Stellar orbits

2nd part

Outlines

Examples of orbits in spherical potentials

- Keplerian orbits
- orbits in an homogeneous sphere
- important remarks

Orbits in axisymmetric potentials

- orbits in the equatorial plane
- orbits outside the equatorial plane
- equations of motion
- orbits in the meridian plane
- examples

Nearly circular orbits

- Epicycle frequences

Spherical Systems

Examples

Examples

① Kepler potential (potential of a mass point)

$$\begin{cases} \phi(r) = -\dfrac{GM}{r} \\ \dfrac{\partial \phi}{\partial r}(r) = \dfrac{GM}{r^2} = GMu^2 \end{cases}$$

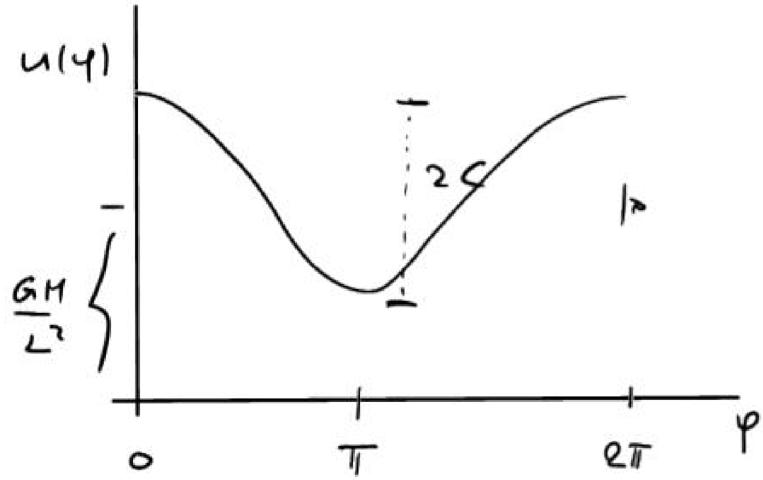
$$\boxed{\frac{d^2u}{d\varphi^2} + u = \frac{1}{L^2u^2}\frac{\partial \phi}{\partial r}\left(\frac{1}{u}\right)} \Rightarrow \boxed{\frac{d^2u}{d\varphi^2} + u = \frac{GM}{L^2}}$$

Harmonic equation, with frequency 1

$$\boxed{\frac{d^2u}{d\varphi^2} + u = \frac{GM}{L^2}}$$

General solution

$$u(\varphi) = \underbrace{C}_{\text{free parameter}} \cos(\varphi - \underbrace{\varphi_0}_{\text{free parameter}}) + \frac{GM}{L^2}$$



period = 2π

In term of r

$$\boxed{r(\varphi) = \frac{1}{C \cos(\varphi - \varphi_0) + \frac{GM}{L^2}}}$$

Introducing

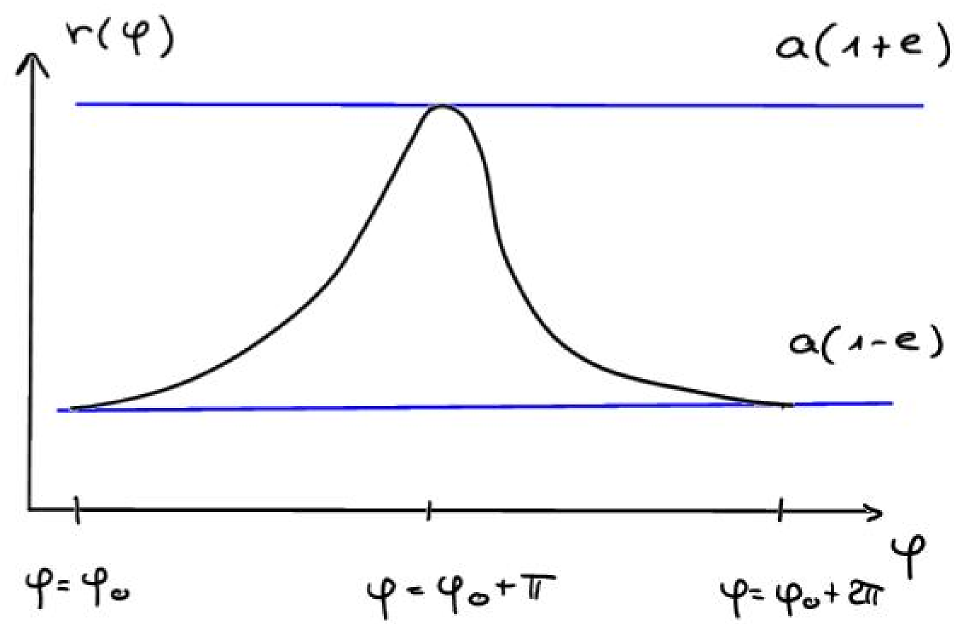
$$\begin{cases} e = \dfrac{CL^2}{GM} & \text{eccentricity} \\[2ex] a = \dfrac{L^2}{GM(1-e^2)} & \text{semi-major axis} \end{cases}$$

evaluate u and $\dfrac{du}{dt}$ for $\varphi = \varphi_0$ $\left(u(\varphi) \overset{\varphi=\varphi_0}{=} C + \dfrac{GM}{L^2} \quad \dfrac{du}{dt}(\varphi) \overset{\varphi=\varphi_0}{=} 0 \right)$

+ using $\dfrac{d^2u}{d\varphi^2} + u = \dfrac{1}{L^2u^2} \dfrac{\partial\phi}{\partial r}\left(\dfrac{1}{u}\right)$

$$\boxed{\begin{cases} r(\varphi) = \dfrac{a(1-e^2)}{1 + e\cos(\varphi - \varphi_0)} \\[2ex] E = -\dfrac{GM}{2a} \end{cases}}$$

↳ from the energy equation



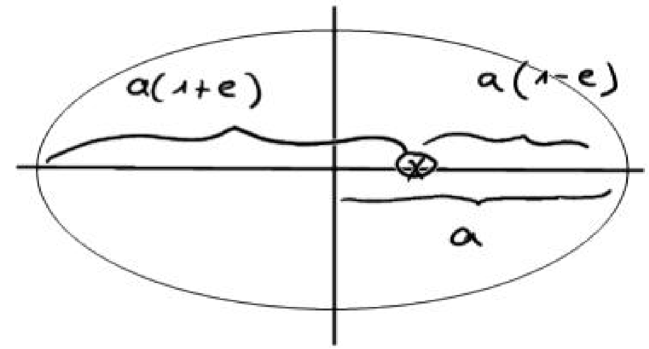
$$r(\varphi) = \frac{a(1-e^2)}{1 + e\cos(\varphi-\varphi_0)}$$

Cases

$e \geq 1$ unbound orbit as $1 + e\cos(\varphi-\varphi_0)$ can be $=0$

$\Rightarrow r \to \infty$

$e < 1$ bound orbit (ellipse)



pericenter / apocenter

$$r_{min} = \frac{a(1-e^2)}{1+e} = a(1-e)$$

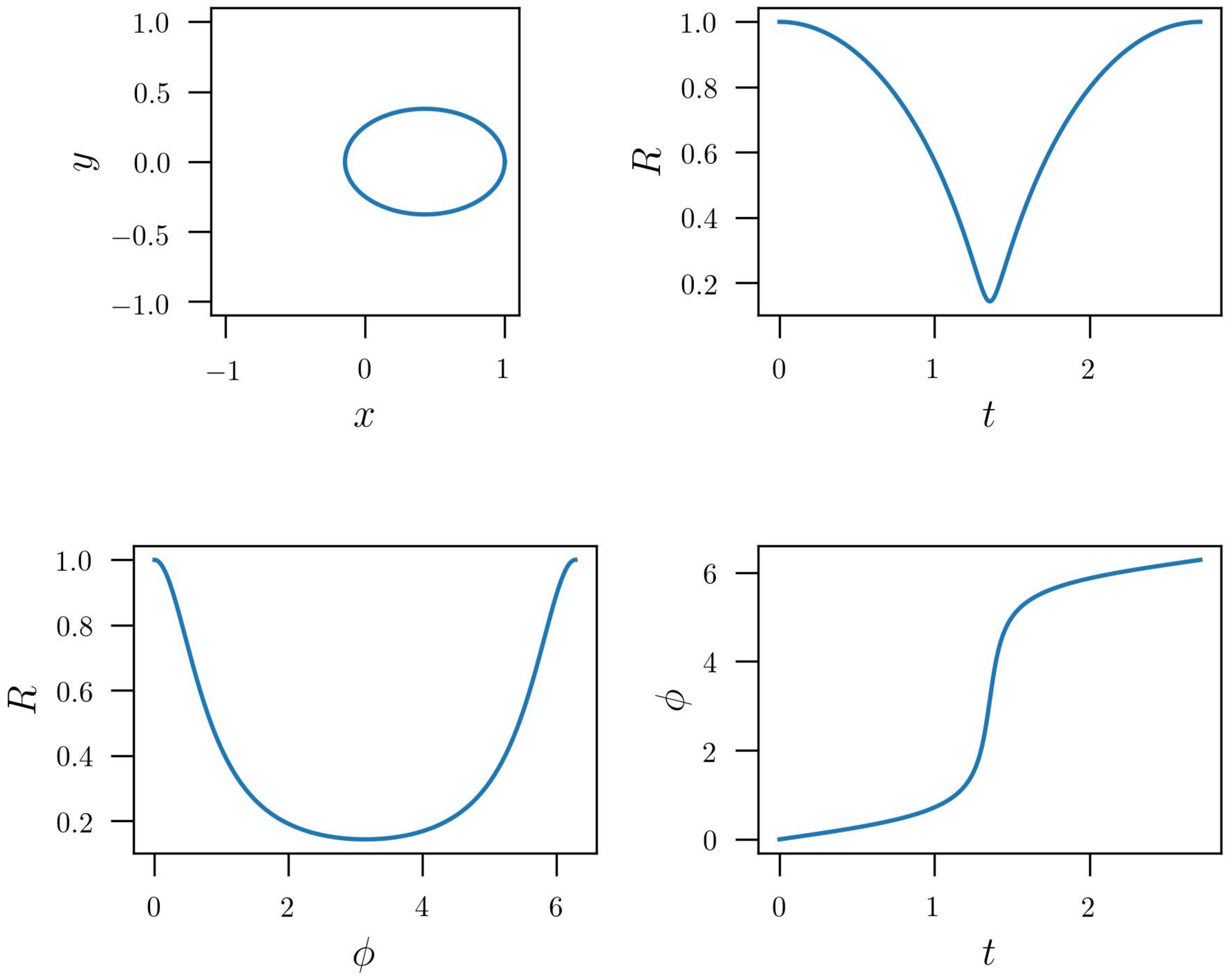
$$r_{max} = \frac{a(1-e^2)}{1-e} = a(1+e)$$

$e = 0$ $r_{min} = r_{max} = a$ (circular orbit)

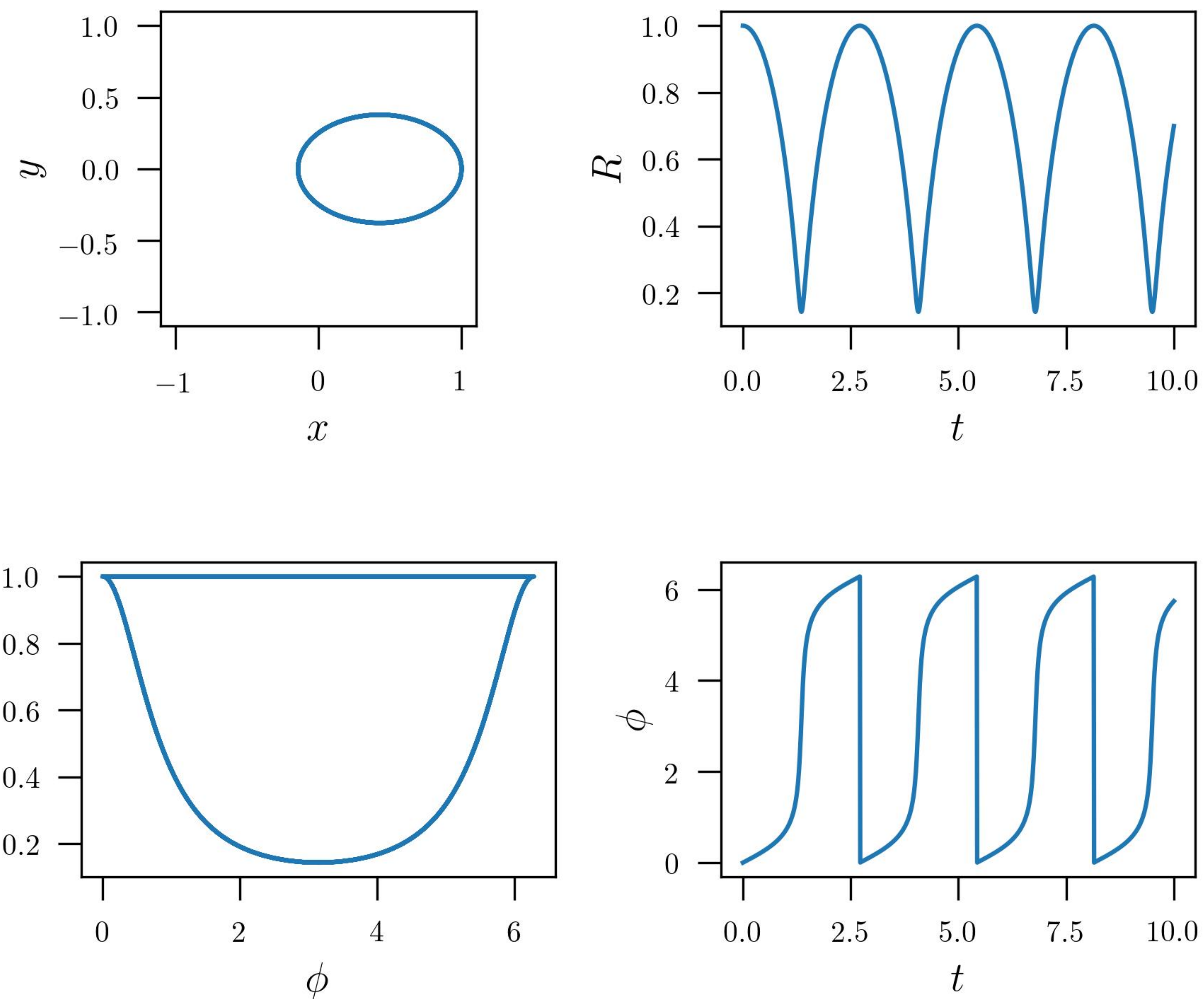
Periods

$$\begin{cases} T_r = 2\displaystyle\int_{r_1}^{r_2} \frac{dr}{\sqrt{2\left(E - \phi(r)\right) - \frac{L^2}{r^2}}} = 2\pi\sqrt{\frac{a^3}{GM}} \\[2em] T_r = T_\varphi \end{cases}$$

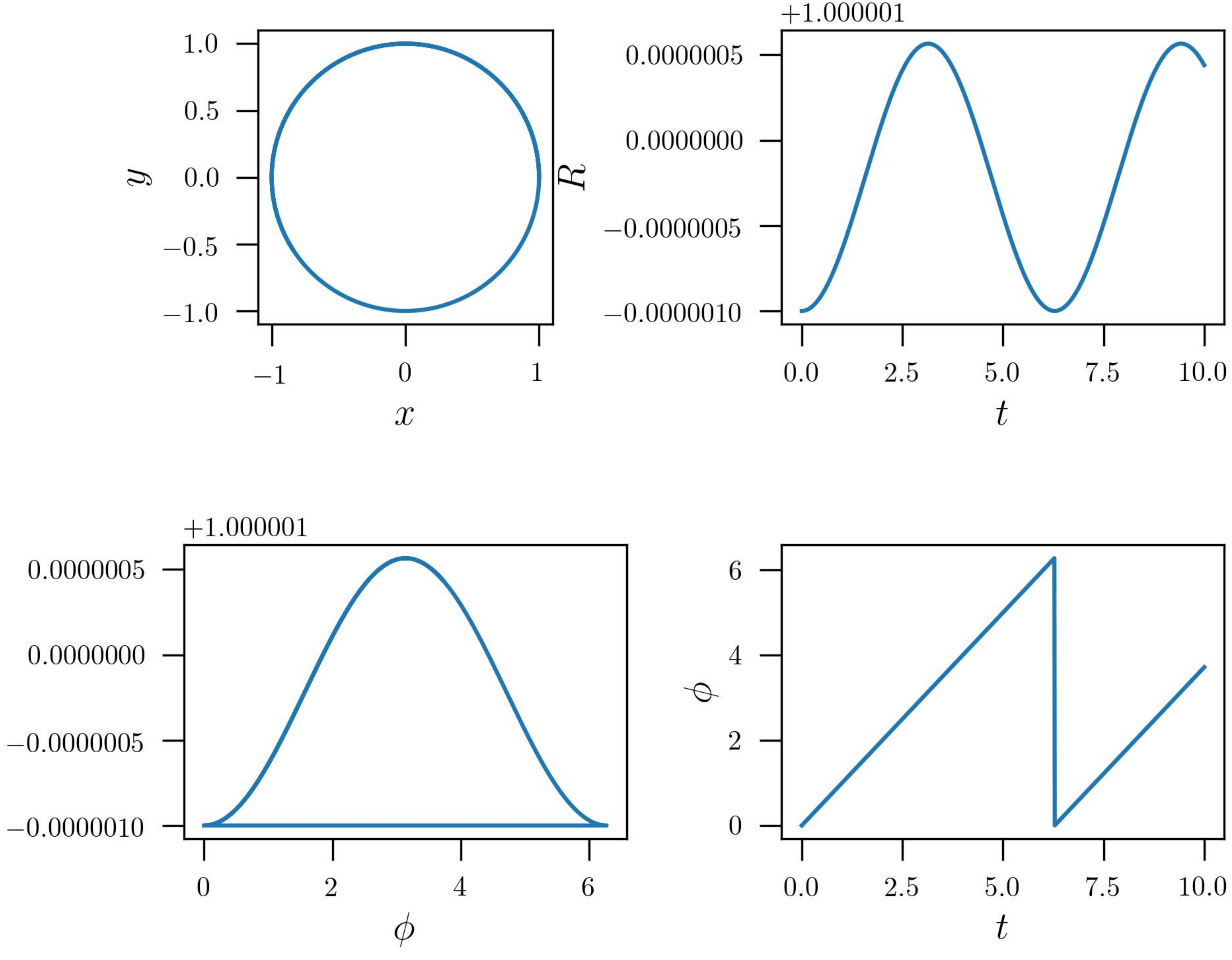
Keplerian orbits (point mass)



Keplerian orbits (point mass)



Keplerian orbits (point mass)



② Homogeneous sphere ρ_0, R_0 (Harmonic oscillations)

$$\phi(r) = \underbrace{-2\pi G \rho_0 R_0^2}_{\text{cte} \to 0} + \frac{2}{3}\pi G \rho_0 r^2$$

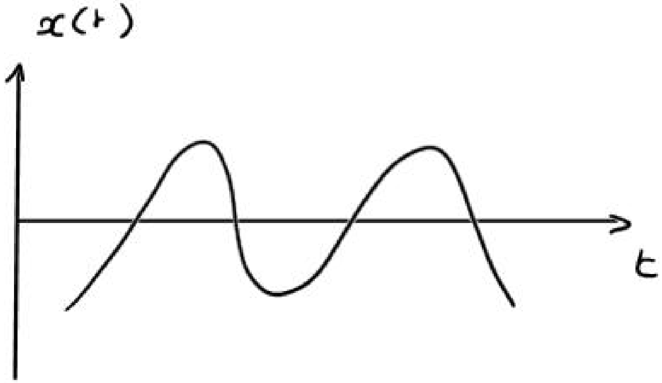
$$\phi(r) = \frac{1}{2}\Omega^2 r^2 \qquad \text{with } \Omega = \sqrt{\frac{4}{3}\pi G \rho_0}$$

Equations of motion (in carthesian coordinates)

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{2}\Omega^2(x^2 + y^2)$$

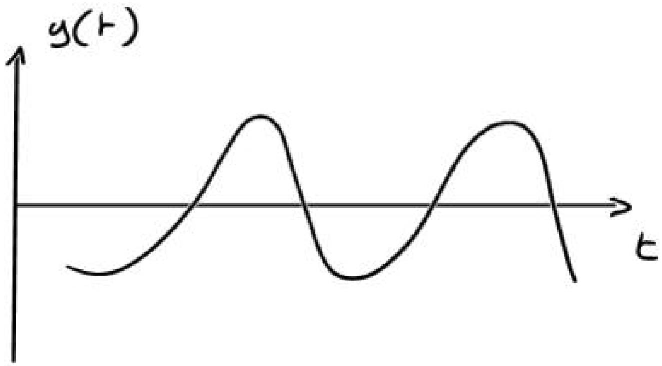
$$\begin{cases} \ddot{x} = -\Omega^2 x \\ \ddot{y} = -\Omega^2 y \end{cases} \qquad \begin{cases} x(t) = X\cos(\Omega t + \varepsilon_x) \\ y(t) = Y\cos(\Omega t + \varepsilon_y) \end{cases}$$

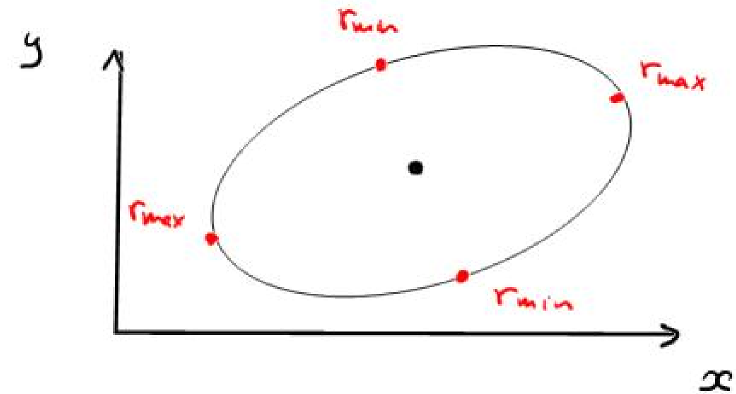
$X, Y, \varepsilon_x, \varepsilon_y$ constants fixed by the initial conditions



same period

$\Rightarrow$ closed orbits (ellipse)

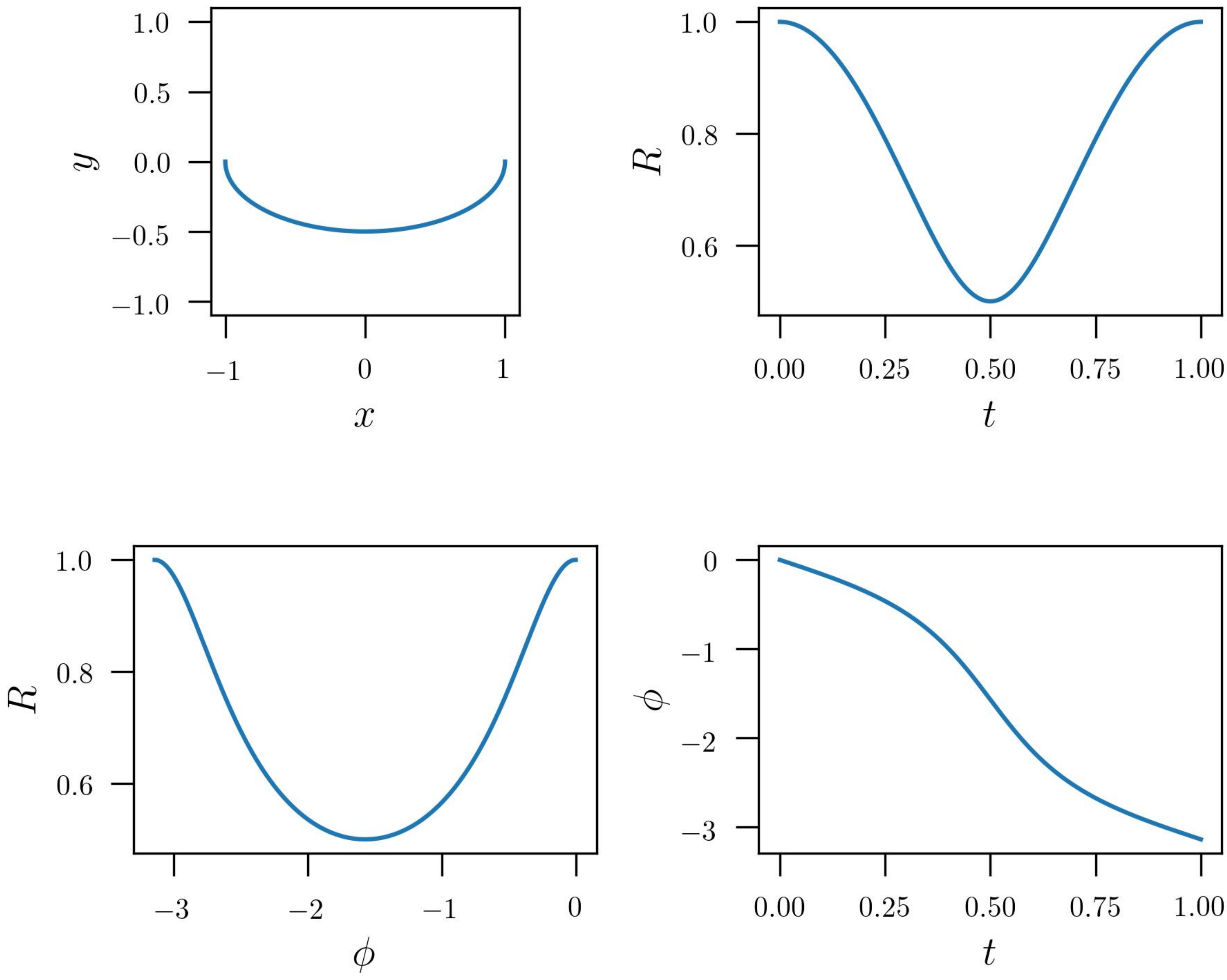




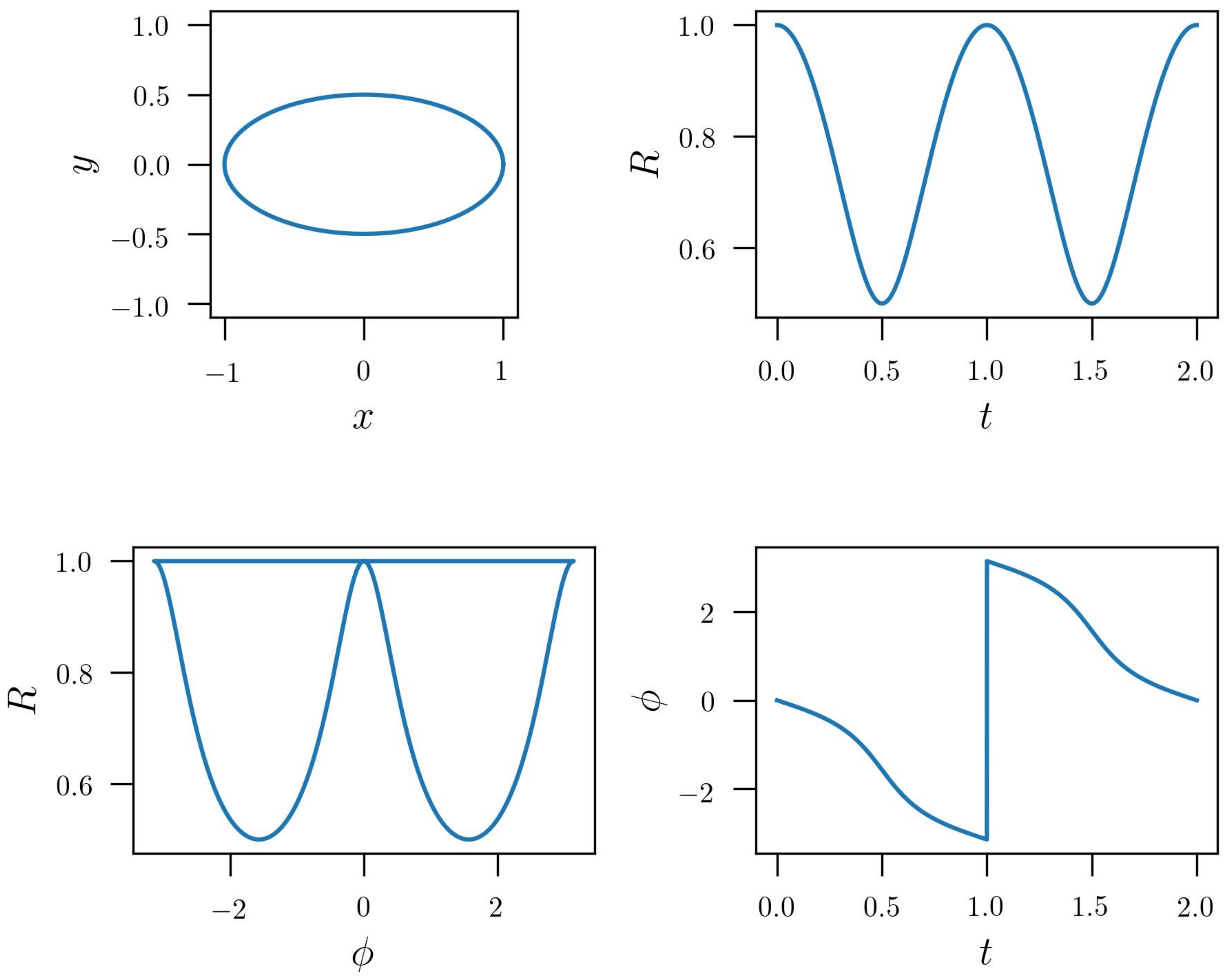
Periods

$$T_\varphi = \frac{2\pi}{\Omega} \qquad T_r = \frac{1}{2} T_\varphi = \frac{\pi}{\Omega}$$

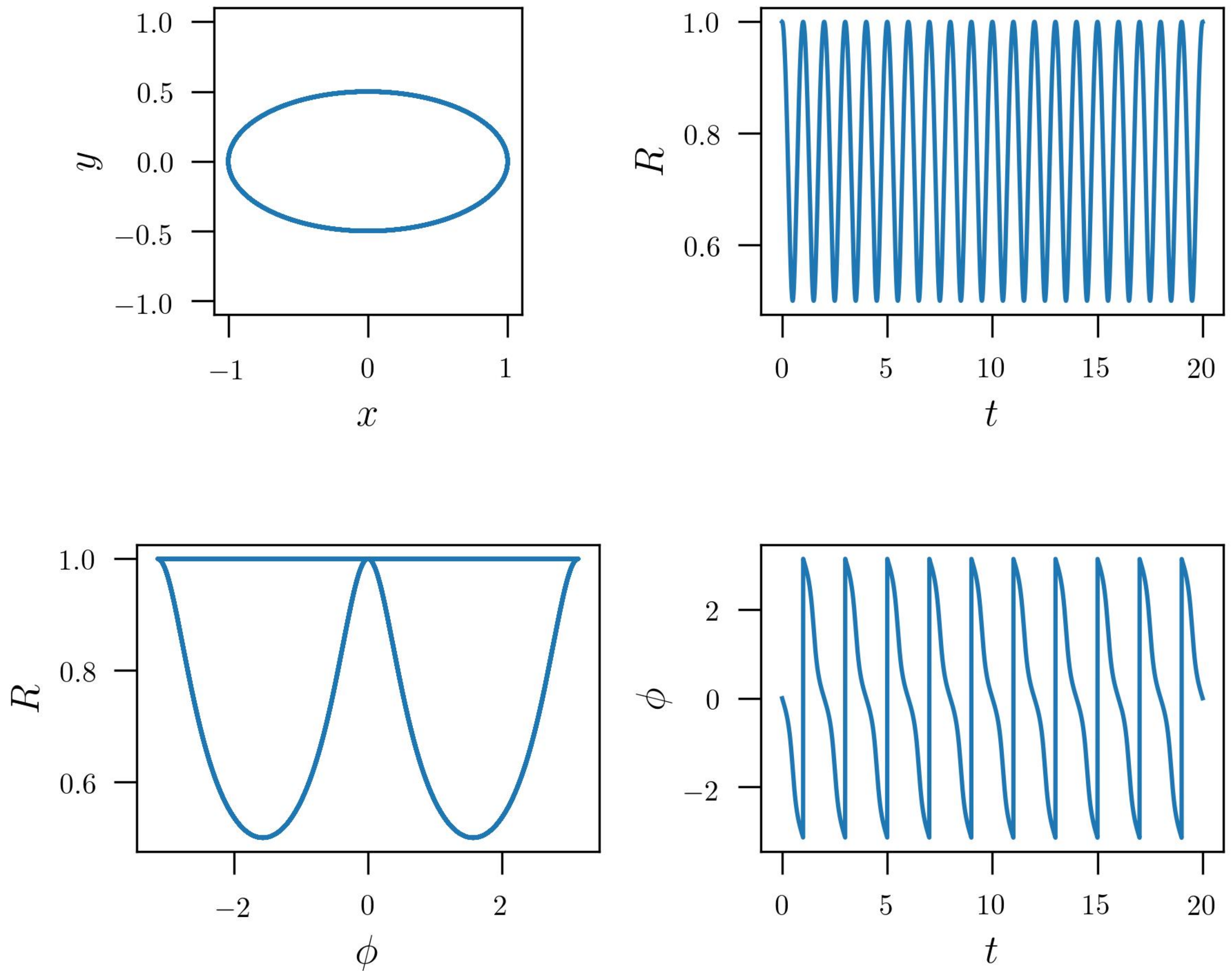
Homegeneous sphere (harmonic)



Homegeneous sphere (harmonic)



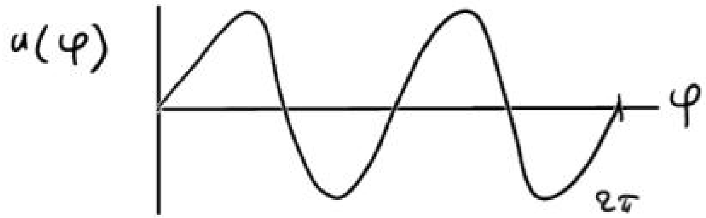
Homegeneous sphere (harmonic)

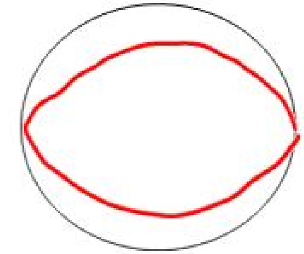


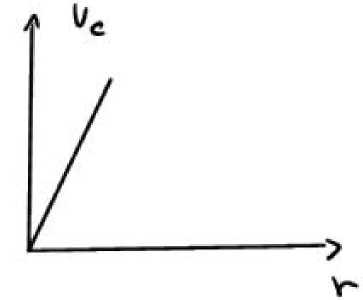
Important Remarks

Homogeneous sphere

$$\boxed{T_r = \frac{1}{2} T_\varphi}$$







Important Remarks

Homogeneous sphere

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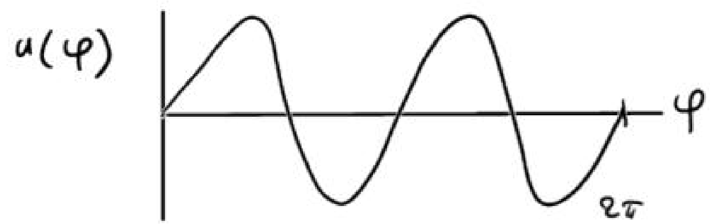
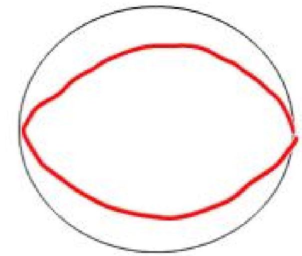
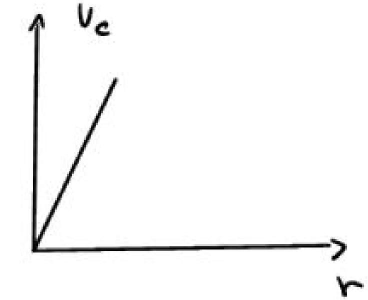
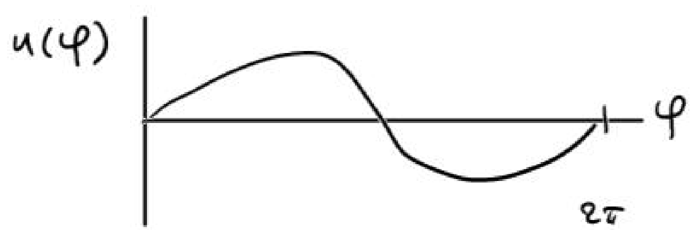


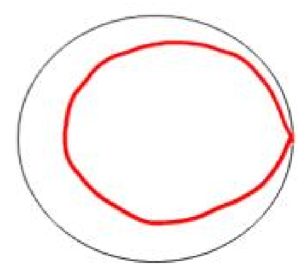


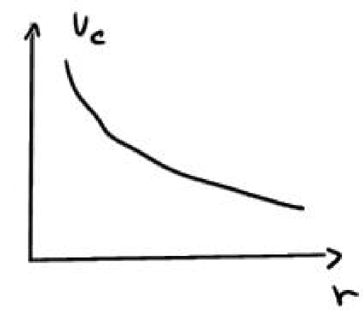


Keplerian potential

$$\boxed{T_r = T_\varphi}$$



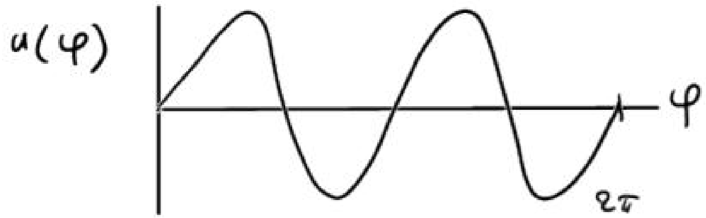


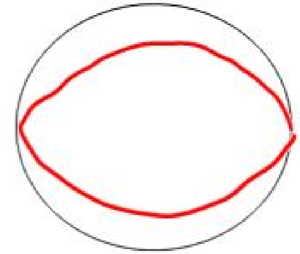


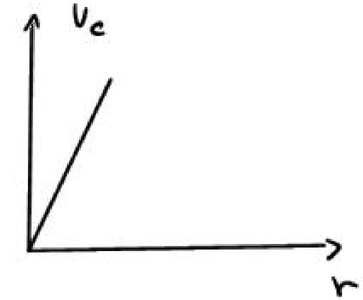
Important Remarks

Homogeneous sphere

$$\boxed{T_r = \frac{1}{2} T_\varphi}$$

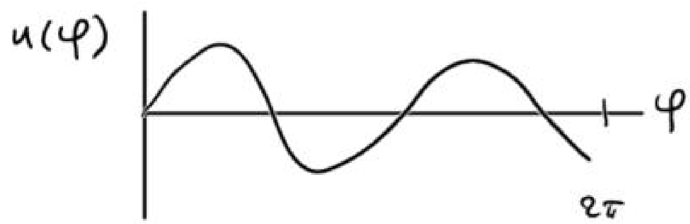


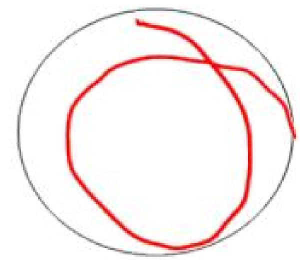


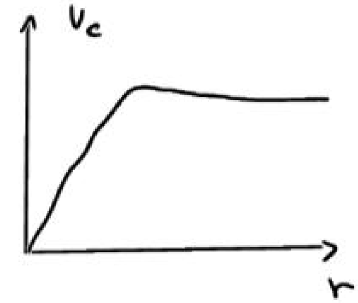


Galaxy

$$\boxed{\frac{1}{2} T_\varphi < T_r < T_\varphi}$$

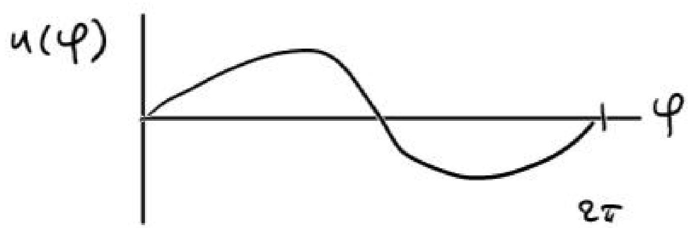


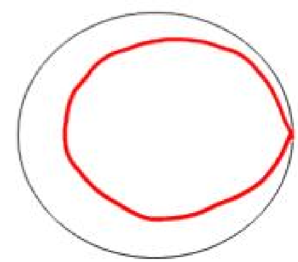


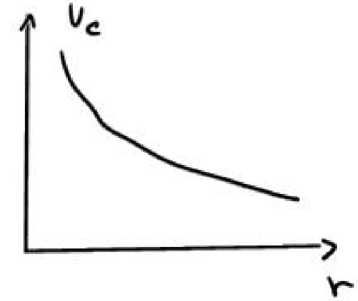


Keplerian potential

$$\boxed{T_r = T_\varphi}$$





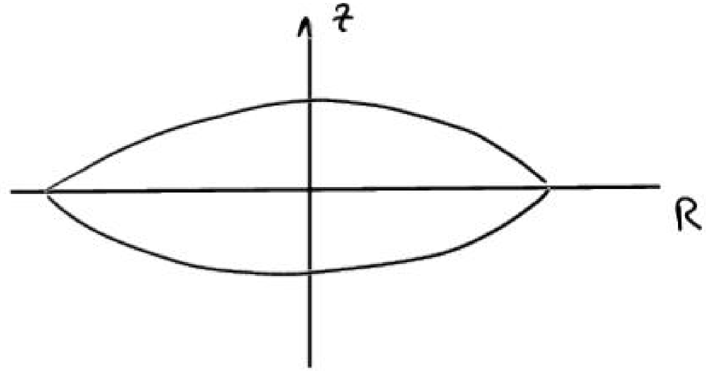


Axisymmetric Systems

Orbits in axisymmetric potentials

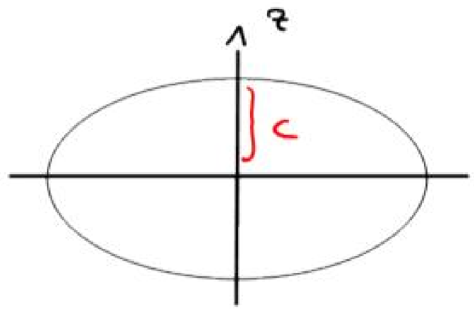
Axisymmetric potential

$$\phi(\vec{x}) = \phi(R, |z|)$$

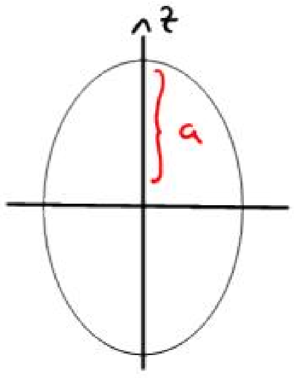


- symmetry of revolution around z
- reflection symmetry with respect to the $z = 0$ plane

Definitions



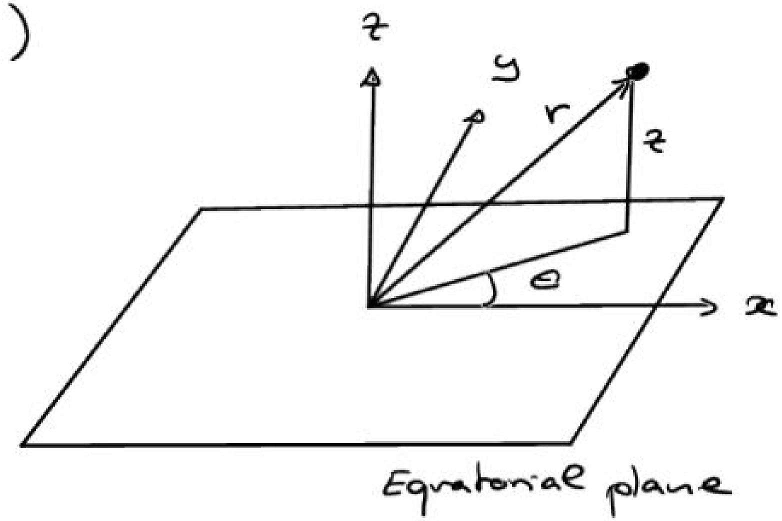
Oblate systems : c, the semi-minor axis is parallel to $\vec{z}$



Prolate systems : a, the semi-major axis is parallel to $\vec{z}$

Description of the dynamics

Cylindrical coordinates (R, θ, z)



Orbits in the equatorial plane

$\forall t, \; \underline{z = 0}$

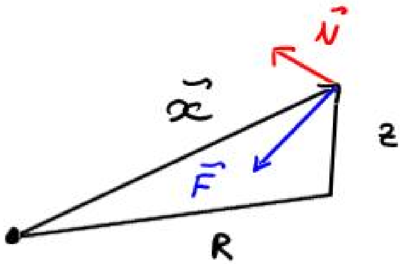
$$\boxed{\phi(R, |z|=0) \equiv \phi(R)}$$

The potential seen by the stars is similar to a spherical potential

- description of the orbits in polar coordinates r, φ
- recycle all results developped for spherical potentials

Tork and angular momentum

$$\begin{cases} \dot{\vec{e}}_R = \dot{\theta}\,\vec{e}_\theta \\ \dot{\vec{e}}_\theta = -\dot{\theta}\,\vec{e}_R \\ \dot{\vec{e}}_z = 0 \end{cases}$$



$$\begin{cases} \vec{x} = R\,\vec{e}_R + z\,\vec{e}_z \\ \vec{v} = \dot{R}\,\vec{e}_R + R\dot{\theta}\,\vec{e}_\theta + \dot{z}\,\vec{e}_z \\ \vec{F} = -\vec{\nabla}\phi = -\frac{\partial}{\partial R}\phi\,\vec{e}_R - \cancel{\frac{1}{R}\frac{\partial}{\partial \theta}\phi}\,\vec{e}_\theta - \frac{\partial}{\partial z}\phi\,\vec{e}_z \end{cases}$$

Tork

$$\vec{N} = \vec{x} \times \vec{F} = \left(z\frac{\partial}{\partial R}\phi - R\frac{\partial}{\partial z}\phi \right) \vec{e}_\theta$$

Angular momentum

$$\vec{L} = L_R\,\vec{e}_R + L_\theta\,\vec{e}_\theta + L_z\,\vec{e}_z$$

$$\frac{d\vec{L}}{dt} = \left(\dot{L}_R - L_\theta\dot{\theta} \right)\vec{e}_R + \left(\dot{L}_\theta + L_R\dot{\theta} \right)\vec{e}_\theta + \dot{L}_z\,\vec{e}_z$$

with $\boxed{\dfrac{d\vec{L}}{dt} = \vec{N}}$

$$\vec{N} = \left(z \frac{\partial}{\partial R} \phi - R \frac{\partial}{\partial z} \phi \right) \vec{e}_\theta$$

$$\frac{d\vec{L}}{dt} = \left(\dot{L}_R - L_\theta \dot{\theta} \right) \vec{e}_R + \left(\dot{L}_\theta + L_R \dot{\theta} \right) \vec{e}_\theta + \dot{L}_z \vec{e}_z$$

We get

$$\begin{cases} \dot{L}_R - L_\theta \dot{\theta} = 0 & (1) \\ \dot{L}_z = 0 & (3) \end{cases}$$

$$\boxed{L_z = cte}$$

With $\vec{L} = \vec{x} \times \vec{v}$

$$\begin{cases} L_R = -zR \\ L_\theta = z\dot{R} - R\dot{z} \\ L_z = R^2 \dot{\theta} \end{cases}$$

$(1) \Rightarrow R^2 \dot{\theta} = cte$

$$\boxed{L_z = R^2 \dot{\theta} = cte}$$

$(3) \Rightarrow R^2 \dot{\theta} = cte$

The z-component of the angular momentum is conserved

Orbits that moves outside the equatorial plane

Cylindrical coordinates

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{cases} \qquad \begin{aligned} \dot{x} &= \dot{R}\cos\theta - R\sin\theta\,\dot{\theta} \\ \dot{y} &= \dot{R}\sin\theta + R\cos\theta\,\dot{\theta} \\ \dot{z} &= \dot{z} \end{aligned}$$

$$\underline{\dot{x}^2 + \dot{y}^2 = \dot{R}^2 + R^2\dot{\theta}^2}$$

Lagrangian (specific) in cylindrical coordinates

$$\mathcal{L} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \phi\left(\sqrt{x^2+y^2}\right) = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2\right) - \phi(R,z)$$

Lagrange equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = 0}$$

$$\vec{q} = \begin{cases} R \\ \theta \\ z \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{R} \\ \dot{\theta} \\ \dot{z} \end{cases}$$

Lagrange equations

$$\begin{cases} \ddot{R} = R\dot{\theta}^2 - \dfrac{\partial \phi}{\partial R} & (1) \\ \dfrac{d}{dt}\left(R^2\dot{\theta}\right) = \cancel{\left(-\dfrac{\partial \phi}{\partial \theta}\right)} = 0 & (2) \\ \ddot{z} = -\dfrac{\partial \phi}{\partial z} & (3) \end{cases}$$

(2) $R^2\dot{\theta} = \text{cte} = L_z$

The z-component of the angurar momentum is conserved

Solution

$$\boxed{\theta(t) = L_z \int_{t_0}^{t_1} \frac{1}{R^2(t)}\, dt}$$

(1) + (3) two coupled through $\phi(R, z)$ equations for R and z

Hamiltonian/Energy

$$\boxed{H(\vec{q},\vec{p},t) := \vec{p}\cdot\dot{\vec{q}} - \mathcal{L}(\vec{q},\dot{\vec{q}},t)}$$

$$\vec{q} = \begin{cases} R \\ \theta \\ z \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{R} \\ \dot{\theta} \\ \dot{z} \end{cases} \qquad \vec{p} = \begin{cases} \frac{\partial \mathcal{L}}{\partial \dot{R}} = \dot{R} \\ \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = R^2\dot{\theta} \\ \frac{\partial \mathcal{L}}{\partial \dot{z}} = \dot{z} \end{cases}$$

$$\boxed{p_\theta = R^2\dot{\theta} = L_z}$$

$$\boxed{H(R,\dot{R},\theta,\dot{\theta},z,\dot{z}) = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2\right) + \phi(R,z) = E}$$

E (Energy) is conserved

as $\mathcal{L}$ is time independant

ϕ

Effective potential

with $L_z = R^2 \dot{\theta}$

$L_z^2 = R^4 \dot{\theta}^2$

Definition

$$\phi_{eff}(R, z) = \phi(R, z) + \frac{L_z^2}{2R^2}$$

$$\begin{cases} \dfrac{\partial \phi_{eff}}{\partial R} = \dfrac{\partial \phi}{\partial R} - \dfrac{L_z^2}{R^3} \equiv \dfrac{\partial \phi}{\partial R} - R\dot{\theta}^2 \\[2ex] \dfrac{\partial \phi_{eff}}{\partial z} = \dfrac{\partial \phi}{\partial z} \end{cases}$$

The equations of motion 1 + 3 becomes

$$\boxed{\begin{cases} \ddot{R} = - \dfrac{\partial \phi_{eff}}{\partial R}(R, z) \\[2ex] \ddot{z} = - \dfrac{\partial \phi_{eff}}{\partial z}(R, z) \end{cases}}$$

The 3D motion of a star in an axisymmetric potential is reduced to a 2D motion in the meridian plane (R, z)

phase space 6D → 4D

Hamiltonian in the meridian plane

Those equations of motion may be derived from the lagrangian

$$\mathcal{L}(R, \dot{R}, z, \dot{z}) = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 - \phi_{eff}(R, z)$$

The corresponding Hamiltonian writes $(p_R = \dot{R}\ ,\ p_z = \dot{z})$

$$\boxed{\begin{aligned} H(R, \dot{R}, z, \dot{z}) &= \underbrace{\frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right)}_{\text{kinetic energy in the orbital plane}} + \phi_{eff}(R, z) \\ &= \frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right) + \phi(R, z) + \frac{L_z^2}{2R^2} \\ &= \frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right) + \phi(R, z) + \frac{1}{2}R^2\dot{\theta}^2 = \underbrace{E}_{\text{orbit's total energy}} \end{aligned}}$$

E is conserved as ϕ_{eff} is time independent

Illustration in the $z = 0$ plane

for $R \to \infty$

$$\phi_{eff} = \phi + \underbrace{\frac{L_z^2}{2}}_{\to 0} \frac{1}{R^2} \sim \phi$$

for $R \to 0$

$$\phi_{eff} = \underbrace{\phi}_{\text{bounded}} + \underbrace{\frac{L_z^2}{2} \frac{1}{R^2}}_{\text{diverges}} \sim \frac{1}{R^2}$$

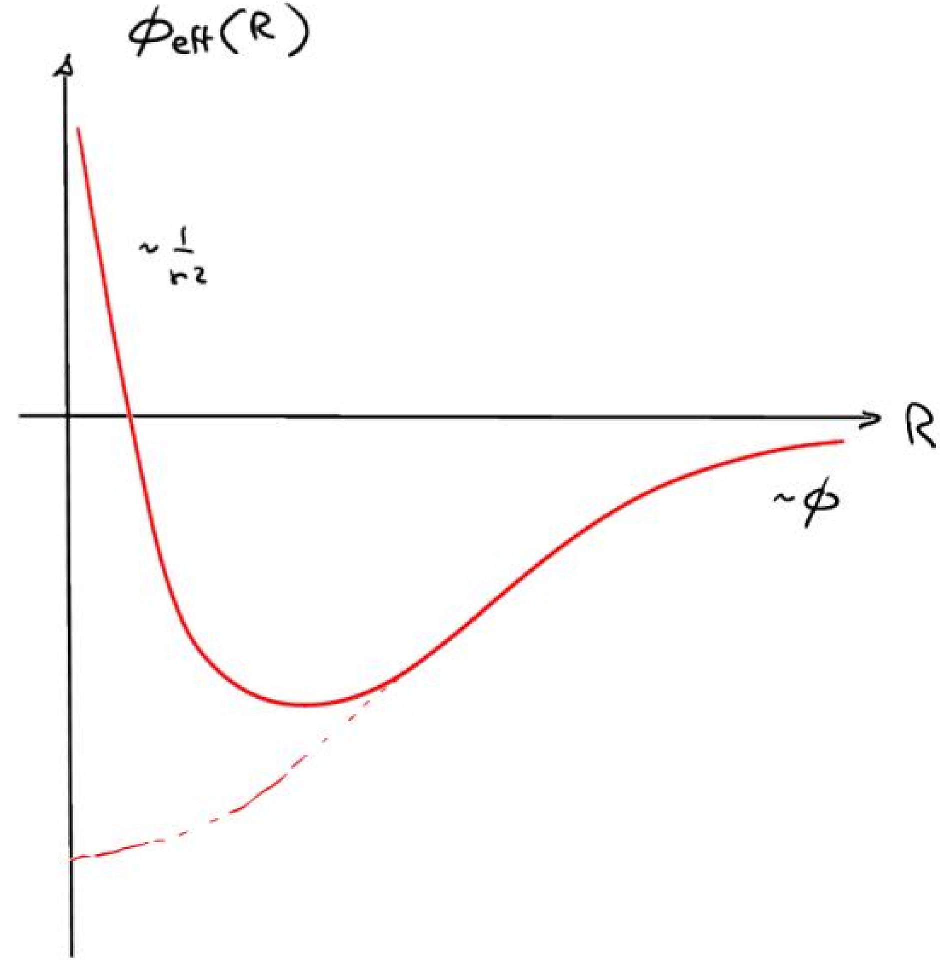


Illustration in the $z = 0$ plane

$$E = \frac{1}{2}\dot{R}^2 + \phi_{eff}(R)$$

4 cases

(1) $E > \phi_{eff}(\infty)$ except at $E = \phi_{eff}$

$\dot{R} \neq 0$ unbound orbits

(2) $E < \min(\phi_{eff}(R))$ $\dot{R}^2 < 0$

impossible

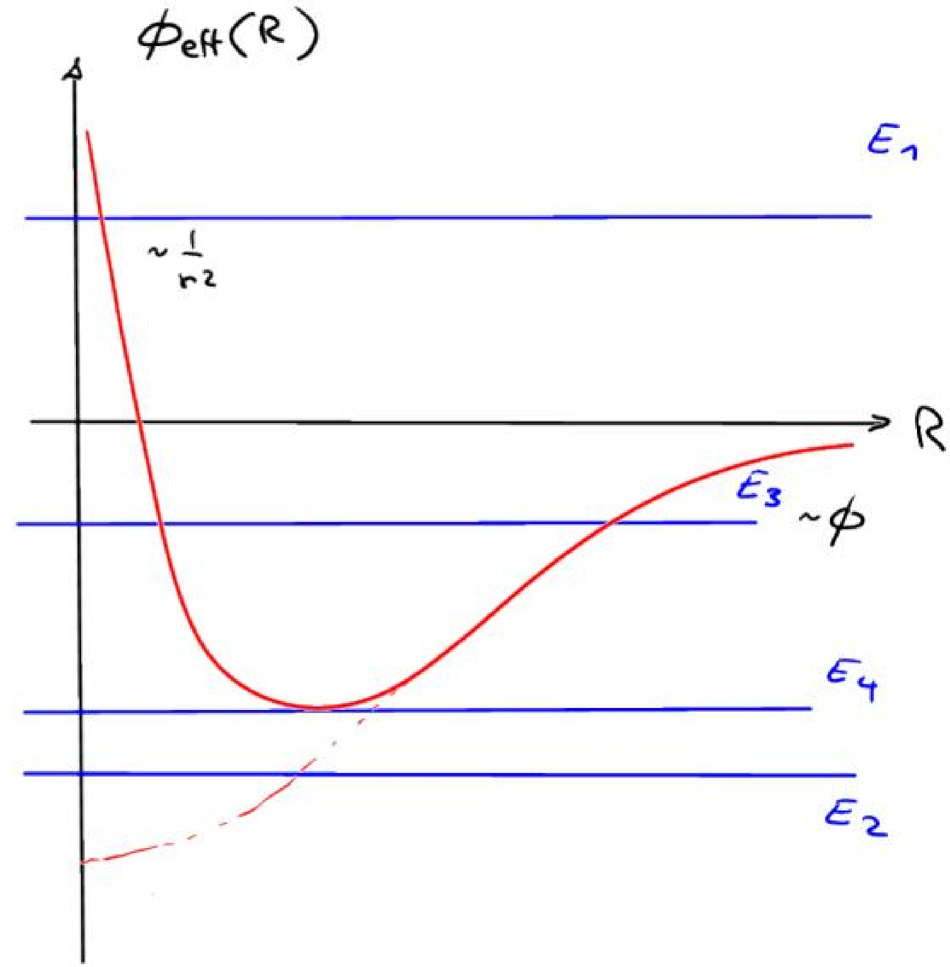
(3) $\min(\phi_{eff}(R)) < E < \phi_{eff}(\infty)$

orbit bounded between

R_1 and R_2 (where $\dot{R} = 0$)

(4) $E = \min(\phi_{eff}(R))$ (stationary point)

$R_1 = R_2$ (circular orbit)

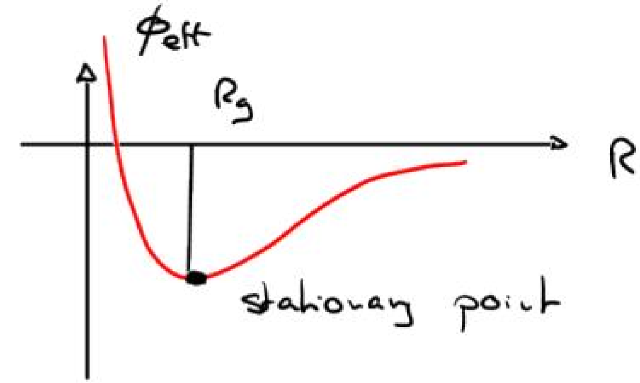


Stationary point

$\dot{R} = \ddot{R} = 0 \qquad \dot{z} = \ddot{z} = 0$

from

$$\boxed{\begin{cases} \ddot{R} = -\dfrac{\partial \phi_{eff}}{\partial R}(R,z) \\ \ddot{z} = -\dfrac{\partial \phi_{eff}}{\partial z}(R,z) \end{cases}}$$



$$\begin{cases} \dfrac{\partial \phi_{eff}}{\partial R} = 0 = \dfrac{\partial \phi}{\partial R} - \dfrac{L_z^2}{R^3} = 0 \\ \dfrac{\partial \phi_{eff}}{\partial z} = 0 = \dfrac{\partial \phi}{\partial z} = 0 \end{cases}$$

$\rightarrow$ by symmetry where $z = 0$

$V_\theta = R\dot{\theta}$

$$\boxed{R_g \text{ such that } \left.\frac{\partial \phi}{\partial R}\right)_{R_g,0} = \frac{L_z^2}{R_g^3} = R_g \dot{\theta}^2 = \frac{V_\theta^2(R_g)}{R_g} = \frac{V_c^2(R_g)}{R_g}}$$

$V_c^2 = R \left.\dfrac{\partial \phi}{\partial R}\right)_{R,0}$

R_g : guiding center

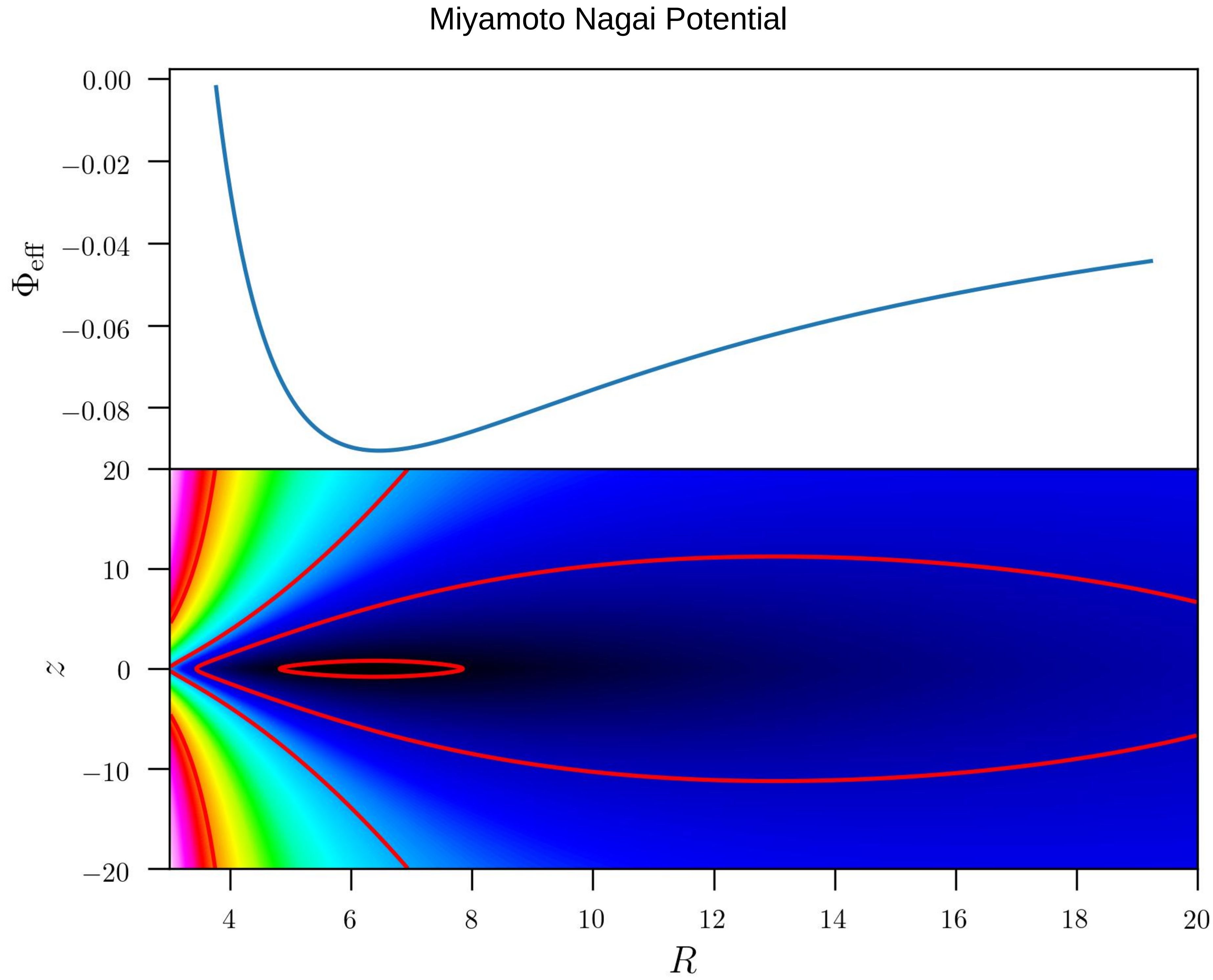
The stationary point in R_g in the meridional plane corresponds to a circular orbit

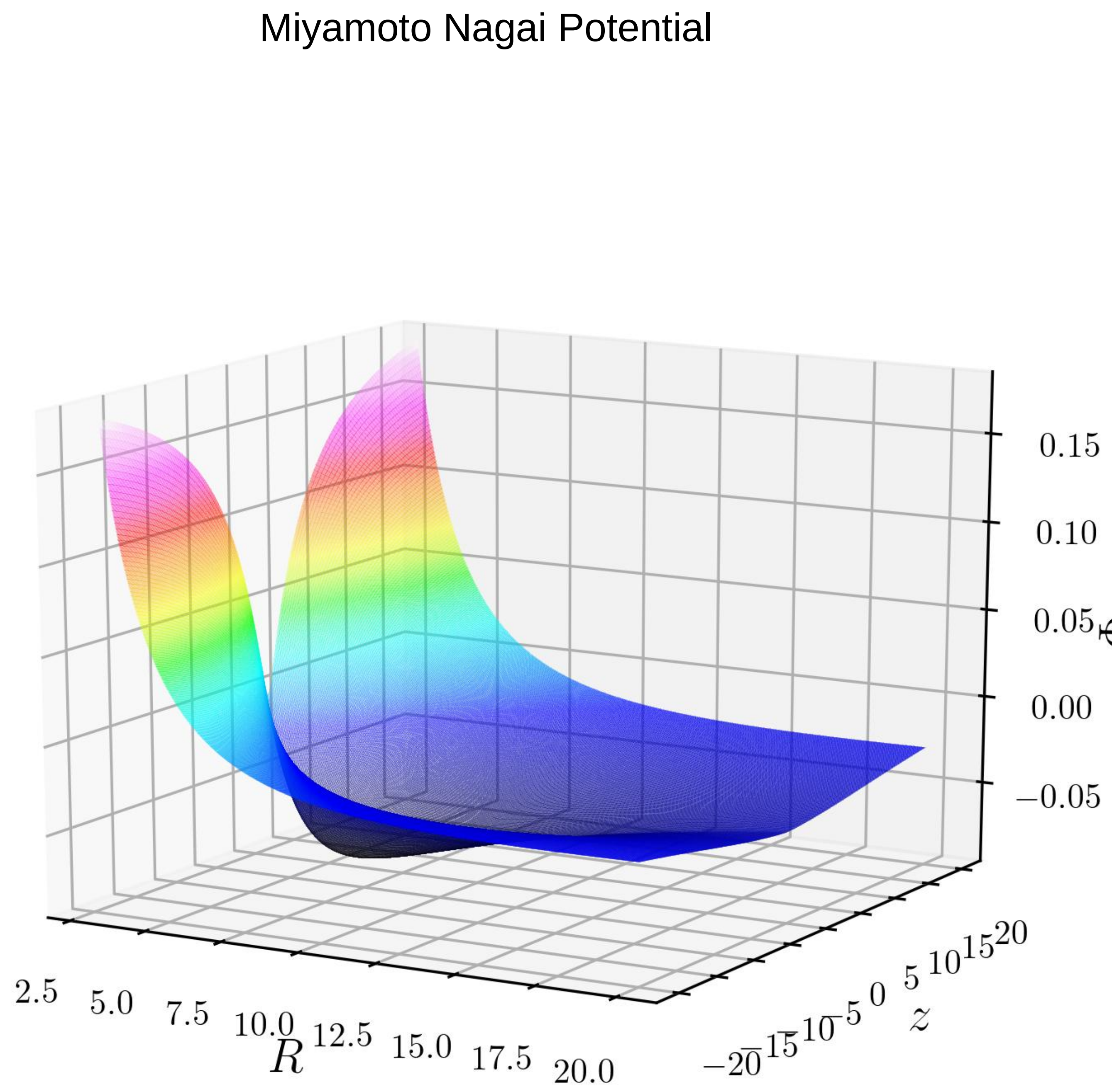
Examples

① Miyamoto - Nagai potential

$$\phi(R, z) = -\frac{GM}{\sqrt{R^2 + \left(a + \sqrt{z^2+b^2}\right)^2}}$$

$$\phi_{eff}(R, z=0) = -\frac{GM}{\sqrt{R^2 + (a+b)^2}} + \frac{L_z^2}{2R^2}$$


Miyamoto Nagai Potential
Φ_{eff}
0.00
−0.02
−0.04
−0.06
−0.08
z
20
10
0
−10
−20
4
6
8
10
12
14
16
18
20
R


Miyamoto Nagai Potential
0.15
0.10
0.05
0.00
−0.05
$\Phi_{\rm eff}$
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
R
−20
−15
−10
−5
0
5
10
15
20
z

Examples

① Miyamoto - Nagai potential

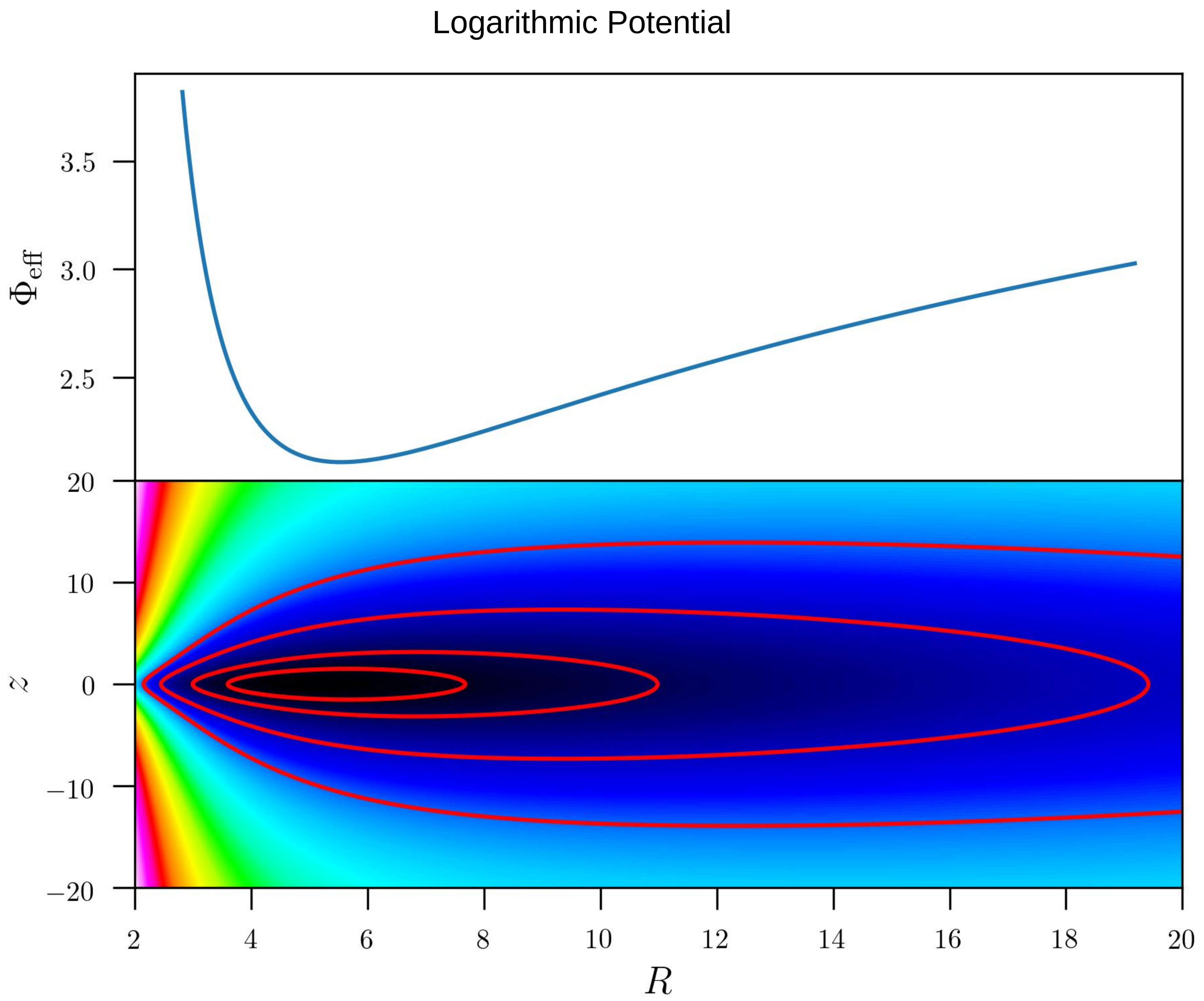
$$\phi(R, z) = - \frac{GM}{\sqrt{R^2 + \left(a + \sqrt{z^2+b^2}\right)^2}}$$

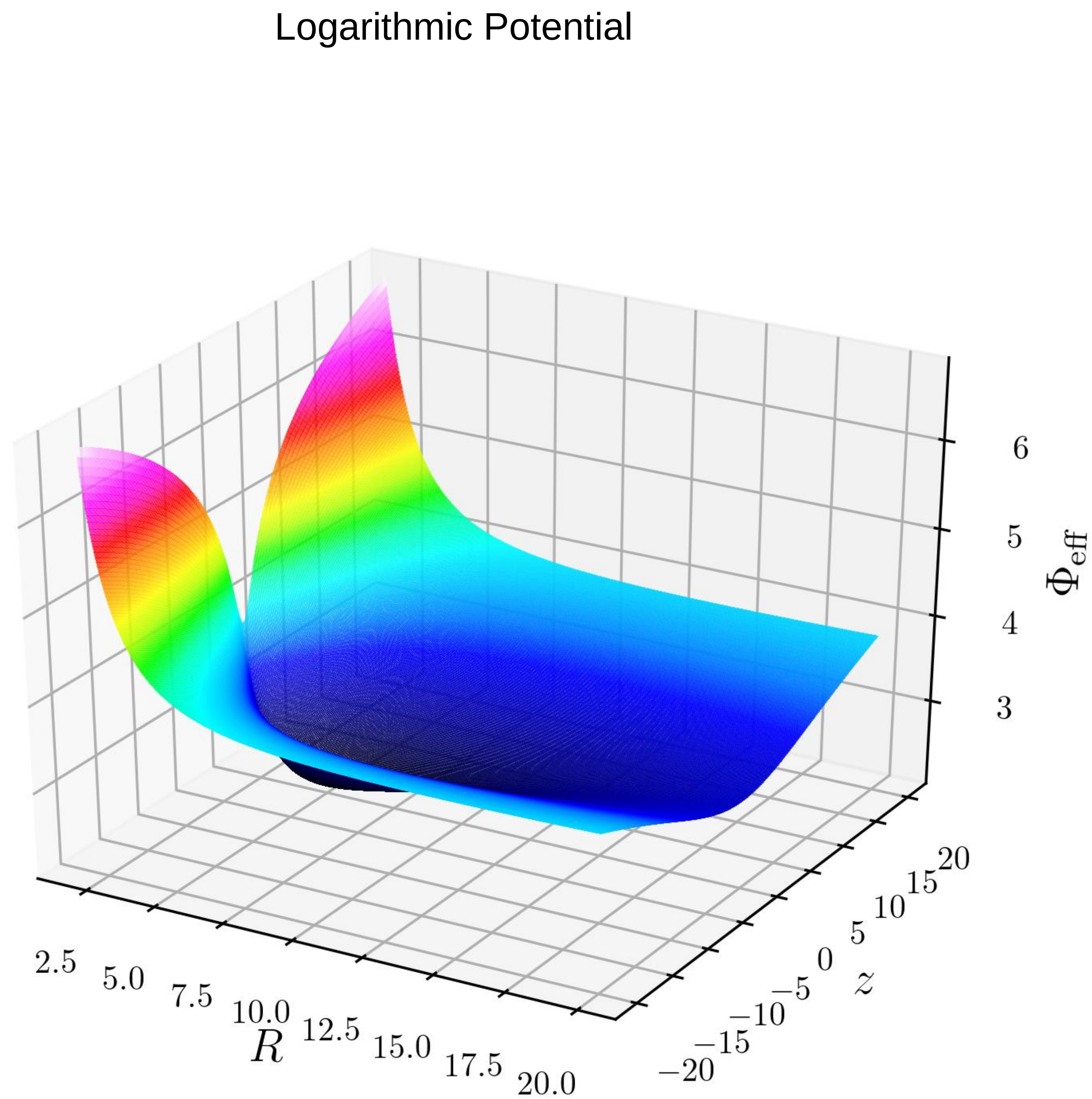
$$\phi_{eff}(R, z=0) = - \frac{GM}{\sqrt{R^2 + (a+b)^2}} + \frac{L_z^2}{2R^2}$$

② Logarithmic potential

$$\phi(R, z) = \frac{1}{2} V_0^2 \ln\left(R^2 + \frac{z^2}{q^2} \right)$$

$$\phi_{eff}(R, z=0) = \frac{1}{2} V_0^2 \ln\left(R^2 \right) + \frac{L_z^2}{2R^2}$$


Logarithmic Potential
$\Phi_{\rm eff}$
3.5
3.0
2.5
20
10
0
−10
−20
z
2
4
6
8
10
12
14
16
18
20
R


Logarithmic Potential
R
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
z
−20
−15
−10
−5
0
5
10
15
20
$\Phi_{\rm eff}$
3
4
5
6

Circular orbits

angular speed $\dot{\theta} = \frac{L_z}{R_g^2}$

angular momentum L_z

energy $\phi_{eff} + \frac{L_z^2}{2R_g}$

Note

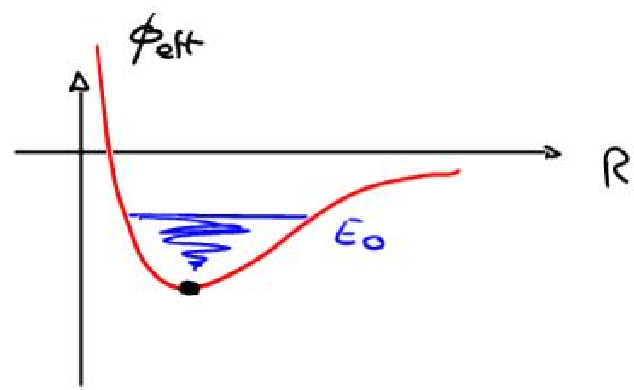
> For a given angular momentum L_z, the circular orbit is the one that minimize the energy.

① $E_0 = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \phi + \frac{L_z^2}{2R}$

② Dissipate energy $\rightsquigarrow$ $L_z = cte$

$\dot{z} \searrow \quad \dot{R} \searrow$

③ circular orbit



General solutions for the equations of motion

$$\boxed{\begin{cases} \ddot{R} = -\dfrac{\partial \phi_{eff}}{\partial R}(R,z) \\ \ddot{z} = -\dfrac{\partial \phi_{eff}}{\partial z}(R,z) \end{cases}}$$

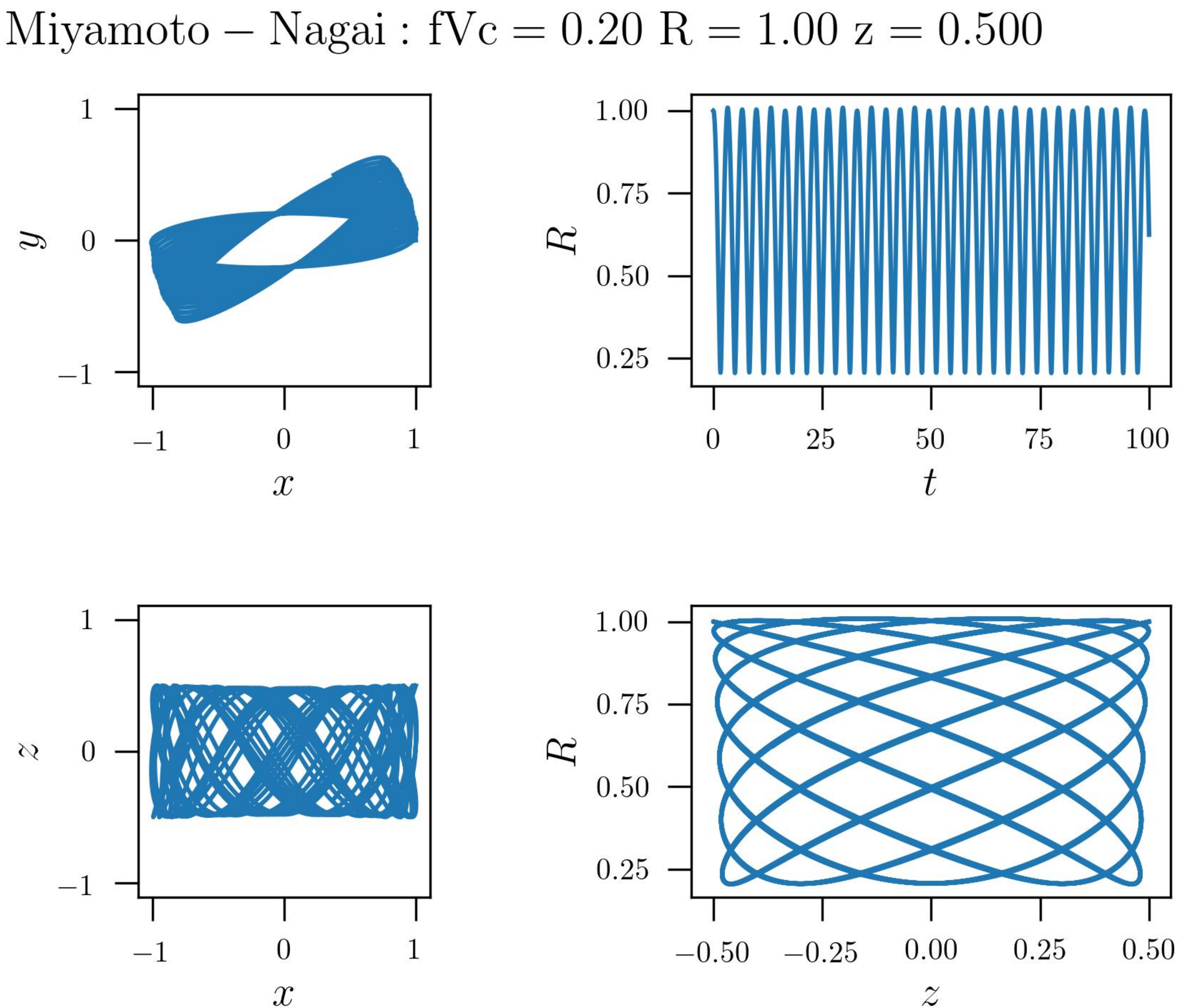
no simple solutions ☹

need numerical integration

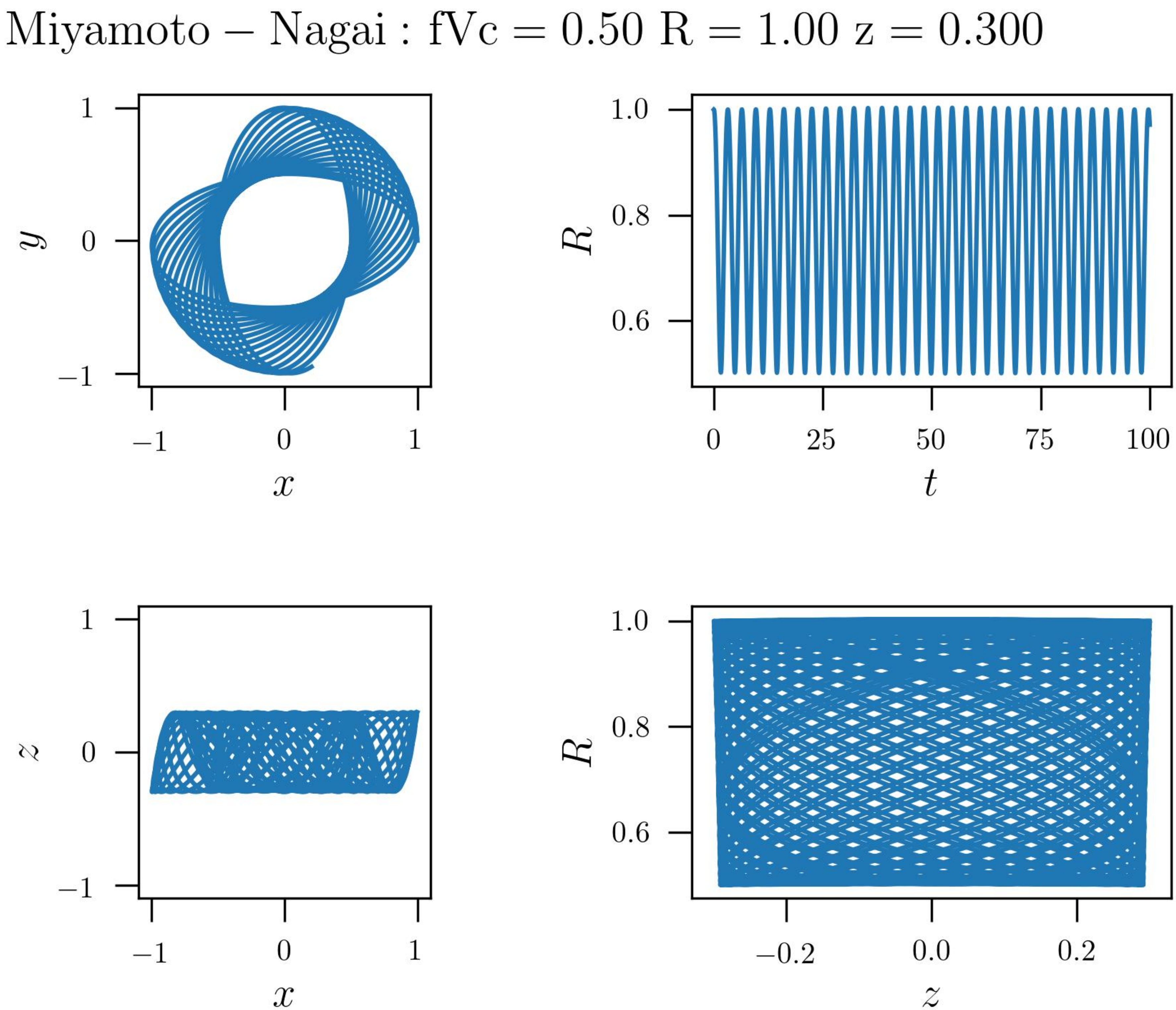
Hamilton's Equations

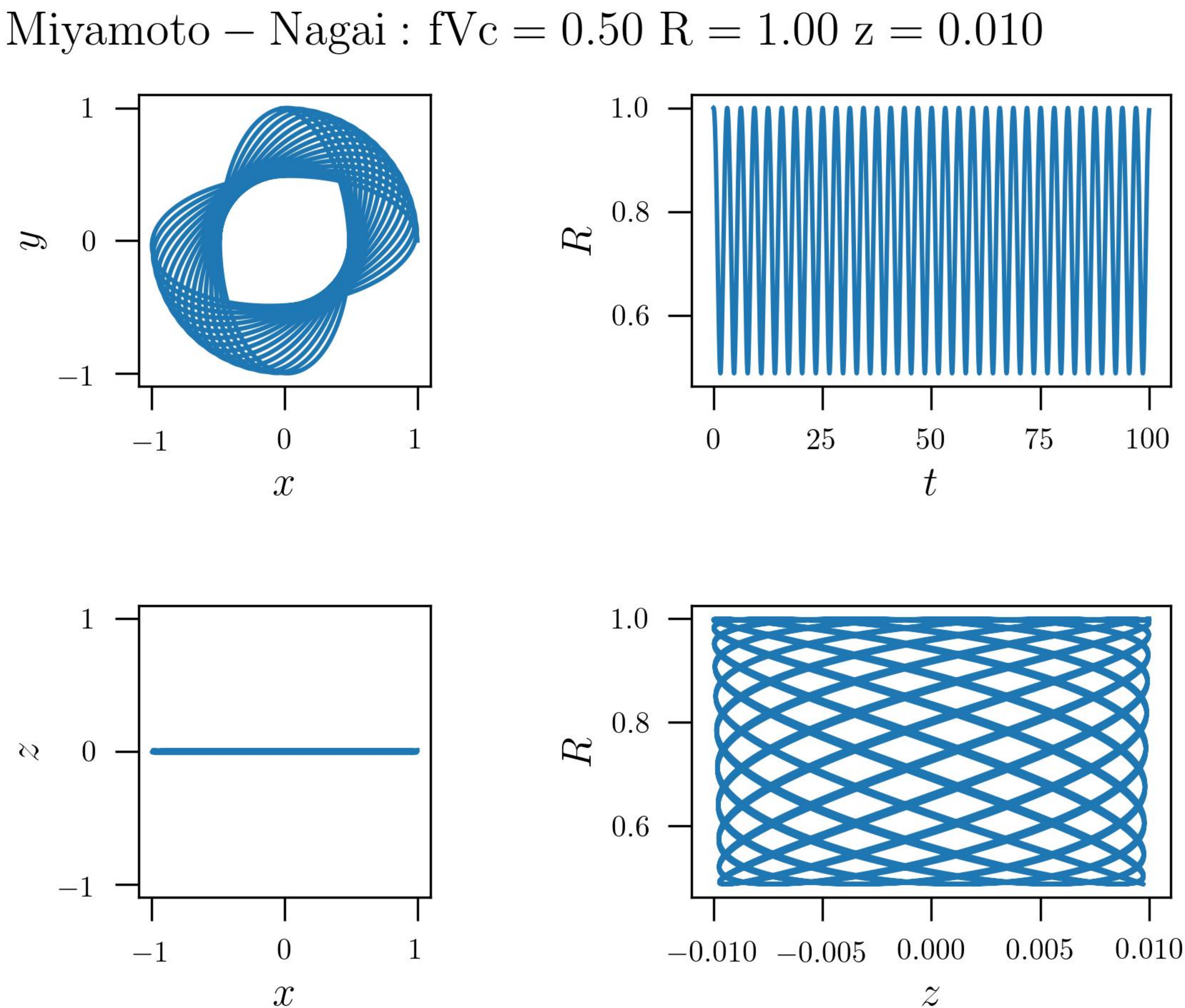
$$\vec{q} = \begin{cases} R \\ z \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{R} \\ \dot{z} \end{cases} \qquad \bar{p} = \begin{cases} \dot{R} \\ \dot{z} \end{cases}$$

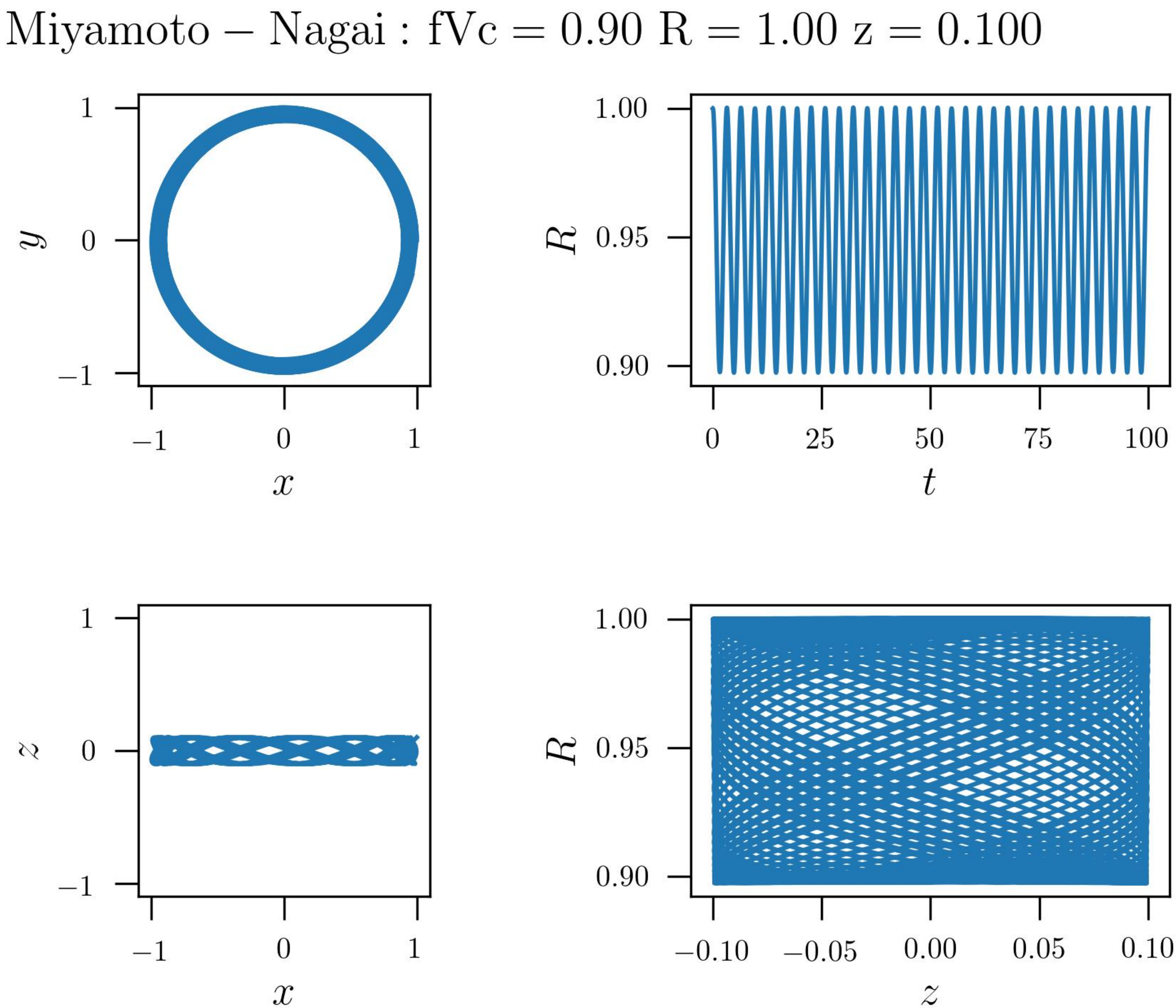
$$\begin{cases} \dot{q}_R = p_R & \equiv \dot{R} \\ \dot{q}_z = p_z & \equiv \dot{z} \\ \dot{p}_R = -\dfrac{\partial \phi_{eff}}{\partial q_R}(q_R, q_z) & \equiv -\dfrac{\partial \phi_{eff}}{\partial R}(R,z) \\ \dot{p}_z = -\dfrac{\partial \phi_{eff}}{\partial q_z}(q_R, q_z) & \equiv -\dfrac{\partial \phi_{eff}}{\partial z}(R,z) \end{cases}$$



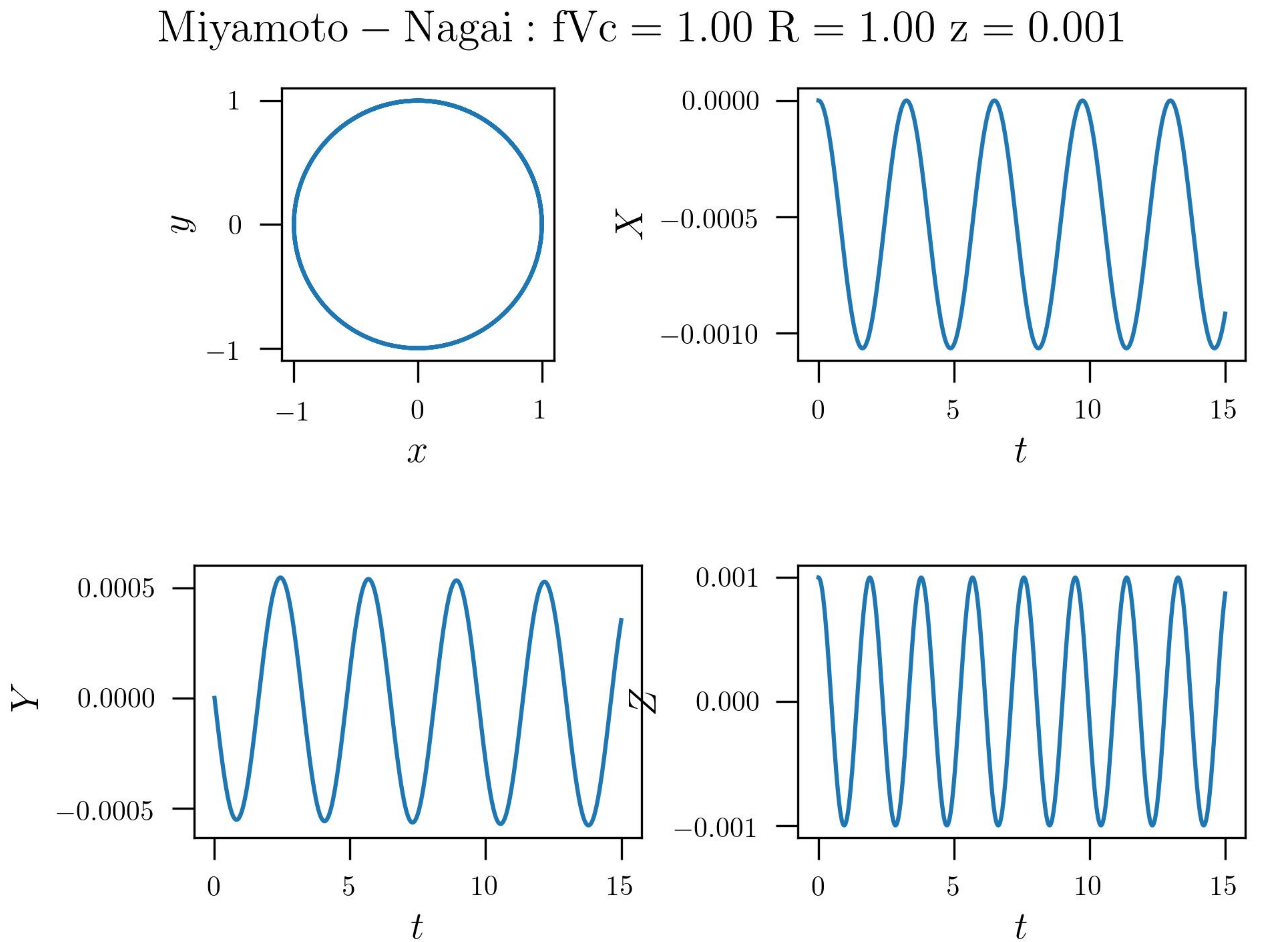
Miyamoto – Nagai : fVc = 0.20 R = 1.00 z = 0.500
y
x
R
t
z
x
R
z
1
0
−1
1.00
0.75
0.50
0.25
0
25
50
75
100
−0.50
−0.25
0.00
0.25
0.50


Miyamoto – Nagai : fVc = 0.50 R = 1.00 z = 0.300
y
x
R
t
z
x
R
z


Miyamoto – Nagai : fVc = 0.50 R = 1.00 z = 0.010
y
x
R
t
z
x
R
z
1
0
−1
1.0
0.8
0.6
25
50
75
100
−0.010
−0.005
0.000
0.005
0.010



Miyamoto – Nagai : fVc = 0.90 R = 1.00 z = 0.100
y
x
R
t
z
x
R
z

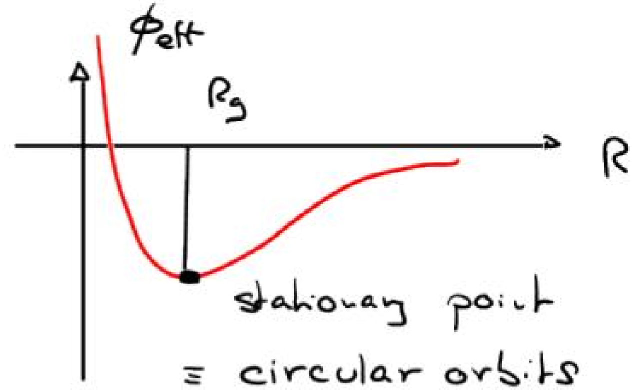


Miyamoto – Nagai : fVc = 1.00 R = 1.00 z = 0.001
y
x
X
t
Y
t
Z
t

Nearly circular orbits

Nearly circular orbits

From the previous study of orbits in axisymmetric potentials



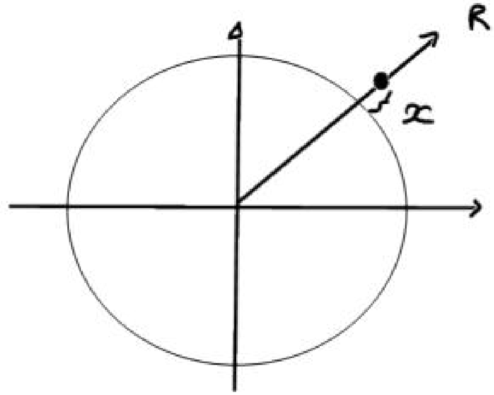
Goal Study orbits in the neighbourhood of circular orbits

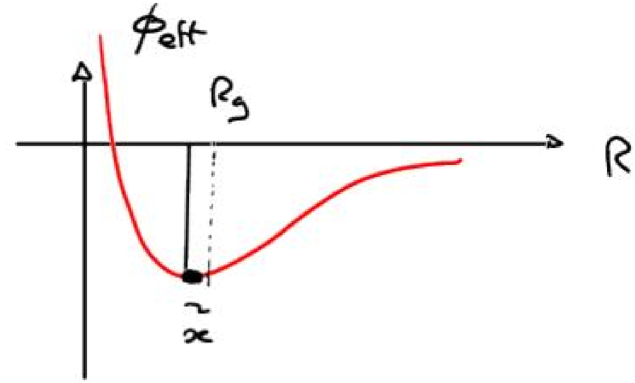
Justifications In a disk galaxy, many stars are found in nearly circular orbits

Recall R_g : the guiding center

$$\boxed{R_g \quad \text{such that} \quad \left.\frac{\partial \phi}{\partial R}\right)_{R_g,0} = \frac{L_z^2}{R_g^3} = R_g \dot{\theta}^2}$$

We define $\quad x := R - R_g \quad$ the distance to the guiding center R_g





Taylor expansion of ϕ_{eff} around $R = R_g$, $z = 0$

$$\phi_{eff}(R,z) \cong \phi_{eff}(R_g,0) + \underbrace{\frac{\partial \phi_{eff}}{\partial R}(R_g,0)}_{=0 \text{ min}} (R - R_g) + \underbrace{\frac{\partial \phi_{eff}}{\partial z}(R_g,0)}_{=0 \text{ sym.}} z$$

$$+ \frac{1}{2} \frac{\partial^2 \phi_{eff}}{\partial R^2}(R_g,0) (R-R_g)^2 + \frac{1}{2} \frac{\partial^2 \phi_{eff}}{\partial z^2}(R_g,0) z^2$$

$$+ \frac{1}{2} \underbrace{\frac{\partial^2 \phi_{eff}}{\partial z \partial R}(R_g,0)}_{=0} (R-R_g) z + O(((R-R_g)z)^3)$$

$= 0 \quad \phi_{eff}(R,z)$ must be sym. with respect to $z = 0$

$$\boxed{\phi_{eff}(R,z) \simeq \phi_{eff}(R_g,0) + \frac{1}{2}\frac{\partial^2\phi_{eff}}{\partial R^2}(R_g,0)\,x^2 + \frac{1}{2}\frac{\partial^2\phi_{eff}}{\partial z^2}(R_g,0)\,z^2}$$

<u>Definition</u>

$$\begin{cases} \kappa^2(R_g) = \left(\dfrac{\partial^2\phi_{eff}}{\partial R^2}\right)_{(R_g,0)} \\[2ex] \nu^2(R_g) = \left(\dfrac{\partial^2\phi_{eff}}{\partial z^2}\right)_{(R_g,0)} \end{cases}$$

$$[\phi] = \left(\frac{m}{s}\right)^2$$

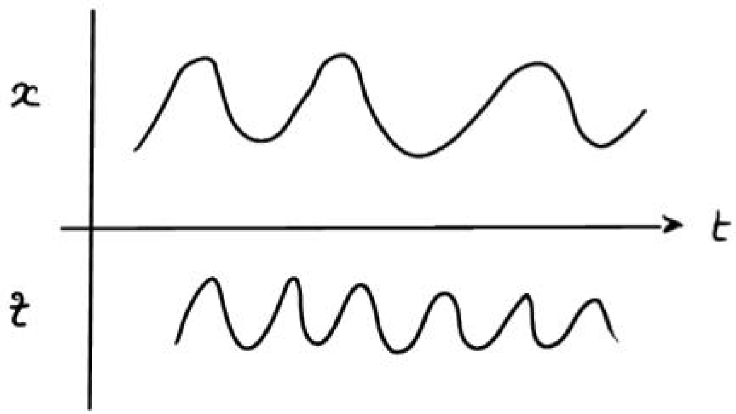
$$\left[\left(\frac{\partial^2\phi}{\partial R^2}\right)^{\frac{1}{2}}\right] \equiv \left[\left(\frac{\partial^2\phi}{\partial z^2}\right)^{\frac{1}{2}}\right] = \frac{1}{s}$$

frequency

<u>Equations of motion near R_g</u>

$$\begin{cases} \ddot{R} = -\dfrac{\partial\phi_{eff}}{\partial R}(R,z) \\[2ex] \ddot{z} = -\dfrac{\partial\phi_{eff}}{\partial z}(R,z) \end{cases} \Rightarrow \boxed{\begin{cases} \ddot{x} = -\kappa^2(R_g)\,x \\[2ex] \ddot{z} = -\nu^2(R_g)\,z \end{cases}}$$

$$\begin{cases} \ddot{x} = -\kappa^2(R_g)\, x \\ \ddot{z} = -\nu^2(R_g)\, z \end{cases}$$



Two decoupled harmonic oscillators with frequencies κ and ν

κ : epicycle (radial) frequency

ν : vertical frequency

Expressions of κ and ν from the total potential

$$\kappa^2(R_g) = \left.\frac{\partial^2 \phi_{eff}}{\partial R^2}\right)_{(R_g,0)} = \left.\frac{\partial^2 \phi}{\partial R^2}\right)_{(R_g,0)} + 3\,\frac{L_z^2}{R_g^4}$$

$$L_z^2 = V_c^2 R_g^2 = R_g^3 \left.\frac{\partial \phi}{\partial R}\right)_{R_g}$$

$$= \left.\frac{\partial^2 \phi}{\partial R^2}\right)_{(R_g,0)} + \frac{3}{R_g}\left.\frac{\partial \phi}{\partial R}\right)_{(R_g,0)}$$

circ. frequency $\Omega^2 = \frac{1}{R_g}\left.\frac{\partial \phi}{\partial R}\right)_{R_g}$

$$= \left.\frac{\partial^2 \phi}{\partial R^2}\right)_{(R_g,0)} + 3\,\Omega^2$$

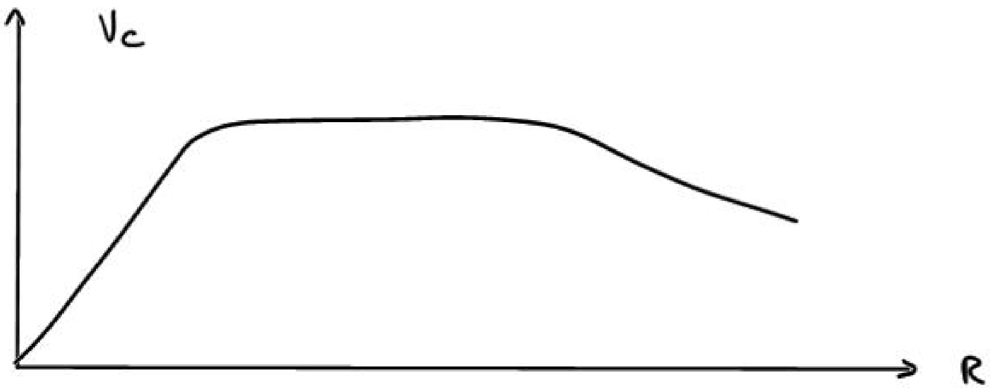
$$\Omega^2 = \frac{V_c^2}{R^2}$$

$$= \left(R\,\frac{\partial(\Omega^2)}{\partial R} + 4\,\Omega^2\right)_{(R_g,0)}$$

$$= \left(\frac{1}{R}\,\frac{\partial(V_c^2)}{\partial R} + 2\,\Omega^2\right)_{(R_g,0)} = \left(\frac{1}{R}\,\frac{\partial(V_c^2)}{\partial R} + 2\,\frac{V_c^2}{R^2}\right)_{(R_g,0)}$$

$$\nu^2(R_g) = \left.\frac{\partial^2 \phi_{eff}}{\partial z^2}\right)_{(R_g,0)} = \left.\frac{\partial^2 \phi}{\partial z^2}\right)_{(R_g,0)}$$

Note : $\varkappa$ depends only on V_c



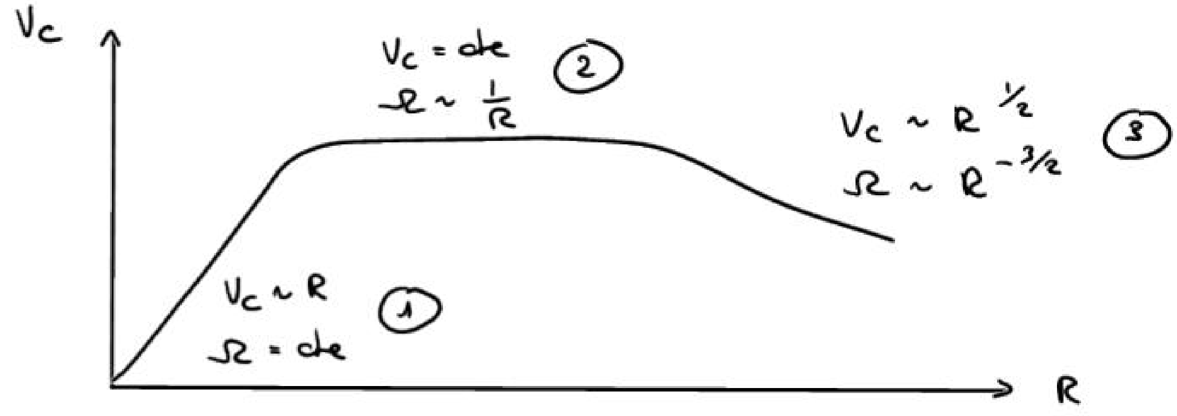
$\varkappa$ obtained by derivating V_c^2

Periods :

$$\begin{cases} \text{radial} & T_R := \dfrac{2\pi}{\varkappa} \\ \text{vertical} & T_z := \dfrac{2\pi}{\nu} \\ \text{azimuthal} & T_\theta := \dfrac{2\pi}{\Omega} \end{cases}$$

Radial dependency of $\varkappa$, ν for a typical galaxy

$$\boxed{\Omega = \frac{V_c}{R}}$$



- 1 near the center

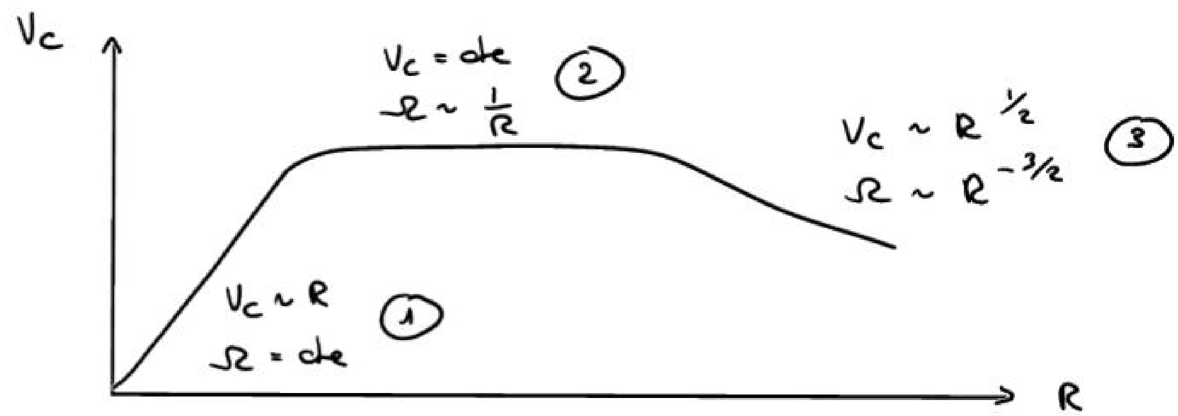
$V_c \sim R$ (rigid rotation) $\Rightarrow$ $\Omega = cte$

$$\varkappa^2 = R\frac{d}{dR}(\Omega^2) + 4\Omega^2 \quad \Rightarrow \quad \varkappa^2 = 4\Omega^2$$

$$\boxed{\varkappa \sim 2\Omega}$$

Radial dependency of κ, γ for a typical galaxy

$$\boxed{\Omega = \frac{V_c}{R}}$$



- 2 flat rotation part

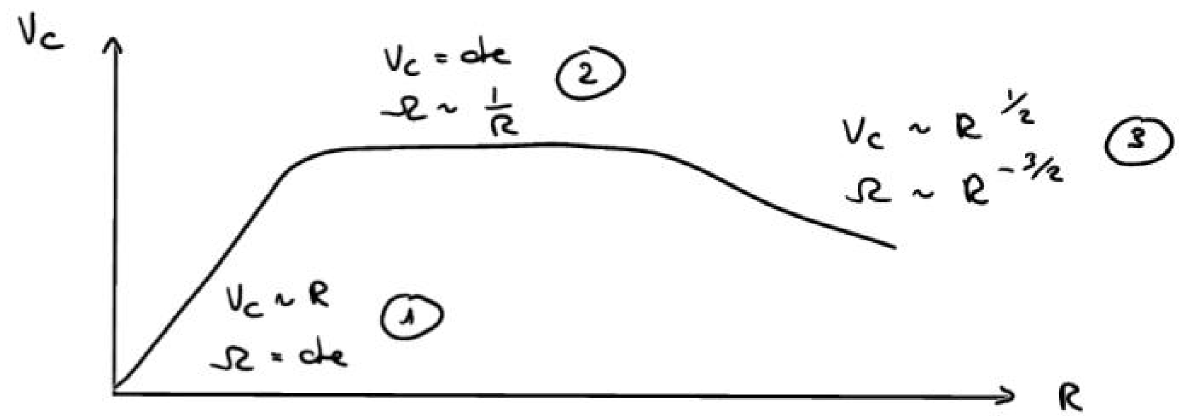
$$V_c = cte \quad \Rightarrow \quad \Omega \sim \frac{1}{R}$$

$$\kappa^2 = \frac{1}{R}\frac{\partial}{\partial R}(V_c^2) + 2\Omega^2 \quad \Rightarrow \quad \kappa^2 = 2\Omega^2$$

$$\boxed{\kappa \sim \sqrt{2}\,\Omega}$$

Radial dependency of κ, ν for a typical galaxy

$$\boxed{\Omega = \frac{V_c}{R}}$$



- 3 further out

$$V_c \sim R^{-1/2} \quad \text{(Keplerian decrease)}$$

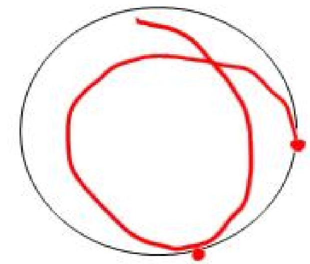
$$\Omega = \frac{V_c}{R} \sim R^{-3/2}$$

$$\kappa^2 = \frac{1}{R}\frac{\partial}{\partial R}(V_c^2) + 2\frac{V_c^2}{R^2} \sim R^{-3}$$

$$\boxed{\kappa = \Omega}$$

Thus, in general

$$\boxed{\Omega \leqslant \mathcal{H} \leqslant 2\,\Omega}$$



Integrals of motions

$$\begin{cases} \ddot{x} = -\varkappa^2(R_g)\, x \\ \ddot{z} = -\nu^2(R_g)\, z \end{cases}$$

$\Rightarrow$ Two integrals of motion (one for each oscillator)

1) $H_R = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\varkappa^2 x^2$

2) $H_z = \frac{1}{2}\dot{z}^2 + \frac{1}{2}\nu^2 z^2$

Thus, if a star oscillates near a circular orbit :

3 integrals of motions L_z, H_R, H_z

Total Hamiltonian (near a circular orbit of radius R_g)

$$\boxed{H(R,\dot{R},\theta,\dot{\theta},z,\dot{z}) = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2\right) + \phi(R,z)}$$

$L_z = R^2\dot{\theta}$

$$= \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \underbrace{\phi(R,z) + \frac{L_z^2}{2R^2}}_{\phi_{eff}(R,z)}$$

$$= \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \phi_{eff}(R_g,0) + \frac{1}{2}\kappa^2(R-R_g)^2 + \frac{1}{2}\nu^2 z^2$$

$$\boxed{H(R,p_R,z,p_z) = H_R(R,p_R) + H_z(z,p_z) + \phi_{eff}(R_g,0)}$$

Orbital motions

$$\begin{cases} \ddot{x} = -\varkappa^2(R_g)\, x \\ \ddot{z} = -\nu^2(R_g)\, z \end{cases}$$

$+ \quad R^2\dot{\theta} = L_z$

Solutions

① motion in z

$$z(t) = Z \cos(\nu t + \xi)$$

② motion in x

$$x(t) = X \cos(\varkappa t + \alpha)$$

Note valid only for small oscillations

as long as $\nu^2 = \dfrac{\partial^2 \phi}{\partial z^2} \simeq$ cte

ie $\rho_{disk} \simeq$ cte $\quad \left(\nu^2 = \dfrac{\partial^2 \phi}{\partial z^2} = 4\pi G \rho \right)$

$\rightarrow$ $z <$ disk scale lenght

~ 300 pc

③ motion in θ

Taylor + $\Omega_g = \dfrac{L_z}{R_g^2}$

$$\dot{\theta} = \frac{L_z}{R^2} \overset{x := R - R_g}{=} \frac{L_z}{(x+R_g)^2} = \frac{L_z}{R_g^2\left(\frac{x}{R_g}+1\right)^2} \cong \Omega_g\left(1 - \frac{2x}{R_g}\right)$$

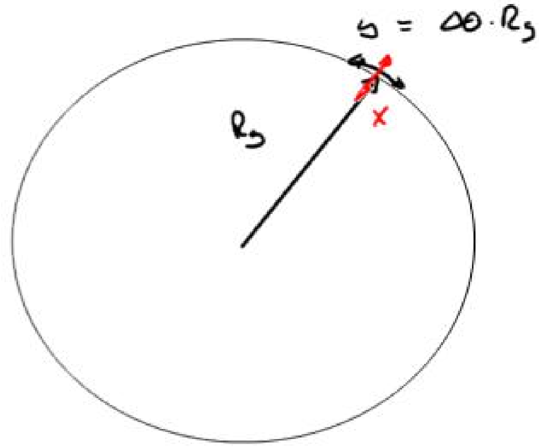
introducing $x(t)$

$$\dot{\theta}(t) = \Omega_g\left(1 - \frac{2X\cos(\varkappa t + \alpha)}{R_g}\right)$$

$$\boxed{\theta(t) = \underbrace{\Omega_g \cdot t}_{\substack{\text{motion of the}\\ \text{guiding center}\\ \text{along the circular}\\ \text{orbit}}} - \underbrace{\gamma \frac{X}{R_g}\sin(\varkappa t + \alpha)}_{\text{oscillations}} + \theta_0}$$

$$\gamma := \frac{2\Omega_g}{\varkappa}$$

New carthesian system x, y, z with an origin that follows the guiding center



$$\begin{cases} R(t) = R_g \\ \Theta(t) = \Omega_g t + \Theta_0 \end{cases}$$

Then, from

$$\Theta(t) = \Omega_g \cdot t \underbrace{- \mu \frac{X}{R_g} \sin(\varkappa t + \alpha)}_{\Delta\Theta = \frac{y}{R_g}} + \Theta_0$$

$$y = -\mu X \sin(\varkappa t + \alpha)$$

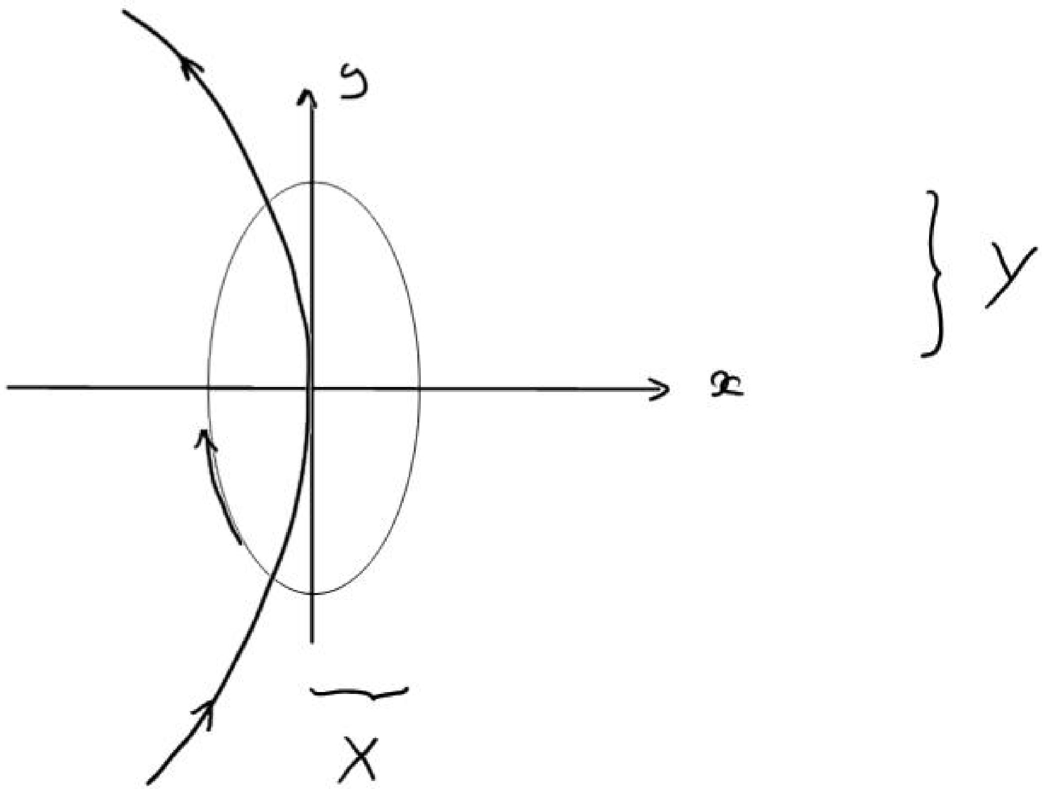
$$\boxed{y(t) = -Y \sin(\varkappa t + \alpha)} \qquad Y := \mu X$$

Complete solution

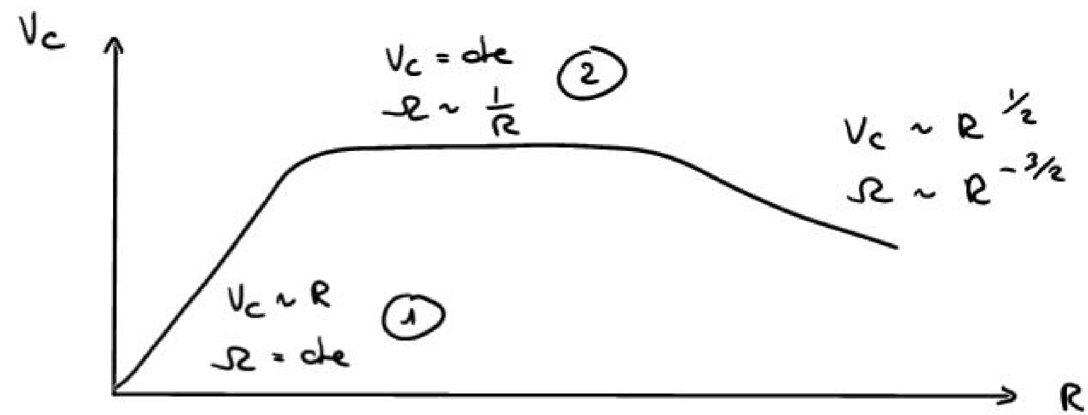
$$\boxed{\begin{cases} x(t) = X\cos(\varkappa t + \alpha) \\ y(t) = -Y\sin(\varkappa t + \alpha) \\ z(t) = Z\cos(\nu t + \xi) \end{cases}}$$

ellipse

$$Y = \frac{2\Omega_y}{\varkappa} X$$



Radial dependency for a typical galaxy



1 near the center

$\varkappa = 2\Omega$ $\qquad \dfrac{X}{Y} = 1$ $\qquad$ circle 

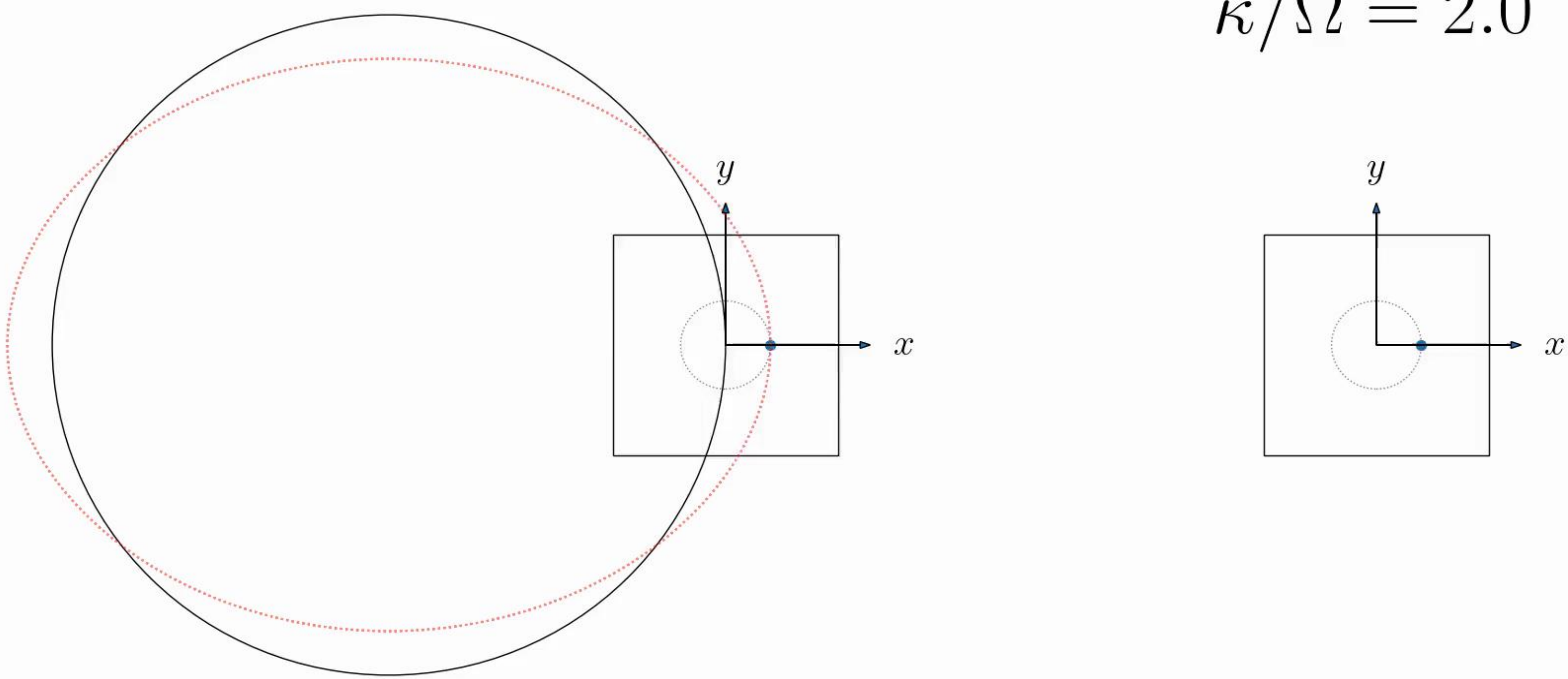
2 flat rotation part

$\varkappa = \sqrt{2}\,\Omega$ $\qquad \dfrac{X}{Y} = \dfrac{\sqrt{2}\,\Omega}{2\Omega}$ $\qquad X < Y$ 

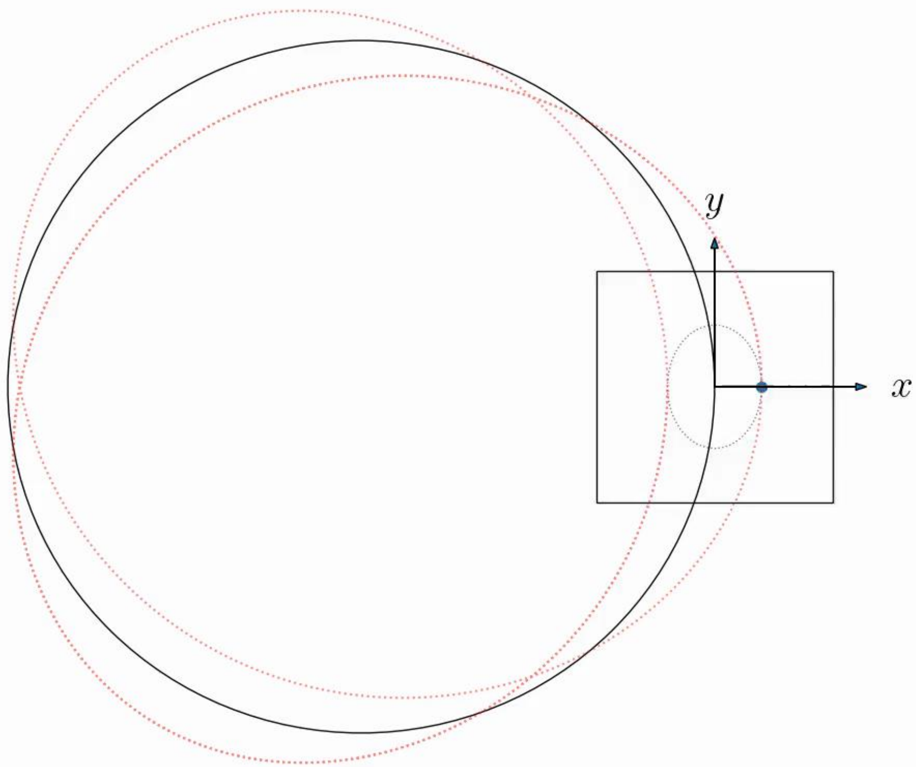
3 further out

$\varkappa = \Omega$ $\qquad \dfrac{X}{Y} = \dfrac{\Omega}{2\Omega}$ $\qquad X < Y$ 

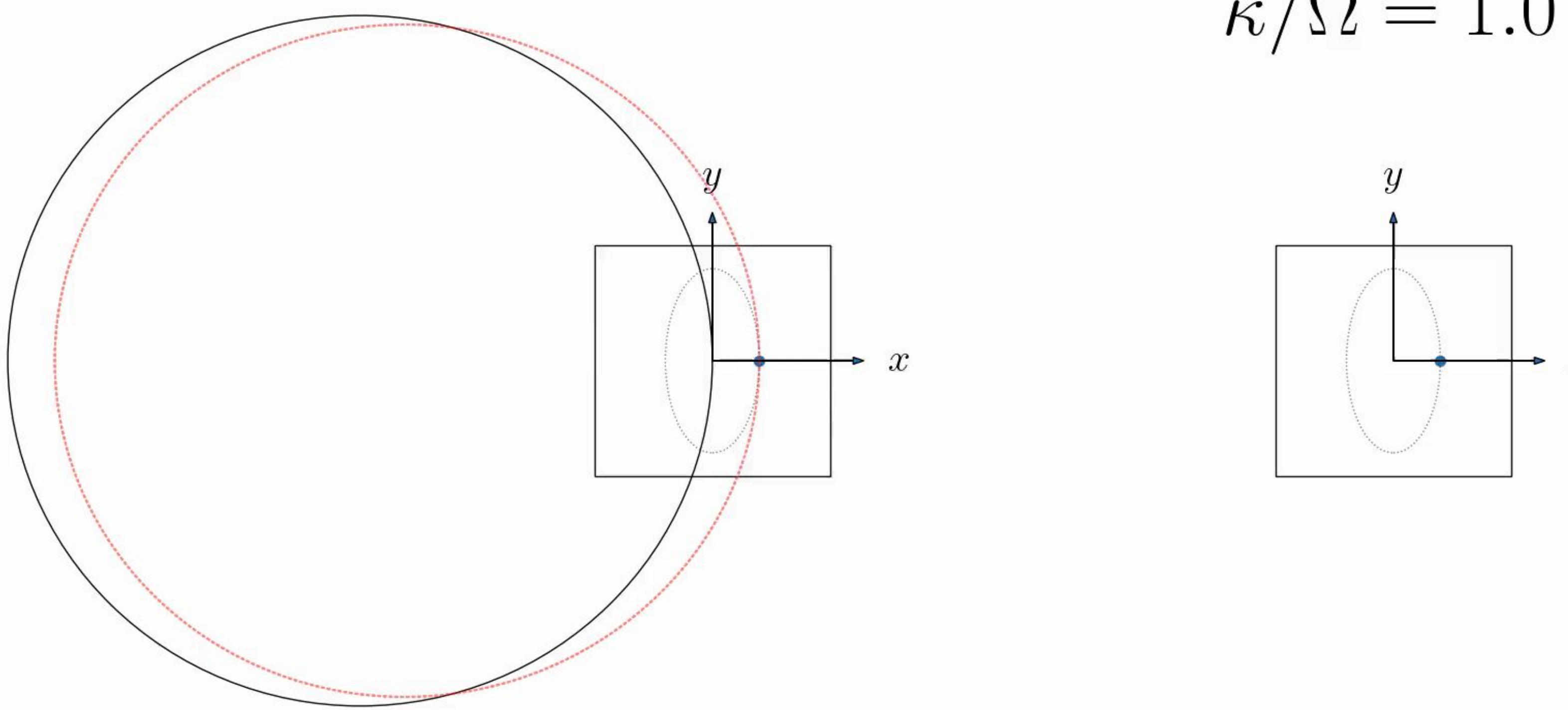
$$\kappa/\Omega = 2.0$$



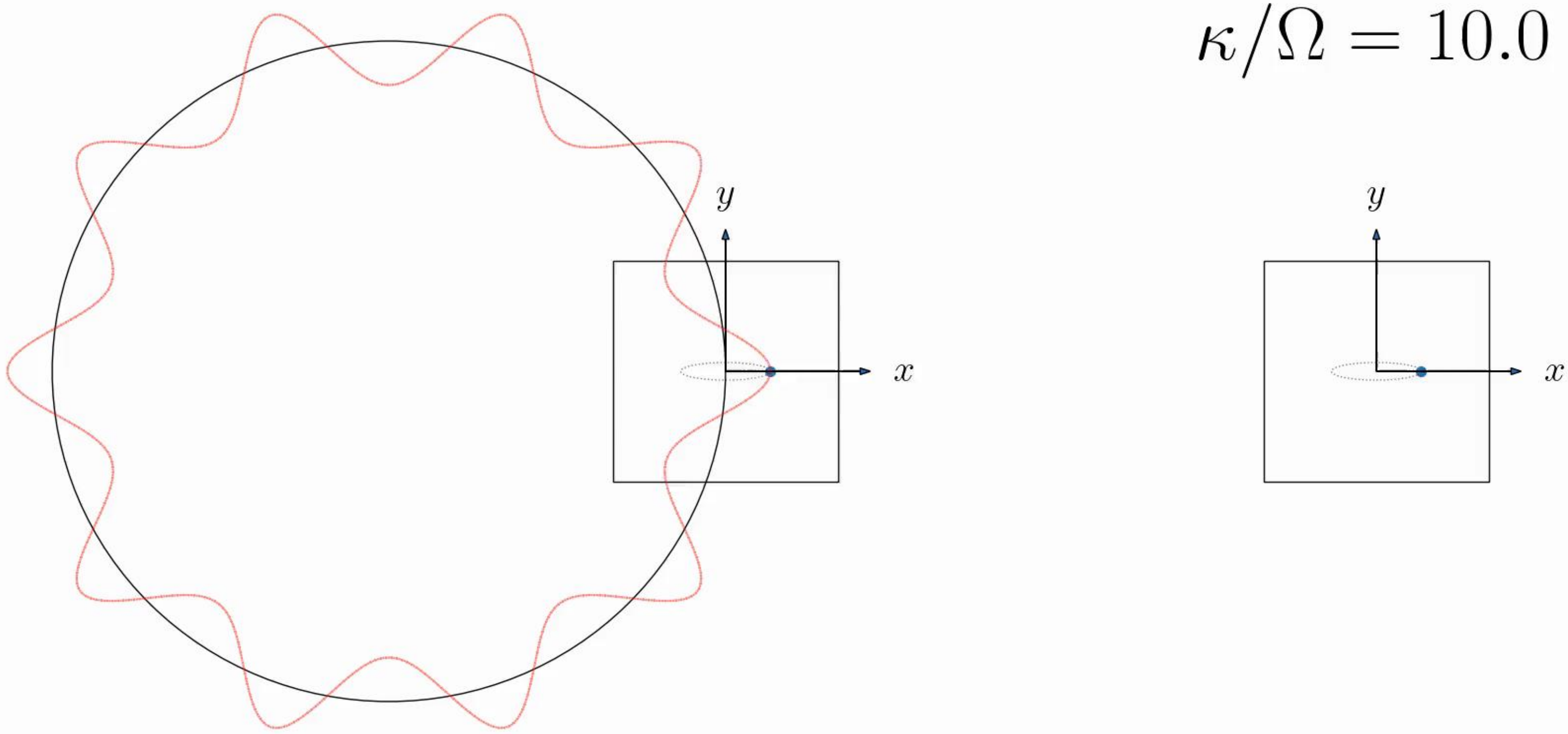
$$\kappa/\Omega = 1.5$$



$$\kappa/\Omega = 1.0$$



$$\kappa/\Omega = 10.0$$



The End