Astrophysics III: Stellar and galactic dynamics

Solutions

Problem 1:

From Poisson's equation in spherical coordinates we get:

$$\nabla^2\Phi = 4\pi G\rho$$

$\nabla^2\Phi$ written in spherical coordinates, and considering a spherical potential we get:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)$$

a lot of straight-forward algebra follows, but finally we get

$$\rho = \frac{v_s^2}{4\pi G r_s^2}\frac{1}{(r/r_s)(1+r/r_s)^2}$$

The circular velocity also follows simply:

$$\begin{aligned} v_c^2 &= r\,\frac{\partial\Phi}{\partial r} = r\,v_s^2\left[-\frac{1}{\frac{r}{r_s}\left(1+\frac{r}{r_s}\right)} + \frac{\ln\left(1+\frac{r}{r_s}\right)}{r_s\left(\frac{r}{r_s}\right)^2}\right] = v_s^2\left[\frac{\ln\left(1+\frac{r}{r_s}\right)}{\frac{r}{r_s}} - \frac{1}{\left(1+\frac{r}{r_s}\right)}\right] \\ &= v_s^2\left[\frac{\left(1+\frac{r}{r_s}\right)\ln\left(1+\frac{r}{r_s}\right) - \frac{r}{r_s}}{\frac{r}{r_s}\left(1+\frac{r}{r_s}\right)}\right] = v_s^2\,\frac{(r_s+r)\;r_s\;\ln\left(1+\frac{r}{r_s}\right) - r\,r_s}{r\;(r_s+r)} \end{aligned}$$

Problem 2:

We consider a wire aligned with the x axis. As the mass distribution is discontinuous, we cannot rely on the Poisson equation to derive the corresponding potential. We instead rely on the Gauss Theorem :

$$\int_S \vec{\nabla}\Phi\cdot d\vec{S} = 4\pi G\,M_S, \tag{1}$$

where S is any surface and M_S is the mass enclosed by the surface S. Lets define S to be the surface of a cylinder of length Δx and radius R, with its symmetry axis being the axis x, i.e., the wire. The surface ds parallel to the axis x is :

$$ds = 2\pi R\cdot\Delta x, \tag{2}$$

and the enclosed mass is :

$$M_S = \lambda_0 \cdot \Delta x. \tag{3}$$

By symmetry (the linear density of the wire is constant) :

$$\vec{\nabla}\Phi = \frac{\partial}{\partial R}\Phi(R) \cdot \vec{e}_R, \tag{4}$$

where $\vec{e}_R$ is perpendicular to the axis x. With (2), (3) and (4), the Gauss theorem becomes :

$$\int_S \vec{\nabla}\Phi \cdot d\vec{S} = 2\pi R \cdot \Delta x \, \frac{\partial}{\partial R}\Phi(R) = 4\pi G \, \lambda_0 \cdot \Delta x, \tag{5}$$

which leads to :

$$\frac{\partial}{\partial R}\Phi(R) = 2\, G\, \frac{\lambda_0}{R}, \tag{6}$$

and after integrating over the radius R :

$$\Phi(R) = 2G\, \lambda_0 \, \ln(R) + C, \tag{7}$$

where C is a constant.

Problem 3:

As per problem 1, the isochrone ρ is straightforward to derive, taking the form:

$$\rho = M \left[\frac{3(b+a)a^2 - r^2(b+3a)}{4\pi(b+a)^3 a^3} \right] \quad \text{with} \quad a \equiv \sqrt{b^2+r^2}$$

The circular velocity is

$$v_c^2 = \frac{GMr^2}{(b+a)^2 a}$$

Problem 4:

Starting from Poisson's equation and applying Gauss' theorem, we have:

$$4\pi\, GM = 4\pi G \int \mathrm{d}^3\mathbf{x}\, \rho = \int \mathrm{d}^3\mathbf{x}\, \nabla^2\Phi = \int \mathrm{d}^2\mathbf{S}\, \nabla\Phi$$

Let's define a unit surface on the disk, corresponding to a mass Σ, which is then the surface density. Defining a slab enclosing the unit surface and making its thickness tend to a vanishing value ($\varepsilon \to 0$, see Fig. 1), the surface integral reduces to twice the gradient of the potential:

$$4\pi\, G\, \Sigma = \int \mathrm{d}^2\mathbf{S}\, \nabla\Phi_\mathrm{K} = 2\frac{\partial\Phi_\mathrm{K}}{\partial z}$$

We have

$$\begin{aligned}\frac{\partial\Phi_\mathrm{K}}{\partial z} &= \frac{\partial}{\partial z}\left[-GM\left[R^2 + (a+|z|)^2\right]^{-1/2}\right] \\ &= GM\left[R^2 + (a+|z|)^2\right]^{-3/2}(a+|z|)\end{aligned}$$

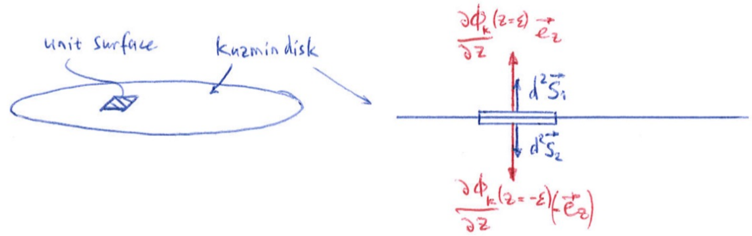

Figure 1: The Kuzmin disk with the unit surface (left) and seen edge-on (right), with the 2ε thick slab, on the surface of which the integration is made.

With $|z| \to 0$, we then have:

$$\begin{aligned} 4\pi\, G\Sigma_{\mathrm{K}} &= 2\frac{\partial \Phi_{\mathrm{K}}}{\partial z} = 2aGM\left[R^2+a^2\right]^{-3/2} \\ \Rightarrow \Sigma_{\mathrm{K}} &= \frac{aM}{2\pi\left(R^2+a^2\right)^{3/2}} \end{aligned}$$

Problem 5:

The velocity curve may be obtained from the formula:

$$v_{\mathrm{c}}^2(R) = -4G\int_0^R \mathrm{d}a \frac{a}{\sqrt{R^2-a^2}}\frac{\mathrm{d}}{\mathrm{d}a}\int_a^\infty \mathrm{d}R' \frac{R'\Sigma(R')}{\sqrt{R'^2-a^2}} \tag{8}$$

Replacing $\Sigma(R')$ using the Mestel's surface density we get:

$$\begin{aligned} \int_a^\infty \mathrm{d}R' \frac{R'\Sigma(R')}{\sqrt{R'^2-a^2}} &= \frac{v_0^2}{2\pi G}\int_a^\infty \mathrm{d}R' \frac{1}{\sqrt{R'^2-a^2}} \\ &= \frac{v_0^2}{2\pi G}\int_a^{R_{\max}} \mathrm{d}R' \frac{1}{\sqrt{(R'/a)^2-1}}\frac{1}{a} \\ &= \frac{v_0^2}{2\pi G}\int_a^{R_{\max}} \mathrm{d}R' \frac{\mathrm{d}}{\mathrm{d}R}\left(\mathrm{arccosh}(R/a)\right) \\ &= \frac{v_0^2}{2\pi G}\left[\mathrm{arccosh}(R_{\max}/a) - \mathrm{arccosh}(1)\right] \\ &= \frac{v_0^2}{2\pi G}\mathrm{arccosh}(R_{\max}/a) \end{aligned} \tag{9}$$

The derivative with respect to a of this latter result writes:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{v_0^2}{2\pi G}\mathrm{arccosh}(R_{\max}/a)\right) &= \frac{v_0^2}{2\pi G}\frac{\mathrm{d}}{\mathrm{d}a}\mathrm{arccosh}(R_{\max}/a) \\ &= -\frac{v_0^2}{2\pi G}\frac{R_{\max}}{\sqrt{R_{\max}^2-a^2}}\frac{1}{a} \end{aligned} \tag{10}$$

which, in the limit $R_{\max} \to \infty$ gives:

$$-\frac{v_0^2}{2\pi G a} \tag{11}$$

This leads to the circular velocity:

$$\begin{aligned} v_c^2(R) &= \frac{2v_0^2}{\pi} \int_0^R \mathrm{d}a \frac{1}{\sqrt{R^2 - a^2}} \\ &= v_0^2 \end{aligned} \tag{12}$$