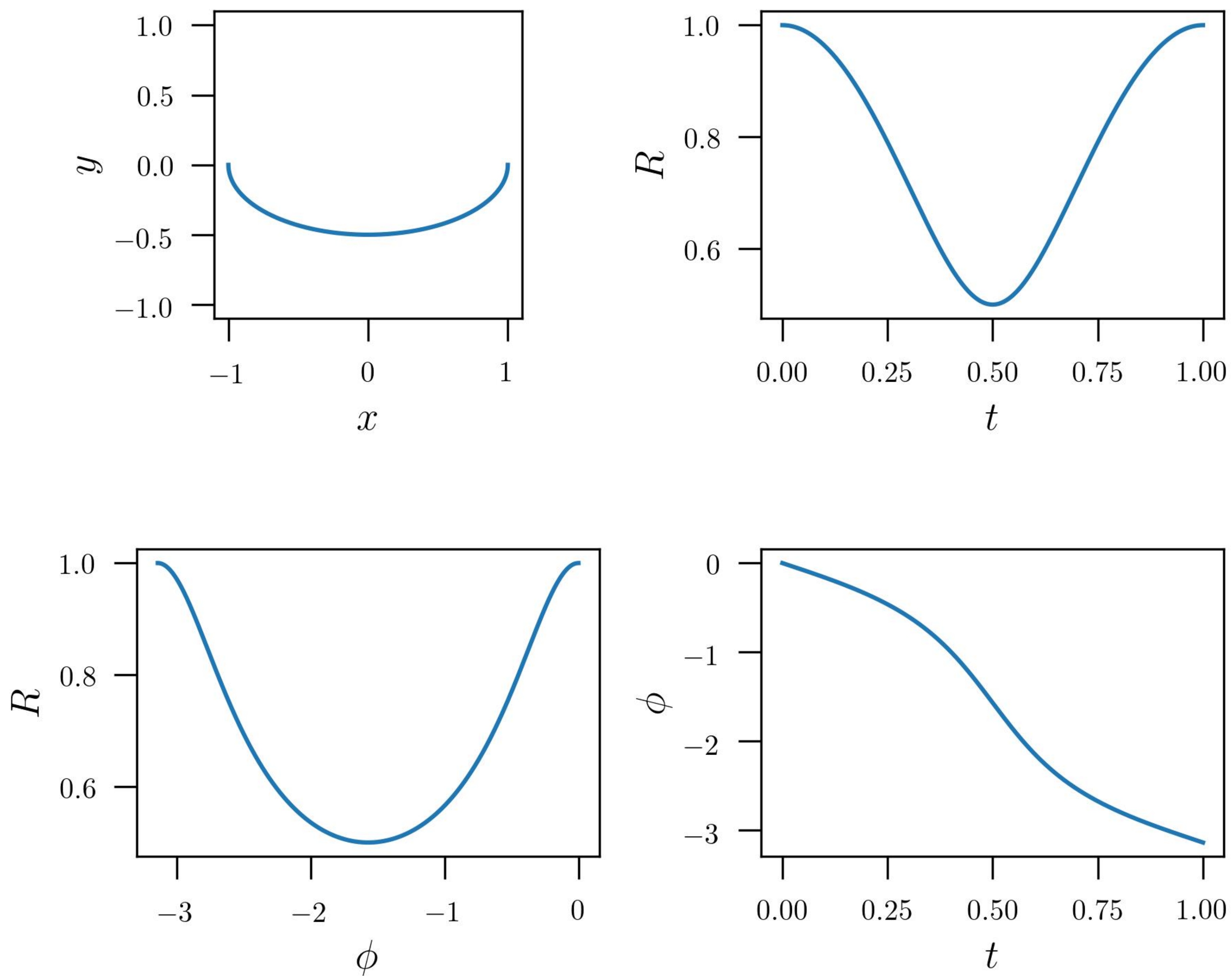




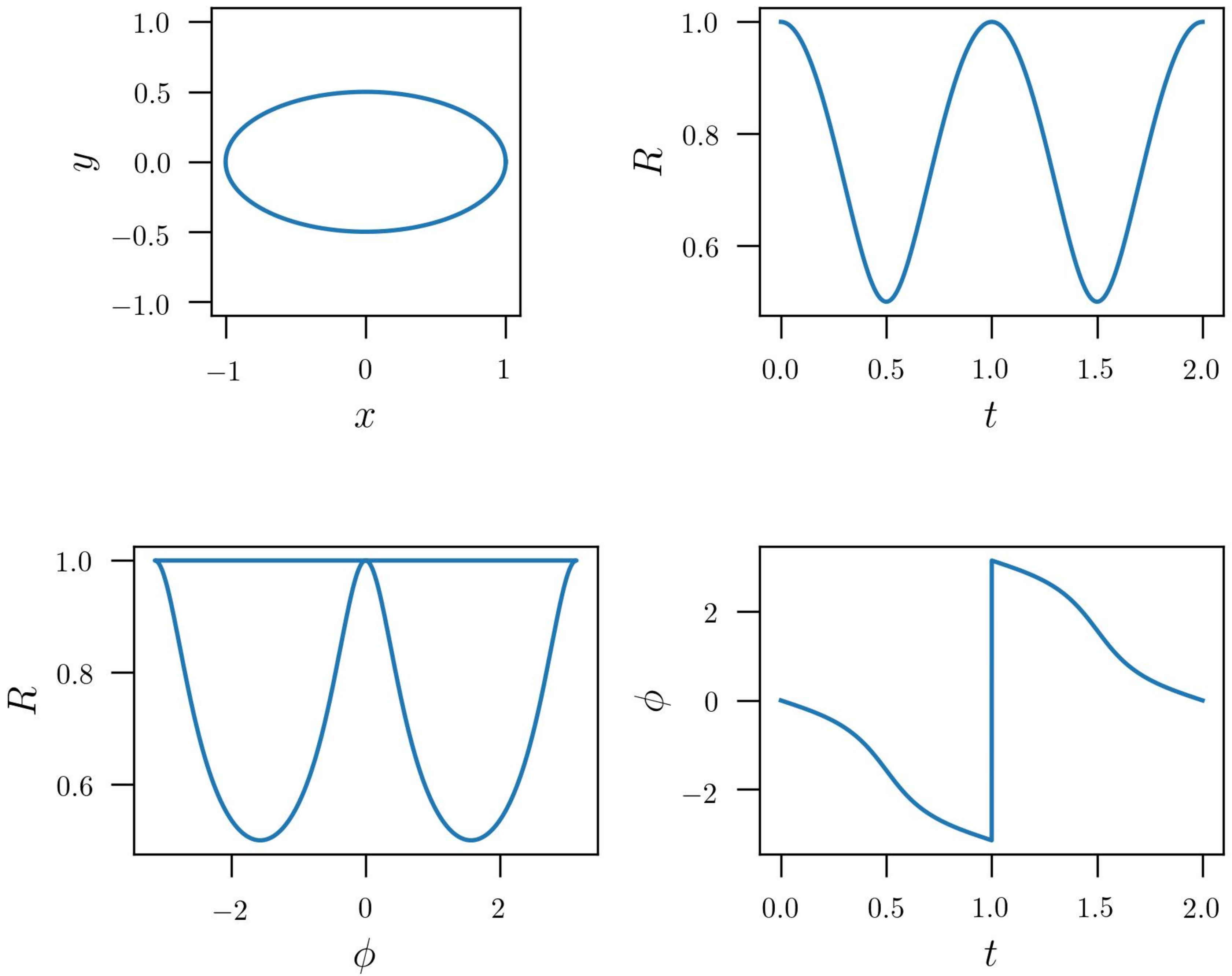
Periods

$$T_\varphi = \frac{2\pi}{\Omega} \qquad T_r = \frac{1}{2} T_\varphi = \frac{\pi}{\Omega}$$

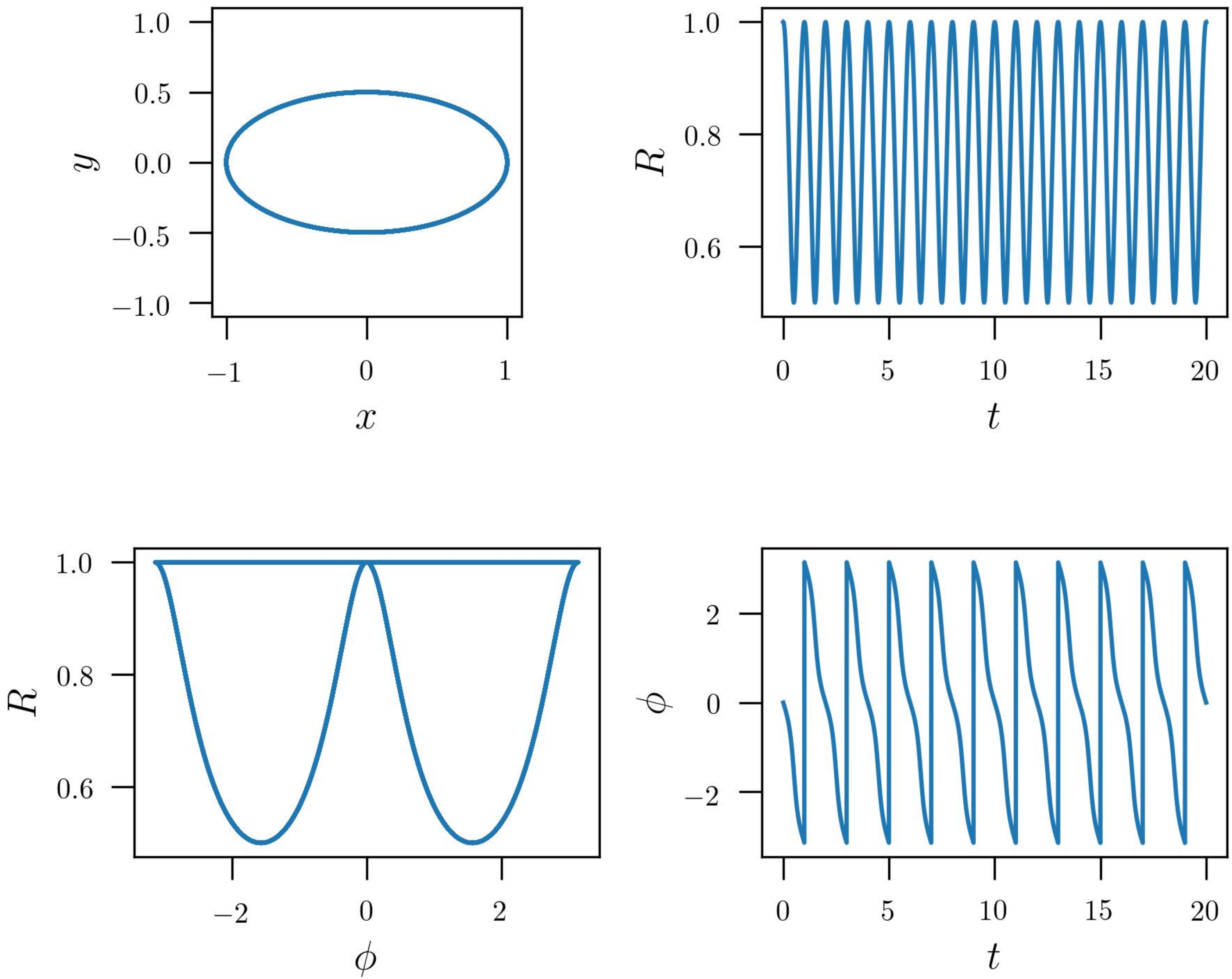
Homegeneous sphere (harmonic)



Homegeneous sphere (harmonic)



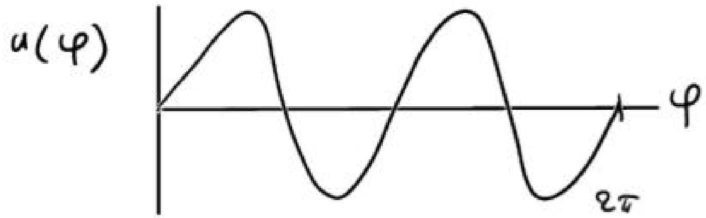
Homegeneous sphere (harmonic)

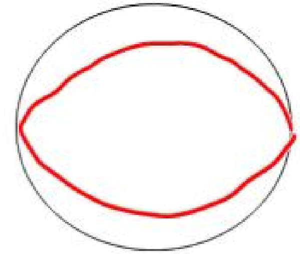


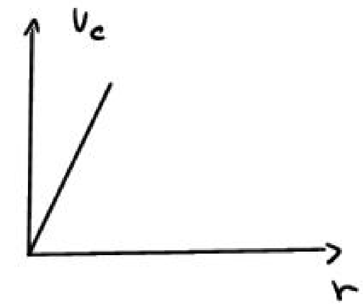
Important Remarks

Homogeneous sphere

$$\boxed{T_r = \frac{1}{2} T_\varphi}$$



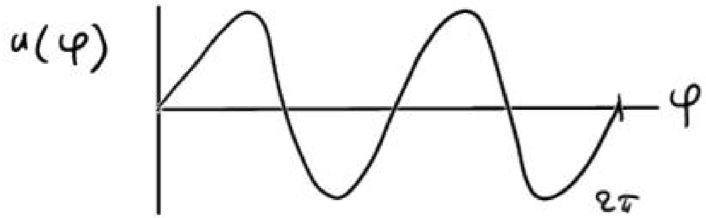


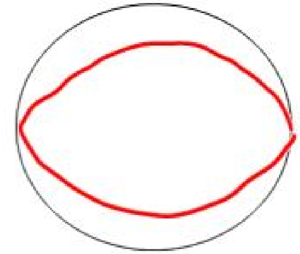


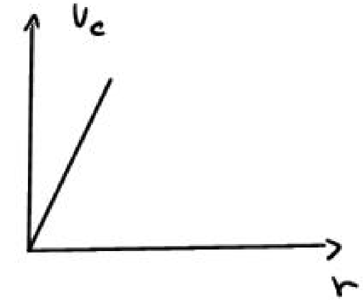
Important Remarks

Homogeneous sphere

$$\boxed{T_r = \frac{1}{2} T_\varphi}$$

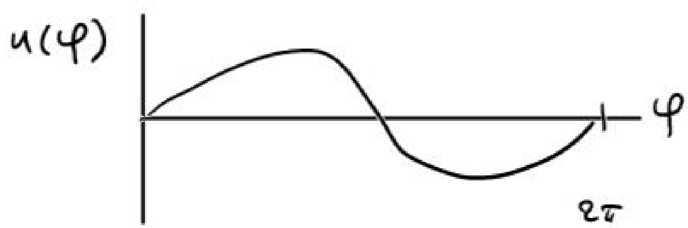


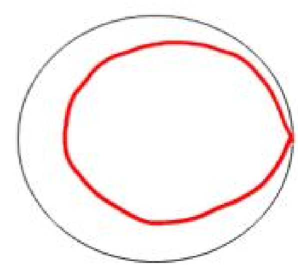


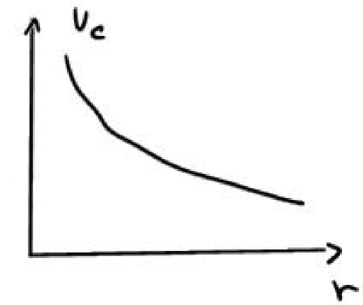


Keplerian potential

$$\boxed{T_r = T_\varphi}$$



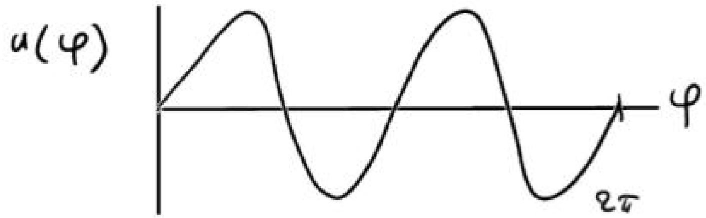


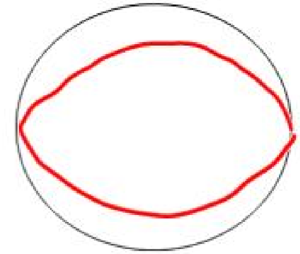


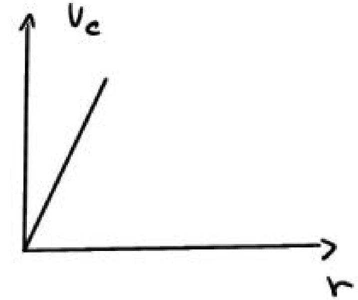
Important Remarks

Homogeneous sphere

$$\boxed{T_r = \frac{1}{2} T_\varphi}$$

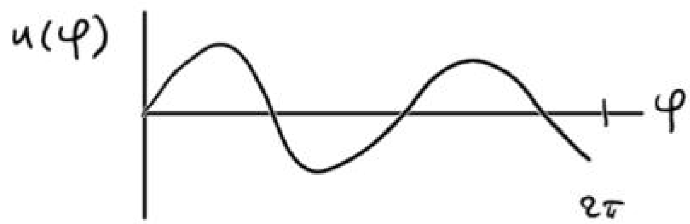


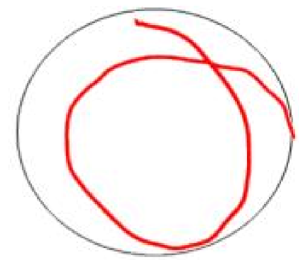


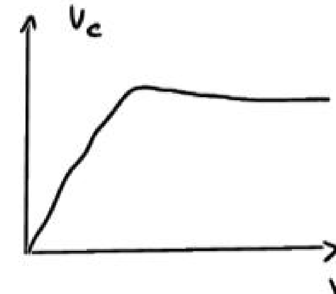


Galaxy

$$\boxed{\frac{1}{2} T_\varphi < T_r < T_\varphi}$$

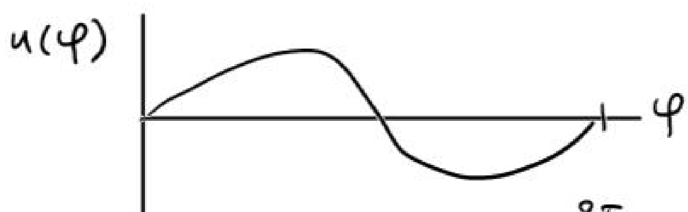


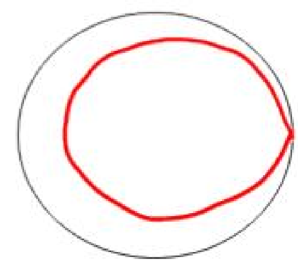


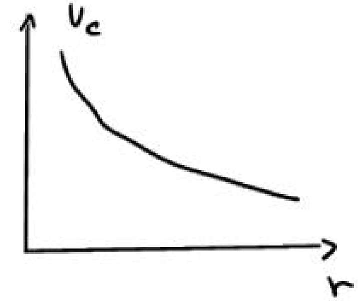


Keplerian potential

$$\boxed{T_r = T_\varphi}$$







Isochrone potential

Good galaxy model that leads to analytical orbits

$$\boxed{\phi(r) = -\frac{GM}{b+\sqrt{b^2+r^2}}}$$

New variable

$$s = -\frac{GM}{b\,\phi(r)} = \frac{b+\sqrt{b^2+r^2}}{b} = 1+\sqrt{1+\frac{r^2}{b^2}}$$

Henon 1959

solution of $s^2 - 2s - \frac{r^2}{b^2} = 0$

$$\Rightarrow \quad \frac{r^2}{b^2} = s^2\left(1-\frac{2}{s}\right)$$

We can write

$$\frac{ds}{dt} = \frac{ds}{dr}\frac{dr}{dt} \quad \Rightarrow \quad \boxed{s(t) = \int_{t_0}^{t} \frac{ds}{dr}\frac{dr}{dt}\,dt}$$ can be integrated

$$\frac{ds}{dt} = \frac{ds}{dr}\frac{dr}{dt} = \left(1+\frac{r^2}{b^2}\right)^{-\frac{1}{2}}\frac{r}{b^2}\sqrt{2(E-\phi)-\frac{L^2}{r^2}}$$

Radial and azimuthal periods

$$T_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E-\phi) - \frac{L^2}{r^2}}} \quad \text{and} \quad \Delta\varphi = 2L\int_{r_1}^{r_2} \frac{dr}{r^2\sqrt{2(E-\phi) - \frac{L^2}{r^2}}}$$

as $\dfrac{dr}{dt} = \sqrt{2(E-\phi) - \dfrac{L^2}{r^2}}$ $\qquad \underbrace{2(E-\phi) - \frac{L^2}{r^2}}_{(r-r_1)(r-r_2)} = 0$ solutions $r_1\, r_2$

We can re-write $\dfrac{dr}{\sqrt{2(E-\phi) - \frac{L^2}{r^2}}}$ in term of s

$$T_r = \frac{2b}{\sqrt{-2E}} \int_{s_1}^{s_2} ds \frac{s-1}{\sqrt{(s_2-s)(s-s_1)}}$$

$$\left\{\begin{array}{l} T_r = \dfrac{2\pi G M}{(-2E)^{3/2}} \longrightarrow \text{⚠} \\[2ex] T_\varphi = \dfrac{4\pi G M}{(-2E)^{3/2}} \, \dfrac{\sqrt{L^2 + 4GMb}}{|L| + \sqrt{L^2 + 4GMb}} \end{array}\right.$$

independent of L

(isochrone)

L'AMAS ISOCHRONE

II. — Calcul des orbites

par M. Hénon
(Institut d'Astrophysique, Paris)

Sommaire. — *On obtient les expressions explicites des orbites stellaires dans un amas isochrone (modèle d'amas globulaire) On calcule la période, l'angle entre apocentres successifs, la densité moyenne le long d'une orbite. Six orbites particulières sont dessinées.*

Abstract. — *One obtains the explicit expressions of the stellar orbits in an isochron cluster (model of a globular cluster). One computes the period, the angle between two successive apocenters, the mean density along an orbit. Six particular orbits are drawned down.*

Резюме. — Автор получает подробные выражения звездных орбит в изохронном скоплении (модель шарового скопления). Вычислен период, угол между последовательными апоцентрами и средняя плотность вдоль орбиты. Даны чертежи шести частных случаев орбит.

L'intégration se fait sans difficultés particulières et mène à :

$$(5)\quad t - t_0 = -\frac{\sqrt{-A^2+2)\,U^2 + (2E+2A^2)\,U - A^2}}{(2-2E)\,(1-U)} + \frac{1}{(2-2E)^{3/2}}\,\text{Arc sin}\,\frac{(2-E)\,U - E}{\sqrt{E^2+2EA^2-2A^2}\,(1-U)}.$$

Cette relation entre U et t, donc entre r et t, définit le mouvement radial de l'étoile.

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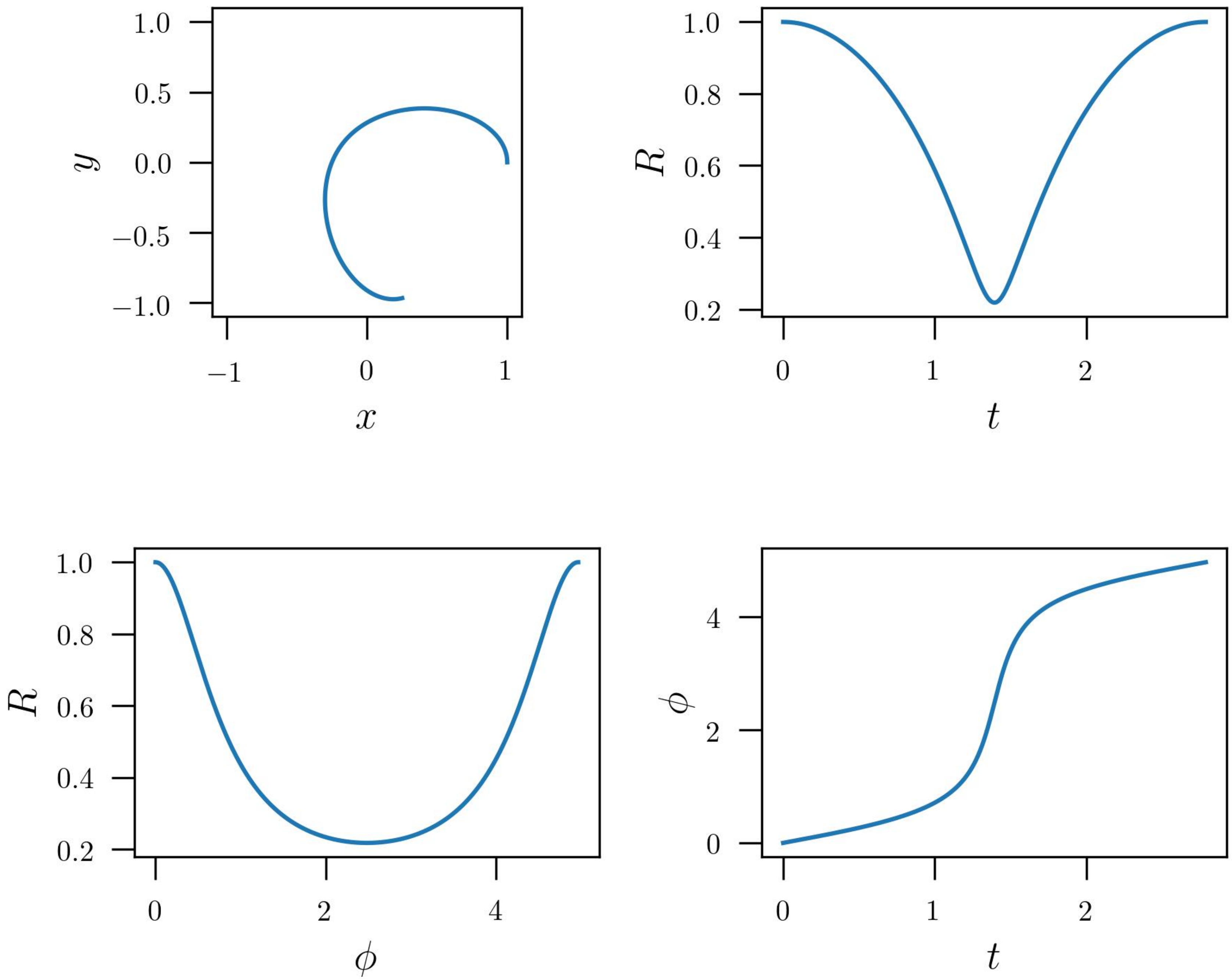

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R
t
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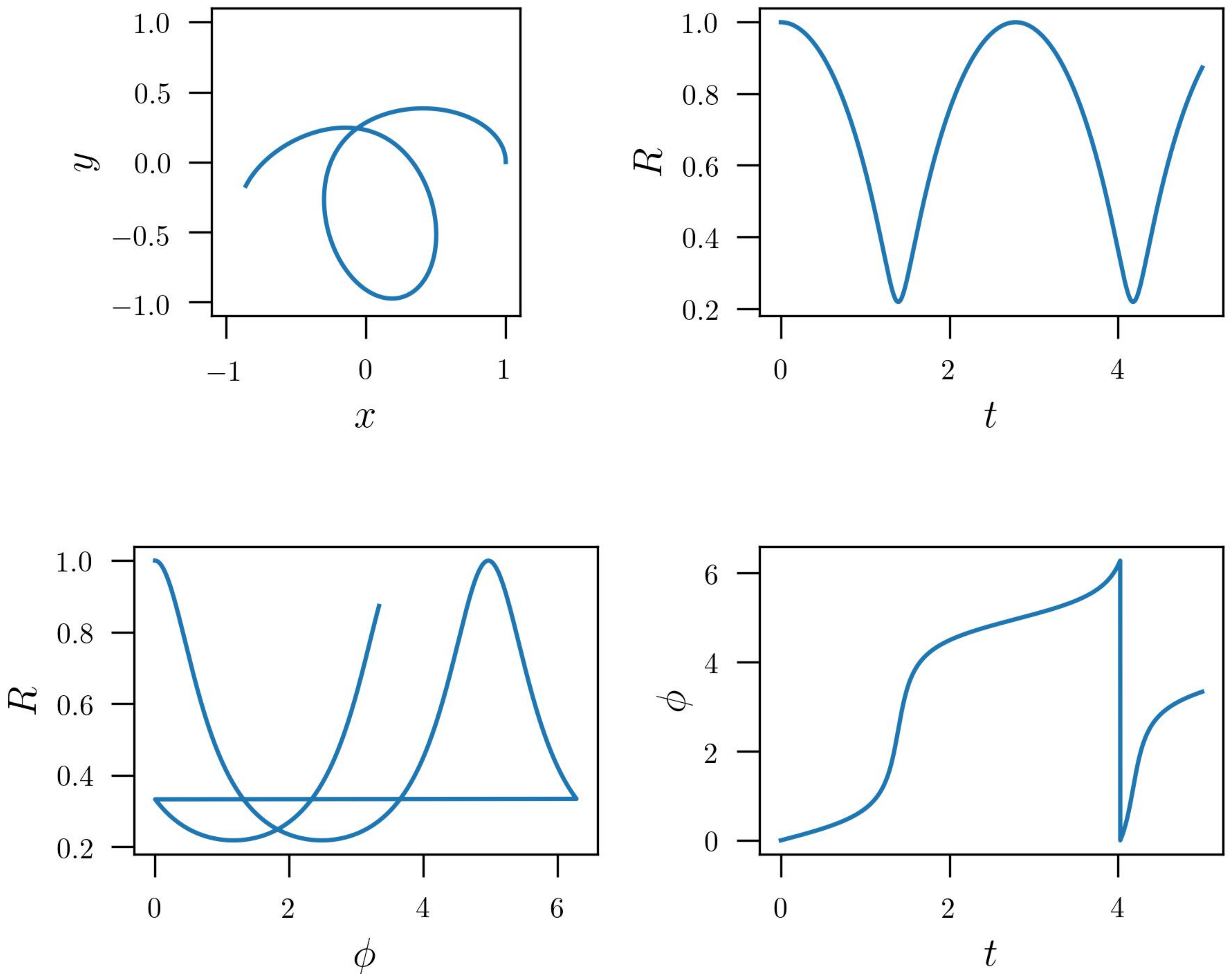

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R
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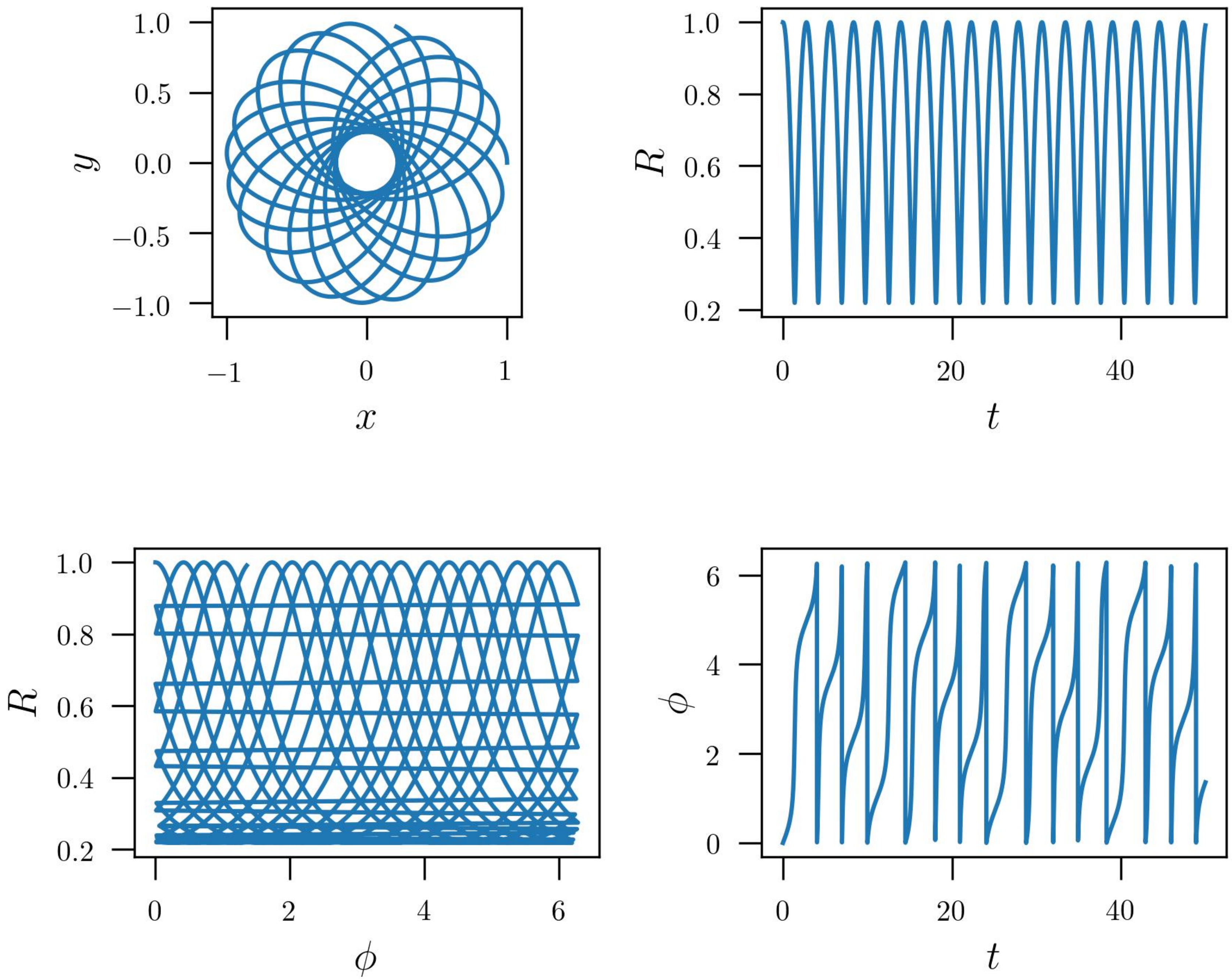

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−1.0
y
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0
1
x
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0.8
0.6
0.4
0.2
R
0
20
40
t
1.0
0.8
0.6
0.4
0.2
R
0
2
4
6
ϕ
6
4
2
0
ϕ
0
20
40
t

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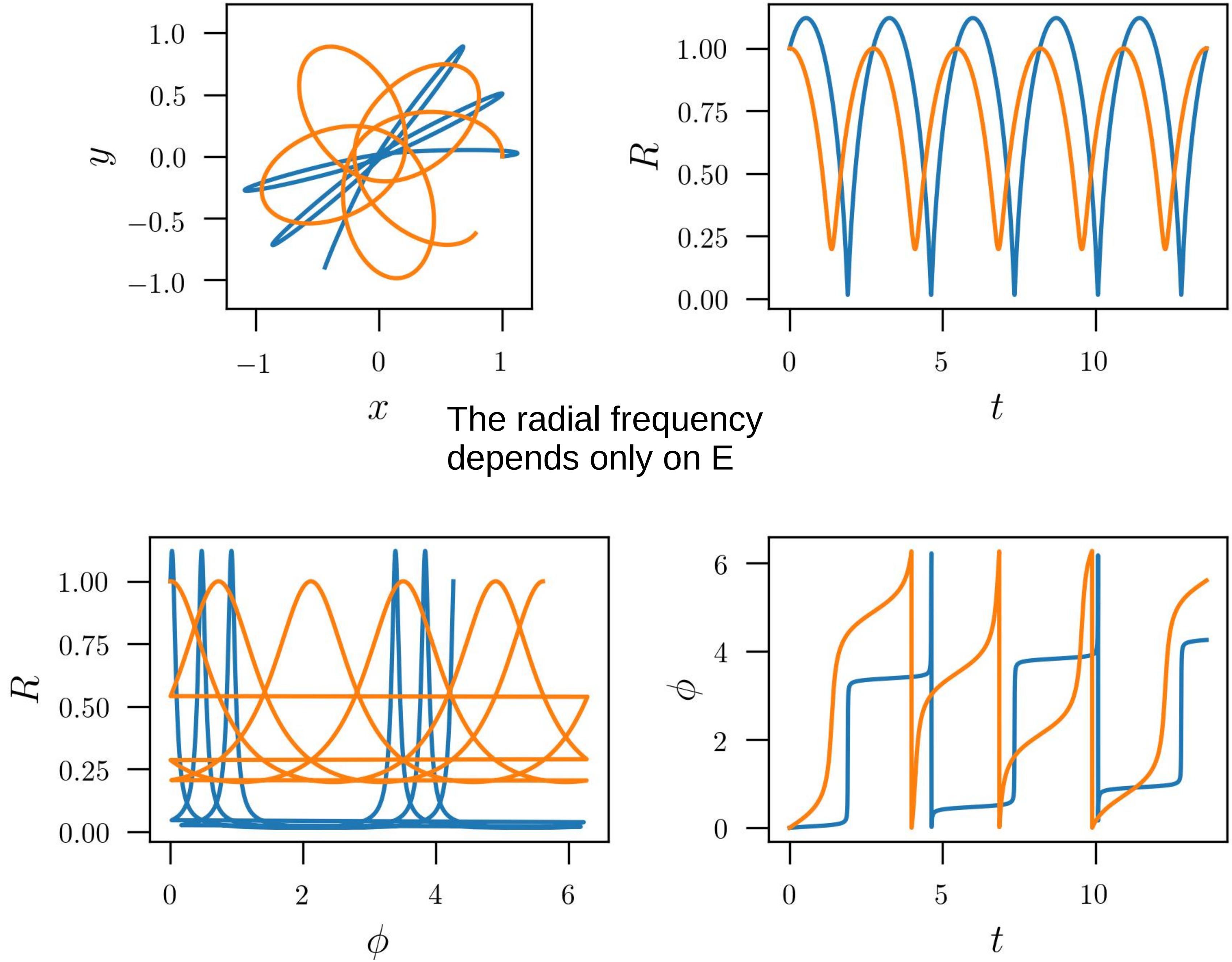

The radial frequency depends only on E

galpy

Welcome to galpy's document

https://galpy.readthedocs.io/en/v1.5.0/

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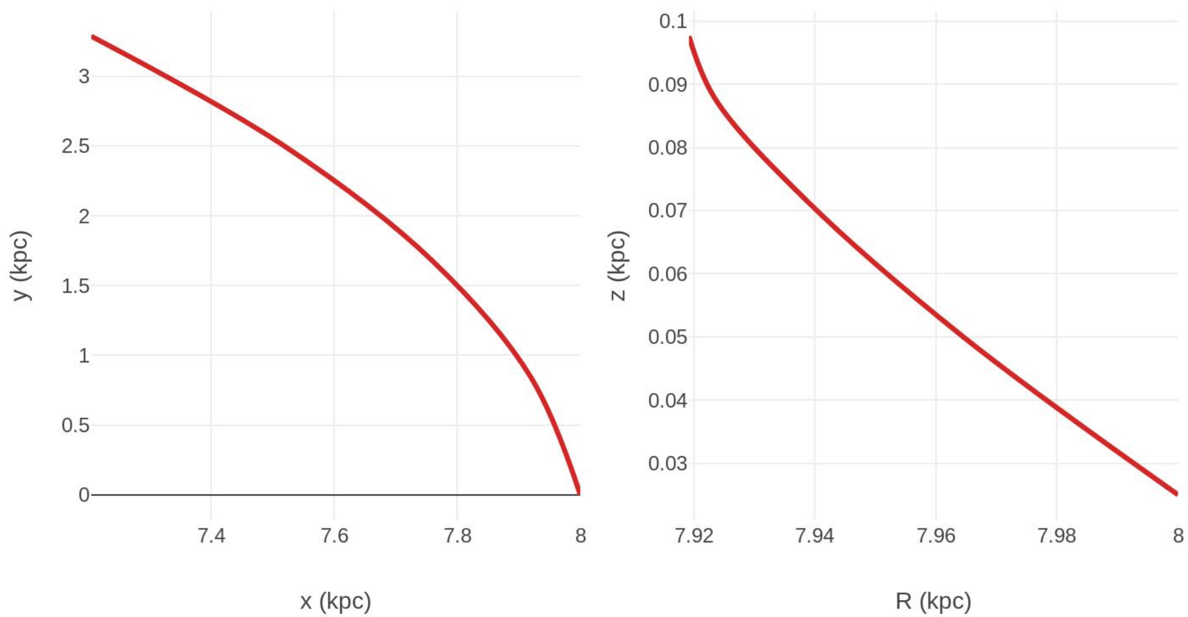
Welcome to galpy's documentation ¶

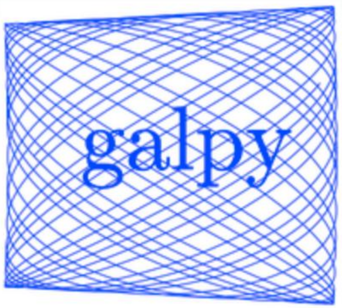
galpy is a Python 2 and 3 package for galactic dynamics. It supports orbit integration in a variety of potentials, evaluating and sampling various distribution functions, and the calculation of action-angle coordinates for all static potentials. galpy is an astropy affiliated package and provides full support for astropy's Quantity framework for variables with units.

galpy is developed on GitHub. If you are looking to report an issue or for information on how to contribute to the code, please head over to galpy's GitHub page for more information.

As a preview of the kinds of things you can do with galpy, here's an animation of the orbit of the Sun in galpy's `MWPotential2014` potential over 7 Gyr:



Play | Pause | Speed x 2 | Speed / 2 | Replay



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