Introduction to Differentiable Manifolds

EPFL – Fall 2021 M. Cossarini, B. Santos Correia

Exercise series 4, with solutions 2021–10–19

Exercise 4.1 (A little heads-up regarding coordinate vectors)**.** Let φ and ψ be smooth charts on a smooth manifold M defined on the same domain U. If the first coordinate functions φ^0 and ψ^0 agree ($\varphi^0 \equiv \psi^0$ on U), this does *not* imply $\frac{\partial}{\partial \varphi^0}|_p = \frac{\partial}{\partial \psi^0}|_p$ for $p \in U$.

Work out a simple example of this fact e.g. on $M = \mathbb{R}^2$ by considering on the one hand the Cartesian coordinates (x, y) and on the other hand the chart (u, v) given by $u = x$, $v = x + y$.

This shows that $\frac{\partial}{\partial \varphi^i}|_p$ *depends on the whole system* $(\varphi^0, \dots, \varphi^{n-1})$*, not only on* φ^i*.*

Solution. The two coordinate charts have the following relationship:

$$\begin{cases} u = x \\ v = x + y \end{cases} \qquad \begin{cases} x = u \\ y = v - u \end{cases}$$

By the chain rule we have

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial u} \cdot \frac{\partial}{\partial y} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}.$$

We consider for example the function $f(x, y) = xy$ on $\mathbb{R}^2$. The coordinate derivatives of f with respect to two different charts are

$$\frac{\partial}{\partial x} f = y$$
$$\frac{\partial}{\partial u} f = y - x \neq \frac{\partial}{\partial x} f$$

Thus the coordinate vectors depends on the whole system.

We consider for example a linear function $f(x, y) = ax + by$ on $\mathbb{R}^2$. The coordinate derivatives of f with respect to two different charts are

$$\frac{\partial}{\partial x} f = a$$
$$\frac{\partial}{\partial u} f = a - b \neq \frac{\partial}{\partial x} f$$

Thus the coordinate vectors depends on the whole system. □

Exercise 4.2 (The tangent space of a vector space)**.** Let V be an n-dimensional vector space.

(a) Let $\mathcal{A}$ be the set of linear isomorphisms $\varphi : V \to \mathbb{R}^n$. Show that there is a topology on V such that all $\varphi \in \mathcal{A}$ are homeomorphisms. Show that $\mathcal{A}$ is a smooth atlas on V. *In other words, any vector space has a natural smooth structure.*

Solution. Pick any isomorphism $\varphi : V \to \mathbb{R}^n$. Since φ is bijective, there is a unique topology on V such that φ is a homeomorphism. We put this topology on V. Then any other isomorphism $\psi : V \to \mathbb{R}^n$ is also a homeomorphism because it can be written as $\psi = (\psi \circ \varphi^{-1})\varphi$, where $\psi \circ \varphi^{-1}$ is a linear isomorphism $\mathbb{R}^n \to \mathbb{R}^n$. The family $\mathcal{A}$ is a smooth atlas since the transition functions are linear isomorphisms $\psi \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, which are smooth. □

(b) Fix $a \in V$. To every $v \in V$ we associate the curve passing through a

$$\gamma_v : \mathbb{R} \to V : t \mapsto a + tv$$

Show that the map $\Phi_a : V \to T_a V : v \mapsto \gamma_v'(0)$ is an isomorphism of vector spaces. *Hence we can identify a vector space with its tangent space in a canonical way.*

Solution. Let us compute $\Phi_a(v)$. We use first the definition of $\gamma_v'(0)$, then the definition of the differential $\mathrm{D}_0\,\gamma$. To compute $D_0\gamma$ we use as chart of V an isomorphism $\phi : V \to \mathbb{R}^n$. We obtain

$$\begin{aligned}\Phi_a(v) &= \gamma_v'(0)\\ &= \mathrm{D}_0\,\gamma[0, \mathrm{id}_{\mathbb{R}}, 1]\\ &= [\gamma(0), \phi, \mathrm{D}_0(\phi \circ \gamma \circ \mathrm{id}_{\mathbb{R}}^{-1}(1)]\\ &= [a, \phi, \frac{\partial}{\partial t}|_{t=0}\phi(a + tv)]\\ &= [a, \phi, \frac{\partial}{\partial t}|_{t=0}(\phi(a) + t\phi(v)]\\ &= [a, \phi, \phi(v)]\end{aligned}$$

On the other hand, we know by a previous exercise that

$$\nu : w \in \mathbb{R}^n \mapsto [a, \phi, w] \in T_aV$$

is an isomorphism. We can conclude that Φ_a is an isomorphism as well, because our computation shows that $\Phi_a = \nu \circ \psi$. □

(c) Let $f : V \to W$ be a *linear* map between vector spaces V, W. Consider the differential $D_af : T_aV \to T_{F(a)}W$ at any point $a \in V$. Identifying $T_aV \cong V$ and $T_{f(a)}W \cong W$ via the isomorphisms Φ_a, $\Phi_{f(a)}$, show that D_af is identified with f. That is, show that the following diagram commutes:

$$\begin{array}{ccc} T_aV & \xrightarrow{D_af} & T_{f(a)}W \\ \Phi_a\uparrow & & \uparrow\Phi_{f(a)} \\ V & \xrightarrow[f]{} & W \end{array}$$

Solution. To check that the diagram commutes, we take two linear isomorphisms $\phi : V \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^n$. Then for any vector $v \in V$ let us show that

$$\begin{array}{ccc} [a, \phi, \phi(v)] & \xrightarrow{D_af} & [f(a), \psi, \psi(f(v))] \\ \Phi_a\uparrow & & \uparrow\Phi_{f(a)} \\ v & \xrightarrow[f]{} & f(v) \end{array}$$

In the previous item we have shown that $\Phi_a(v) = [a, \phi, \phi(v)]$, and in the same way we have $\Phi_{f(a)}(f(v)) = [f(a), \psi, \psi(f(v))]$. To finish, we use the definition of $\mathrm{D}_a\,f$ to check that

$$\begin{aligned}\mathrm{D}_a\,f[a, \phi, \phi(v)] &= [f(a), \phi, \mathrm{D}_{\phi(a)}(\psi\, f\, \phi^{-1})(\phi(v))\\ &= [f(a), \phi, (\psi\, f\, \phi^{-1})(\phi(v))\\ &= [f(a), \psi, \psi(f(v))].\end{aligned}$$

Here, we used the fact that $\mathrm{D}_{\phi(a)}(\psi\, f\, \phi^{-1}) = \psi\, f\, \phi^{-1}$ since $\psi\, f\, \phi^{-1}$ is a linear map $\mathbb{R}^m \to \mathbb{R}^n$. □

Exercise 4.3 (Differential of the determinant function)**.** Consider the determinant function $\det : M_n(\mathbb{R}) \to \mathbb{R}$, where $M_n(\mathbb{R}) \simeq \mathbb{R}^{n\times n}$ is the vector space of real $n \times n$, with its natural smooth structure. We want to compute its differential transformation $D_A \det$ at any matrix $A \in \mathrm{GL}_n(\mathbb{R})$ (i.e. at any invertible matrix),

$$D_A \det : T_AM_n(\mathbb{R}) \to T_{\det(A)}\mathbb{R}$$

(Note that we may identify $T_AM_n(\mathbb{R})$ with $M_n(\mathbb{R})$ and $T_{\det(A)}\mathbb{R}$ with $\mathbb{R}$.)

(a) Verify that det is a smooth function.
Hint: Write the determinant as a sum over all n-permutations.

Solution. The determinant can be written as

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{0 \le i < n} a_{i,\sigma(i)}.$$

Each of the terms $f_\sigma(A) := \operatorname{sgn}(\sigma)\prod_{0\le i<n} a_{i,\sigma(i)}$ is a monomial, hence a smooth function. □

(b) Show that the differential of det at the identity matrix $I \in M_n(\mathbb{R})$ is

$$D_I \det(B) = \operatorname{tr}(B).$$

where tr denotes the trace.

Solution. Let's define a curve $\gamma_B : \mathbb{R} \to GL(n) : t \to I + tB$, for $B \in GL(n)$. Using the identification $\Phi_I : GL(n) \to T_I(GL(n)) : B \to \gamma_B'(0)$ (and the usual identification $T_1\mathbb{R} \cong \mathbb{R}$) we have

$$\begin{aligned} D_I \det(B) &= D_I \det(\gamma_B'(0)) \\ &= (\det \circ \gamma_B)'(0) \\ &= \frac{d}{dt}\Big|_{t=0} (\det(I + tB)) \\ &= \sum_{\sigma\in S_n} \frac{d}{dt}\Big|_{t=0} f_\sigma(I + tB) \end{aligned}$$

Let us derivate each of the monomials f_σ.

The coefficients of the matrix $A = I + tB$ are $a_{i,j} = \delta_{i,j} + t\, b_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta. Note that at $t = 0$ all the coefficients that are not on the diagonal vanish.If $\sigma \neq \mathrm{id}_n$, then the monomial f_σ has at least two coefficients that are not on the diagonal, hence we have $\frac{d}{dt}\Big|_{t=0} (f_\sigma(I + tB)) = 0$. Thus the only term which survives is the one corresponding to the permutation $\sigma = \mathrm{id}_n$, and we have

$$\begin{aligned} D_I \det(B) &= \frac{d}{dt}\Big|_{t=0} f_{\mathrm{id}_n}(I + tB) \\ &= \frac{d}{dt}\Big|_{t=0} \sum_{0\le i<n} (1 + t\, b_{i,i,}) \\ &= \frac{d}{dt}\Big|_{t=0} (1 + t\operatorname{tr}(B) + t^2 \dots) \\ &= \operatorname{tr} B \end{aligned}$$

□

(c) Show that for arbitrary $A \in \mathrm{GL}_n(\mathbb{R}), B \in M_n(\mathbb{R})$.

$$D_A \det(B) = (\det A)\operatorname{tr}(A^{-1}B)$$

Hint: Write $\det(A + tB) = (\det A)(\det(I + tA^{-1}B))$.

Solution. Similarly, we define $\gamma_B : \mathbb{R} \to GL(n) : t \to A+tB$, for $B \in GL(n)$. With the identification $\Phi_A : GL(n) \to T_A(GL(n)) : B \to \gamma_B'(0)$ we have

$$\begin{aligned} D_A \det(B) &= D_A \det(\gamma_B'(0)) \\ &= (\det \circ \gamma_B)'(0) \\ &= \frac{d}{dt}\Big|_{t=0} (\det(A + tB)) \\ &= \det(A) \lim_{t\to 0} \frac{1 + t\ \operatorname{tr}(A^{-1}B) + O(t^2)) - 1}{t} \\ &= \det(A)\ \operatorname{tr}(A^{-1}B) \end{aligned}$$

□

(d) Show that $D_A \det$ is the null linear transformation if $A = 0$ and $n \geq 2$.

Solution. It suffices to check that $f_B'(t) = 0$ when $t = 0$ for the function

$$f_B(t) = \det(A + tB) = \det(tB) = t^n \det(B).$$

Now, $f_B'(t) = n\, t^{n-1} \det(B)$, thus $f_B'(0) = 0$ as required. □

Exercise 4.4 (Diffeomorphic manifolds have the same dimension)**.** Let M and N be nonempty diffeomorphic manifolds. Show that $\dim M = \dim N$.

Solution. Let m, n be the dimensions of M, N respectively. Let $f : M \to N$ be a diffeomorphism. Take any point $p \in M$ and let $q = f(p) \in N$. The differential transformation

$$D_p f : T_pM \to T_qN$$

is a linear isomorphism, because it admits as inverse the map

$$D_p(f^{-1}) : T_qN \to T_pM.$$

Therefore T_pM is isomorphic to T_qN. On the other hand, we have $T_pM \simeq \mathbb{R}^m$ and $T_qN \simeq \mathbb{R}^n$. It follows that $m = n$. □

Exercise 4.5 (Tangent vectors as derivations)**.** Let M be a $\mathcal{C}^k$ manifold, $k \geq 1$, and let $p \in M$. Show that the map $\nu_p : X \in T_pM \mapsto D_X \in Der_pM$ defined by

$$D_X : f \in C^k(M, \mathbb{R}) \mapsto D_pf(X) \in T_{f(p)}\mathbb{R} \cong \mathbb{R}$$

is linear and injective.

Hint: To prove injectivity, take a chart ϕ that is defined at p. Any vector $X \in T_pM$ can be written as $X = \sum_i X^i \frac{\partial}{\partial \phi^i}|_p$. Show that $D_X(\phi^j) = X^j$.

Solution. The equation $D_X(\phi^j) = X^j$, which we will prove below, implies that ν_p is injective because it says that the components X^j of the vector X (and hence, the vector X) can be determined from D_X.

The vector X is of the form $X = [p, \phi, v]$ for some $v \in \mathbb{R}^n$. The components of v are the coefficients X^j, because

$$X = [p, \phi, \sum_i v^i e_i] = \sum_i v^i [p, \phi, e_i] = \sum_i v^i \frac{\partial}{\partial \phi^i}|_p.$$

Thus it is sufficient to prove that $D_X(\phi^j) = v^j$.

By definition of D_X we have

$$D_x(\phi^j) = D_p\phi^j(X),$$

thus we have to compute the differential of ϕ^j. Note that $\varphi^j = \pi^j \circ \varphi$, where $\pi^j : U \in \mathbb{R}^n \mapsto x^j \in \mathbb{R}$ is the projection on the j-th axis. To compute the differential of ϕ^j we use the local expression

$$\phi^j|_{\phi^j}^{\mathrm{id}_\mathbb{R}} = \phi^j \circ \phi^{-1} = \pi^j \circ \phi \circ \phi^{-1} = \pi^j$$

Hence we have

$$\begin{aligned}
D_x(\phi^j) &= D_p\phi^j[p, \phi, v] \\
&= [\phi^j(p), \mathrm{id}_\mathbb{R}, D_{\phi(p)}(\phi^j|_{\phi^j}^{\mathrm{id}_\mathbb{R}})(v)] \\
&= [\phi^j(p), \mathrm{id}_\mathbb{R}, D_{\phi(p)}(\pi^j)(v)] \\
&= [\phi^j(p), \mathrm{id}_\mathbb{R}, \pi^j(v)] \\
&= [\phi^j(p), \mathrm{id}_\mathbb{R}^n, v^j] \\
&\equiv v^j \in \mathbb{R} \quad \text{by the identification } T_{\phi^j(p)}\mathbb{R} \equiv \mathbb{R}.
\end{aligned}$$

□

Exercise 4.6 (Nonvectorial derivations* – optional)**.** Let M be a $\mathcal{C}^k$-differentiable n-manifold and let Der_pM be the vector space of derivations at some point $p \in M$.

(a) If M is a smooth manifold, show that $Der_p(M)$ has dimension n.
Hint: Prove Hadamard's lemma: any $f \in \mathcal{C}^{1+k}(M)_p$ can be locally written as $f(p) + \sum_i \varphi^i f_i$, with $f_i \in \mathcal{C}^k(M)$, φ a chart satisfying $\varphi(p) = 0$.
Solution. Proof of Hadamard's lemma: Consider the local expression $g = f_\varphi = f \circ \varphi^{-1}$. For x near the origin, we can write

$$\begin{aligned} g(x) - g(0) &= \int_0^1 \frac{\partial}{\partial t} g(tx)\,\mathrm{d}t \\ &= \int_0^1 \sum_i \partial_i g(tx)\, x^i\,\mathrm{d}t \\ &= \sum_i x^i\, g_i(x)\mathrm{d}t, \end{aligned}$$

where

$$g_i(x) = \int_0^1 \partial_i g(tx)\,\mathrm{d}t$$

is $\mathcal{C}^k$. Then we define $f_i = g_i \circ \varphi$.

It follows that any derivation $X \in Der_p M$ is a tangent vector, because it is determined by the n numbers $X(\varphi^i)$. Indeed, for a function f we can write $f(0) + \sum_i \varphi^i f_i$ using Hadamard's lemma; then we compute

$$X(f) = \underbrace{X(f(p))}_{=0} + \sum_i X(\varphi^i) f_i(p) + \sum_i \underbrace{\varphi^i(p)}_{=0} X(f_i).$$

□

(b) Let $I \subseteq C^k(M, \mathbb{R})$ be the ideal of functions that vanish at p. Show that a linear map $X : \mathcal{C}^k(M) \to \mathbb{R}$ satisfies the Leibniz identity iff it vanishes on I^2 and on $\mathbb{R}$. Conclude that $Der_p(M) \equiv (I/I^2)^*$.
Solution. If X satisfies Leibniz, we see that $X(h) = 0$ for any $h \in I^2$ by writing $h = fg$, with $f, g \in I$, and computing

$$X(h) = X(fg) = X(f)\,g(p) + f(p)\,X(g) = 0.$$

We also see that $X(1) = 0$ because

$$X(1) = X(1^2) = X(1)\,1 + 1\,X(1) = X(1) + X(1)$$

It follows that $X(c) = cX(1) = 0$ for all $c \in \mathbb{R}$.

Reciprocally, suppose X vanishes on I^2 and on $\mathbb{R}$. To prove the Leibniz identity

$$X(fg) = X(f)g(p) + f(p)\,X(g)$$

we do as follows. Write $f = f_0 + \overline{f}$, $g = g_0 + \overline{g}$, where $\overline{f} = f - f_0$ and $g = \overline{g} - g_0$ are elements of I. Note that $X(f) = X(\overline{f})$ because $X(f(0)) = 0$. Then we compute

$$\begin{aligned} X(fg) &= X\big((f(0) + \overline{f})(g(0) + \overline{g})\big) \\ &= X(f(0)\,g(0)) + X(\overline{f}\,g(0)) + X(f(0)\,\overline{g}) + X(\overline{f}\,\overline{g}) \\ &= 0 + g(0)\,X(\overline{f}) + f(0)\,X(\overline{f}) \\ &= X(f)\,g(0) + f(0)\,X(g) \end{aligned}$$

□

(c) (Newns–Walker, 1956) If $k < \infty$, show that I/I^2 is infinite dimensional if $k < \infty$. Conclude that $Der_p M$ is infinite dimensional.
Hint: (From Laird E Taylor (1972), "The tangent space of a C^k manifold") For the case $M = \mathbb{R}$, $p = 0$, show that the functions $f_\sigma(t) = |t|^\sigma$ with $k < \sigma < k + 1$, taken modulo I^2, are linearly independent. To distinguish these functions, define the vanishing order $\mathrm{ord}(f)$ of a function $f \in I$ as the maximum $\alpha \geq 0$ such that $\lim_{t \to 0} \frac{f(t)}{|t|^\alpha} = 0$ and use Taylor's theorem to show that $\mathrm{ord}(f) \notin (k, k+1)$ if $f \in I^2$.

Solution. A function $g \in I$ can be written, using Taylor's theorem, as $g = \sum_{1 \leq i < k} a_i t^i + r_g(t)t^k$, with $a_i \in \mathbb{R}$ and r_g a continuous function. Therefore a function $h \in I^2$ can be written as $h = \sum_{2 \leq j \leq k} b_j + r_h(t)t^{k+1}$, with r_h continuous. It follows that $\operatorname{ord}(h) \geq k+1$ if all $b_i = 0$, otherwise $\operatorname{ord}(h)$ is the smallest i such that $b_i \neq 0$.

Now let us prove that the functions f_σ are linearly independent modulo I^2. Consider a finite linear combination $f = \sum_i c_i f_{\sigma_i} \in I^2$ where σ_i are different numbers in the interval $(k, k+1)$. We see that all c_i are 0, because otherwise

$$\operatorname{ord}(f) = \max_{c_i \neq 0} \sigma_i \in (k, k+1).$$

For a general manifold M we use a chart φ satisfying $\varphi(p) = 0$ and a curve $\gamma(t) = \varphi^{-1}(te_0)$. The functions $g_\sigma = f_\sigma(\varphi^0)$ are linearly independent modulo the I^2 of M at p because the functions $f_\sigma = g_\sigma \circ \gamma$ are linearly independent modulo the I^2 of $\mathbb{R}$. □