Introduction to Differentiable Manifolds

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Exercise series 6 **2021–11–02**

Exercise 6.1. (a) If $f : M \to N$ is an immersion, show that a continuous map $h : L \to M$ is $\mathcal{C}^k$ if the composite $f \circ h$ is $\mathcal{C}^k$.

(b) If $f : M \to N$ is an embedding, show that a function $h : M \to L$ is $\mathcal{C}^k$ if the composite $f \circ h$ is $\mathcal{C}^k$.

(c) If $f_0 : M_0 \to N$ and $f_1 : M_1 \to N$ are $\mathcal{C}^k$ embeddings with the same image, show that there is a diffeomorphism $h : M_0 \to M_1$ such that $f_1 \circ h = f_0$.

Exercise 6.2. If S, T are embedded submanifolds of M, N respectively, then $S \times T$ is an embedded submanifold of $M \times N$.

Exercise 6.3. (a) Show that a subset $S \subseteq \mathbb{R}^n$ is a $\mathcal{C}^r$-embedded k-submanifold if each point $x \in S$ has an open neighborhood W such that the set $S \cap W$ is the graph of a $\mathcal{C}^r$ function that expresses some $n-k$ coordinates in terms of the remaining k coordinates. (More precisely, the function is of the form $f : U \subseteq \mathbb{R}^I \to \mathbb{R}^{I'}$, where I is a k-element subset of $n := \{0, \dots, n-1\}$, I' is its complement, and $U \subseteq \mathbb{R}^I$ is an open set.)

(b) Let S be the set of real $m \times n$ matrices of rank k. Show that S is a smooth submanifold of $\mathbb{R}^{m\times n}$. What is its dimension ?

Hint: A rank-k matrix $A \in \mathbb{R}^{m\times n}$ has an invertible $k \times k$ submatrix $A|_{I\times J}$ (where $I \subseteq m$, $J \subseteq n$ are k-element sets). Show that the coefficients $A_{i',j'}$ with $i' \notin I$ and $j' \notin J$ can be expressed as a smooth function of the other coefficients of A.

Exercise 6.4. * If M is connected and $f : M \to M$ is a $\mathcal{C}^k$ map such that $f \circ f = f$, then $f(M)$ is an embedded submanifold of M.

Hint: Show that f has constant rank. Use what you know about a linear projector $P : V \to V$ and the complementary projector $\mathrm{id}_V - P$.