Astrophysics III, Dr. Yves Revaz — EPFL

4th year physics — Exercises week 6

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Astrophysics III: Stellar and galactic dynamics

Solutions

Problem 1:

$$\vec{a}(\vec{r}) = -\sum_i \frac{G m_i}{|\vec{r} - \vec{r_i}|^3}(\vec{r} - \vec{r_i}) \tag{1}$$

Problem 2:

$$V_c^2 = r \left|\frac{\partial \Phi}{\partial r}\right| = r\,|\vec{a}_r| \tag{2}$$

Problem 3:

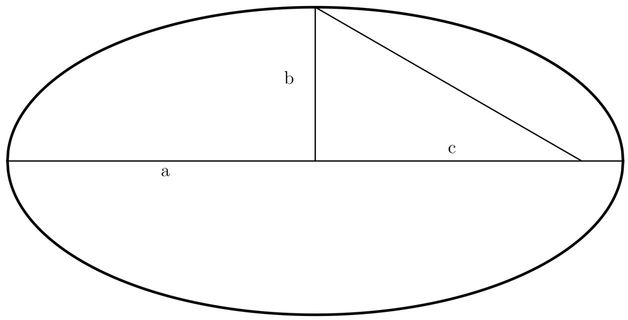
The exact same algorithm can be implemented for all models:

```
dphi = np.zeros(len(r), np.float)

for i in range(len(r)):
        # create coordinate array of current position r where
        # we compute vc2(r). We only use coordinate zero because
        # it makes life simple in spherical coordinates, but
        # Nbody() creates 3 dimensions for positions by default.
        pos = np.array([r[i], 0, 0])
        # get array of || r - r_i ||^2
        rr2 = np.sum((nb.pos-pos)**2, axis=1)
        # get array of all accelerations
        # G m_i / | r - r_i |^3 * (r - r_i)
        accx = nb.mass/ (rr2 + eps**2)**(3./2) *
                 (nb.pos[:, 0] - pos[0])
        dphi[i] = - np.sum(accx, axis=0)

vc2_acc = r * dphi
```

Problem 6:



The ellipse equation is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3}$$

the focii are at

$$c = \pm\sqrt{a^2 - b^2}$$

and the eccentricity is defined as

$$e = \frac{c}{a}$$

Using these relations, we write

$$e^2 = \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2}$$
$$y^2 = b^2 - \frac{b^2}{a^2}x^2 = \frac{b^2}{a^2}(a^2 - x^2) = (1 - e^2)(a^2 - x^2)$$

We apply a coordinate transformation now: Let $x = x' + ae(= x' + c)$. This gives

$$y^2 = (1 - e^2)\left(a^2 - (x' + ae)^2\right) \tag{4}$$

Now we show that the equation of Keplerian orbits (5) can be written in the same form as (4). The Keplerian orbits are defined as

$$r(\varphi) = \frac{a(1 - e^2)}{1 + e\cos(\varphi)} \tag{5}$$

with $x' = r\cos(\varphi)$, $y = r\sin(\varphi)$

$$
\begin{aligned}
r(1 + e\cos(\varphi)) &= r + er\cos(\varphi) = r + ex' \\
&= a(1 - e^2) \\
r^2 &= a^2(1 - e^2)^2 + e^2x'^2 - 2a(1 - e^2)ex' \\
&= x'^2 + y^2 \\
y^2 &= a^2(1 - e^2) + x'^2(e^2 - 1) - 2a(1 - e^2)ex' \\
&= (1 - e^2)[a^2(1 - e^2) - x'^2 - 2aex'] \\
&= (1 - e^2)[a^2 - a^2e^2 - (x' + ae)^2 + a^2e^2] \\
&= (1 - e^2)[a^2 - (x' + ae)^2]
\end{aligned}
$$

which is exactly equation (4) again.