Astrophysics III, Dr. Yves Revaz | EPFL

4th year physics | Exercises week 7

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Astrophysics III: Stellar and galactic dynamics

Solutions

Problem 1:

For a given potential $\Phi(\vec{r})$, the acceleration reads:

$$\vec{a}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}) \tag{1}$$

General cartesian:
$\Phi(\vec{r}) = \Phi(x,y,z)$; $\vec{\nabla}\Phi(\vec{r}) = \frac{\partial\Phi(x,y,z)}{\partial x}\vec{e_x} + \frac{\partial\Phi(x,y,z)}{\partial y}\vec{e_y} + \frac{\partial\Phi(x,y,z)}{\partial z}\vec{e_z}$.

Spherical symmetry:
$\Phi(\vec{r}) = \Phi(r)$; $\vec{\nabla}\Phi(\vec{r}) = \frac{d\Phi(r)}{dr}\vec{e_r}$, and $\vec{e_r} = \frac{x}{r}\vec{e_x} + \frac{y}{r}\vec{e_y} + \frac{z}{r}\vec{e_z}$.

$$\begin{aligned} a_x &= -\frac{d\Phi(r)}{dr} \times \frac{x}{r} && (2)\\ a_y &= -\frac{d\Phi(r)}{dr} \times \frac{y}{r} && (3)\\ a_z &= -\frac{d\Phi(r)}{dr} \times \frac{z}{r} && (4)\\ & && (5) \end{aligned}$$

Cylindrical symmetry:
$\Phi(\vec{r}) = \Phi(R,z)$; $\vec{\nabla}\Phi(R,z) = \frac{\partial\Phi(R,z)}{\partial r}\vec{e_r} + \frac{\partial\Phi(R,z)}{\partial z}\vec{e_z}$, and $\vec{e_r} = \frac{x}{R}\vec{e_x} + \frac{y}{R}\vec{e_y}$.

$$\begin{aligned} a_x &= -\frac{\partial\Phi(R,z)}{\partial R} \times \frac{x}{R} && (6)\\ a_y &= -\frac{\partial\Phi(R,z)}{\partial R} \times \frac{y}{R} && (7)\\ a_z &= -\frac{\partial\Phi(R,z)}{\partial z} && (8) \end{aligned}$$

The derivative of the potentials are the following:

a) Point mass:

$$\frac{d\Phi(r)}{dr} = \frac{GM}{r^2} \tag{9}$$

b) Plummer-Schuster potential:

$$\frac{d\Phi(r)}{dr} = \frac{GMr}{(r^2+e^2)^{3/2}} \tag{10}$$

c) Miyamoto-Nagai potential:

$$\frac{\partial \Phi(R,z)}{\partial R} = \frac{GMR}{\left(R^2 + \left[a + \sqrt{z^2+b^2}\right]^2\right)^{3/2}} \tag{11}$$

$$\frac{\partial \Phi(R,z)}{\partial z} = \frac{GMz}{\left(R^2 + \left[a + \sqrt{z^2+b^2}\right]^2\right)^{3/2}} \left(1 + \frac{a}{\sqrt{z^2+b^2}}\right) \tag{12}$$

$$\tag{13}$$

d) Harmonic potential:

$$\frac{\partial \Phi(x,y,z)}{\partial x} = \omega_x^2 x \tag{14}$$

$$\frac{\partial \Phi(x,y,z)}{\partial y} = \omega_y^2 y \tag{15}$$

$$\frac{\partial \Phi(x,y,z)}{\partial z} = \omega_z^2 z \tag{16}$$

The circular velocity follows trivially :

$$V_c = \sqrt{R \frac{\partial}{\partial R} \Phi(R)} \tag{17}$$

So does the period :

$$T = 2\pi \frac{R}{V_c(R)} \tag{18}$$

Problem 2:

a) Point mass:

```
def Vc(r):

    return sqrt( GM /r )

r = sqrt(x**2+y**2+z**2)

cte = -GM/r**3

Fx = cte * x ; Fy = cte * y ;  Fz = cte * z
```

b) Plummer-Schuster potential:

```
def Vc(r):

    return sqrt( GM*r**2/(r**2+e**2)**1.5 )

r = sqrt(x**2+y**2+z**2)
```

```
cte = -GM/(r**2+e**2)**1.5

Fx = cte * x ; Fy = cte * y ; Fz = cte * z
```

c) Miyamoto-Nagai potential:

```
def Vc(r):

    return  sqrt(GM* r**2/(r**2+ (a+b)**2 )**1.5)

R2 = x**2 + y**2 ; Z2 = z*z ; C1 = sqrt(Z2+b**2)

F  = -GM*(R2+(a+C1)**2)**(-1.5)

Fx = F*x ; Fy = F*y ; Fz = F*(1.+a/C1) * z
```

d) Harmonic potential:

```
def Vc(r):

    return  wx*r

Fx = -wx**2 * x ; Fy = -wy**2 * y ; Fz = -wz**2 * z
```

And then, for all potentials:

```
YP[0] = Y[3] ; YP[1] = Y[4] ; YP[2] = Y[5]

YP[3] = Fx ; YP[4] = Fy ; YP[5] = Fz

def Period(r):

  return 2*pi*r/Vc(r)
```

Problem 3:

a) circular orbit:

All we need to do is set our star to go around a plane (the polar plane for axisymmetric potentials) with the circular velocity. For simplicity, you can just set

```
y = z = vy = vz = 0
```

, and then compute the circular velocity for some arbitrary x that you like. Luckily, this is the default behaviour of orbits.py, so you can just launch the program with the appropriate potential file, don't need to specify any positions nor velocities, and let it run for one orbit:

point mass:

```
./orbit.py  --norbit 1  --potentialfile=pm.py
```

Plummer:

```
./orbit.py  --norbit 1  --potentialfile=plummer.py
```

Miyamoto-Nagai:

```
./orbit.py  --norbit 1  --potentialfile=miyamoto.py
```

b) quasi periodic orbit:

You can only obtain quasi periodic orbits in axisymmetric potentials, not in the spherically symmetric ones. To get a quasi periodic orbit, we need to perturb the circular motion just a bit. The orbits still need to be bounded though, i.e. have total energy ≤ 0. orbit.py will print out the circular orbit velocity if you give it no velocity command line arguments. For example, with x=3, it will give vy = 0.533484, so try modifying the velocity slightly, e.g. like so:

Plummer (in the plane):

```
./orbit.py  --norbit 10  --potentialfile=plummer.py  --x=3 --vy=0.5
```

Or you can give it a small velocity in z-direction:

Miyamoto-Nagai (oscillations in z):

```
./orbit.py  --norbit 10  --potentialfile=miyamoto.py  --x=4 --vz=0.2
```

c) resonant orbit:

Resonant orbits are orbits where at leas two of the the orbital periods have a ratio of a rational number. This is easily done for the harmonic oscillator, where we can specify the frequencies in each coordinate direction explicitly. However, we also need to give the particle an initial velocity in z direction, otherwise it won't leave the $x - y$ plane.

harmonic:

```
./orbit.py  --norbit 3  --potentialfile=harmonique.py --vy=1.5 --vz=1.3
```

Similarly, we also need to give the particle a small initial velocity in z direction for the Miyamoto-Nagai potential as well. To demonstrate that this will be a resonant orbit, more work is needed. A clean way of showing it is by computing the epicycle frequencies (which will be done in the next exercise session). We arrive at For the Miyamoto-Nagai potential:

$$\Omega^2(R) = \frac{GM}{(R^2 + (a+b)^2)^{3/2}} \tag{19}$$

$$\nu^2(R) = \frac{GM(a+b)}{b(R^2 + (a+b)^2)^{3/2}} \tag{20}$$

Now we choose a and b such that e.g. $\frac{\nu}{\Omega}$ is a rational number. For example, let $a = 5$ and $b = 4$. Then

$$\frac{\nu}{\Omega} = \sqrt{\frac{a+b}{b}} = \pm\frac{3}{2} \tag{21}$$

Make sure to set these values for a and b in `miyamoto.py` and don't forget to give the script some small initial value for `vz`, otherwise you'll only get circular orbits, as that's what the script does by default.

miyamoto-nagai:

```
python3 orbit.py --norbit 10 --potentialfile=miyamoto.py --x=1 --vz=0.01
```