Stellar orbits

3rd part

Outlines

Nearly circular orbits

- Epicycle frequencies

Motions of stars in the Sun neighbourhood

- The Oort constants
- Probing the mass in the stellar disk

Surfaces of section

- Integral of motions
- Poincaré maps

Stellar orbits

Nearly circular orbits

Orbital motions

$$\begin{cases} \ddot{x} = -\varkappa^2(R_g)\, x \\ \ddot{z} = -\nu^2(R_g)\, z \end{cases}$$

$+ \quad R^2\dot{\theta} = L_z$

Solutions

① motion in z

$$\boxed{z(t) = Z\cos(\nu t + \xi)}$$

② motion in x

$$\boxed{x(t) = X\cos(\varkappa t + \alpha)}$$

Note valid only for small oscillations

as long as $\nu^2 = \dfrac{\partial^2\phi}{\partial z^2} \approx$ cte

ie $\rho_{disk} \approx$ cte $\quad \left(\nu^2 = \dfrac{\partial^2\phi}{\partial z^2} = 4\pi G\rho \right)$

→ $z <$ disk scale lenght

~ 300 pc

③ motion in θ

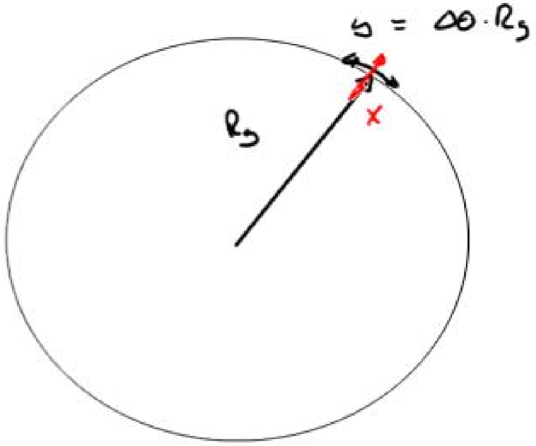
$$\dot{\theta} = \frac{L_z}{R^2} \overset{x := R - R_g}{=} \frac{L_z}{(x+R_g)^2} = \frac{L_z}{R_g^2\left(\frac{x}{R_g}+1\right)^2} \overset{\text{Taylor} \;+\; \Omega_g = \frac{L_z}{R_g^2}}{\simeq} \Omega_g\left(1 - \frac{2x}{R_g}\right)$$

introducing $x(t)$

$$\dot{\theta}(t) = \Omega_g\left(1 - \frac{2X\cos(\kappa t + \alpha)}{R_g}\right)$$

$$\boxed{\theta(t) = \underbrace{\Omega_g \cdot t}_{\substack{\text{motion of the}\\ \text{guiding center}\\ \text{along the circular}\\ \text{orbit}}} - \underbrace{\gamma\frac{X}{R_g}\sin(\kappa t + \alpha)}_{\text{oscillations}} + \theta_0} \qquad \gamma := \frac{2\Omega_g}{\kappa}$$

New carthesian system x, y, z with an origin that follows the guiding center



$$\begin{cases} R(t) = R_g \\ \theta(t) = \Omega_g t + \theta_0 \end{cases}$$

Then, from

$$\theta(t) = \Omega_g \cdot t \underbrace{- \gamma \frac{X}{R_g} \sin(\varkappa t + \alpha)}_{\Delta\theta = \frac{y}{R_g}} + \theta_0$$

$$y = -\gamma X \sin(\varkappa t + \alpha)$$

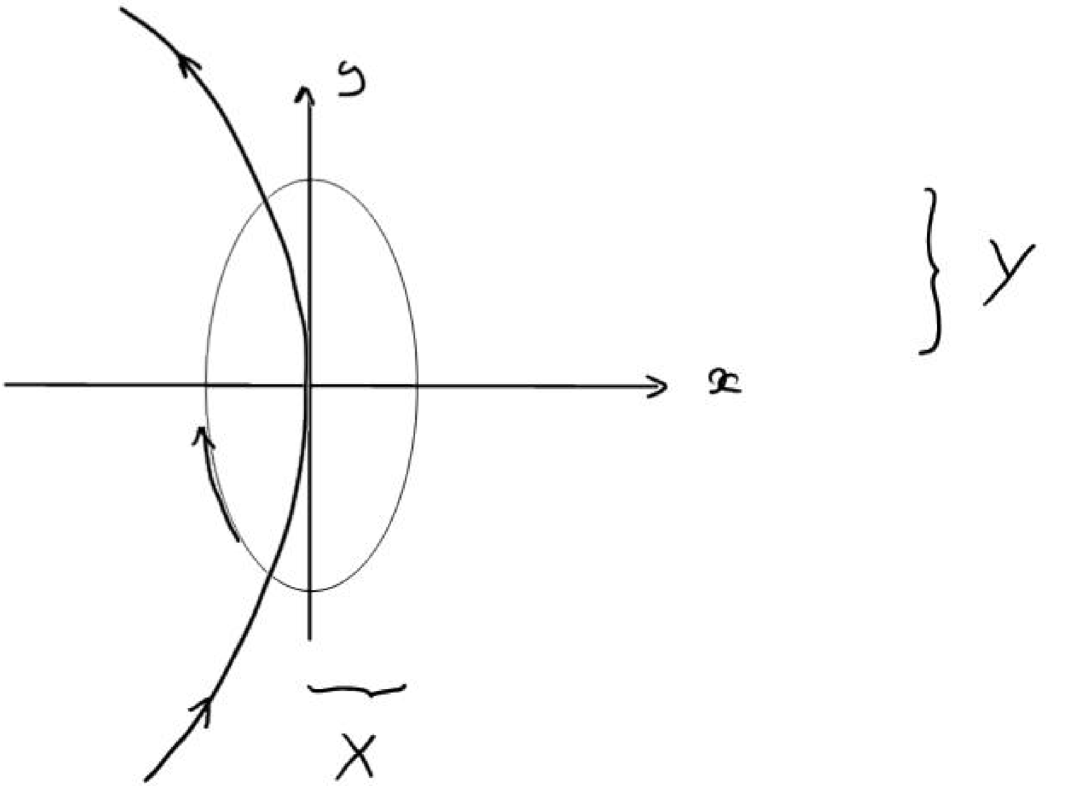
$$\boxed{y(t) = -Y \sin(\varkappa t + \alpha)} \qquad Y := \gamma X$$

Complete solution

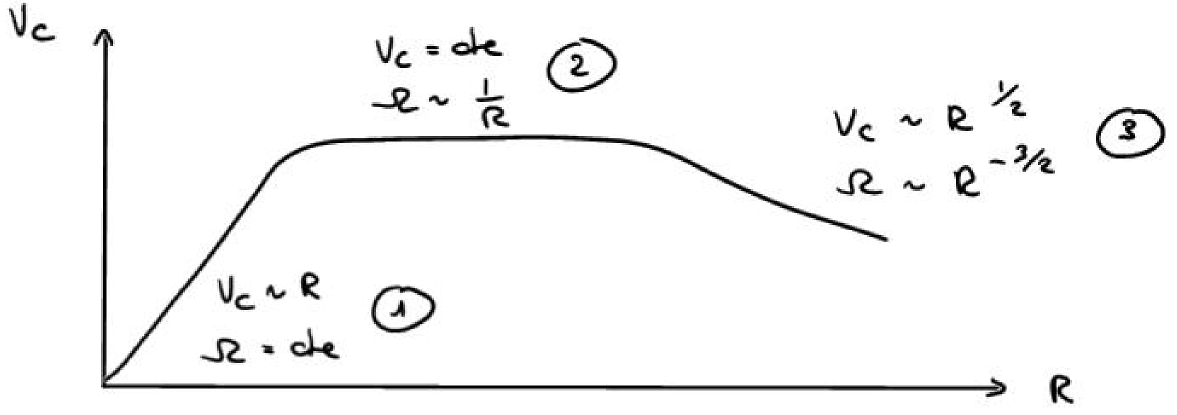
$$\boxed{\begin{cases} x(t) = X\cos(\varkappa t + \alpha) \\ y(t) = -Y\sin(\varkappa t + \alpha) \\ z(t) = Z\cos(\nu t + \xi) \end{cases}}$$

$\Big\}$ ellipse

$$Y = \frac{2\Omega_y}{\varkappa} X$$



Radial dependency for a typical galaxy



① **near the center**

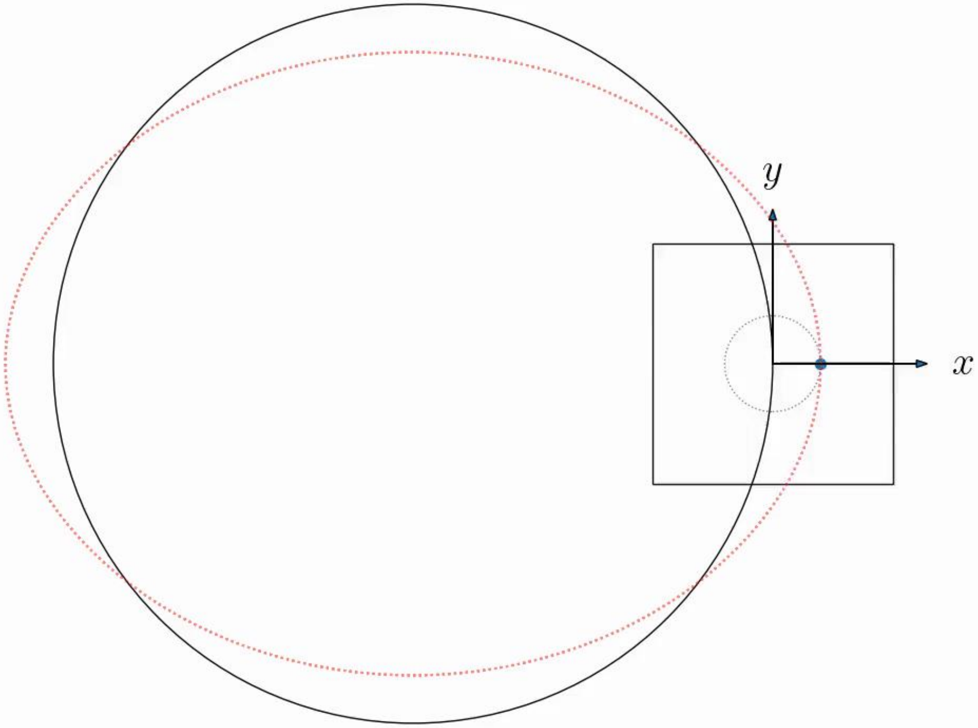
$\varkappa = 2\Omega$ $\quad \frac{X}{Y} = 1$ $\quad$ circle

② **flat rotation part**

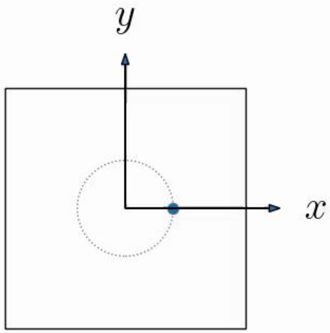
$\varkappa = \sqrt{2}\,\Omega$ $\quad \frac{X}{Y} = \frac{\sqrt{2}\,\Omega}{2\Omega}$ $\quad X < Y$

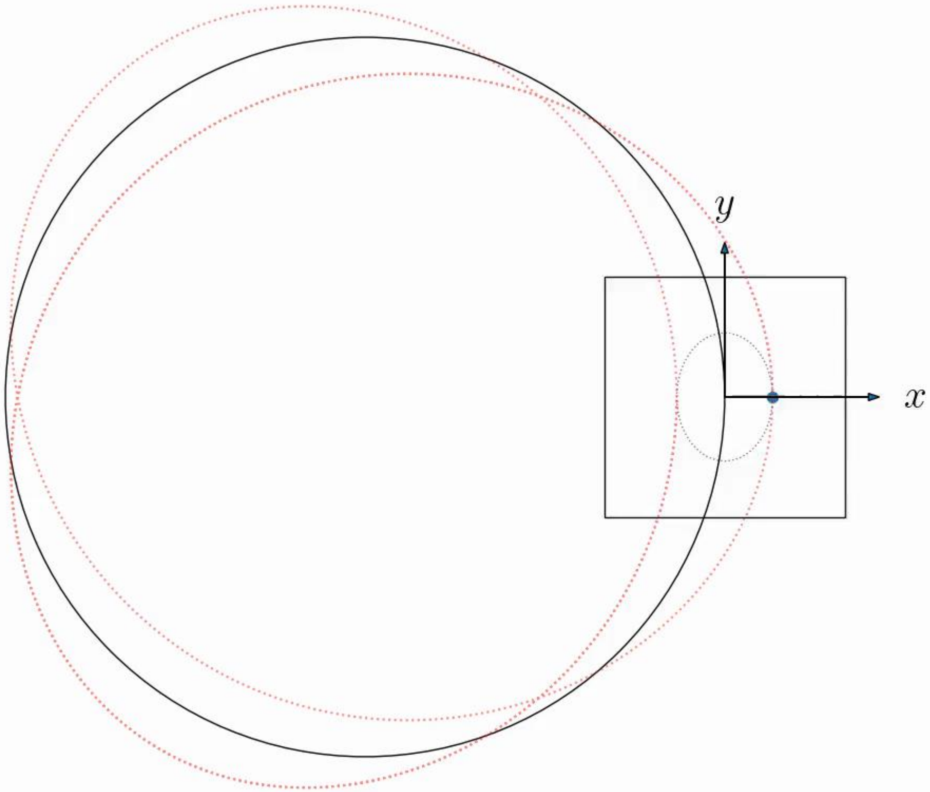
③ **further out**

$\varkappa = \Omega$ $\quad \frac{X}{Y} = \frac{\Omega}{2\Omega}$ $\quad X < Y$

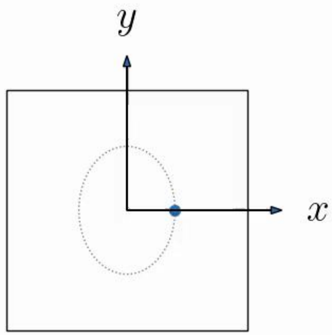


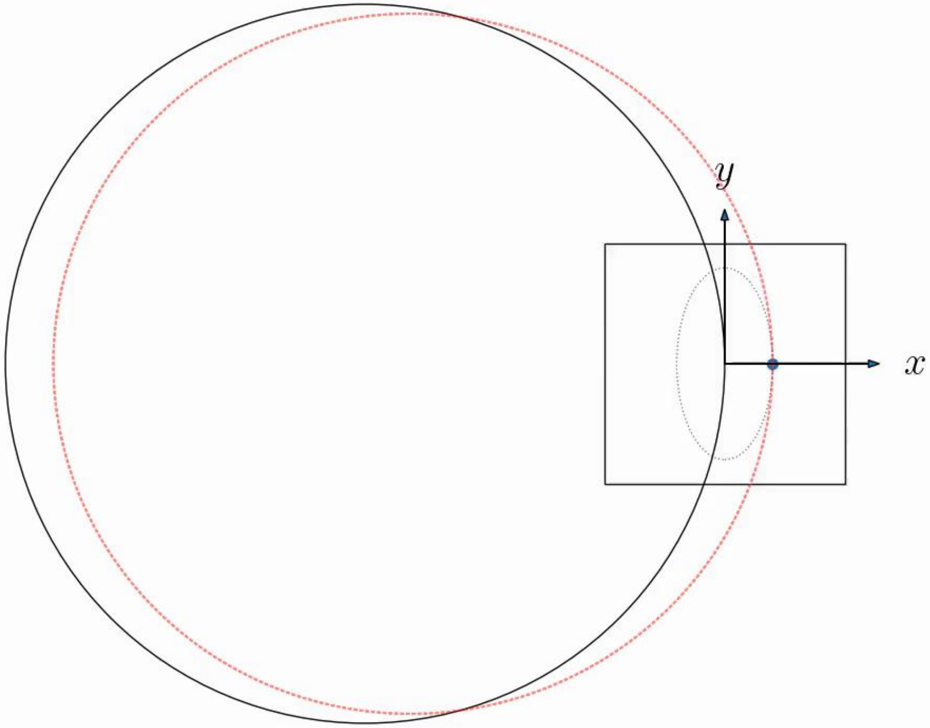
$$\kappa/\Omega = 2.0$$



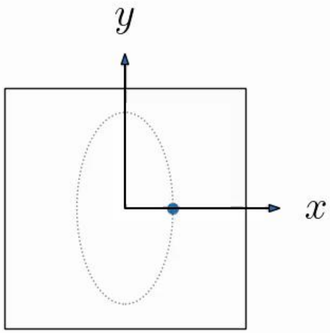


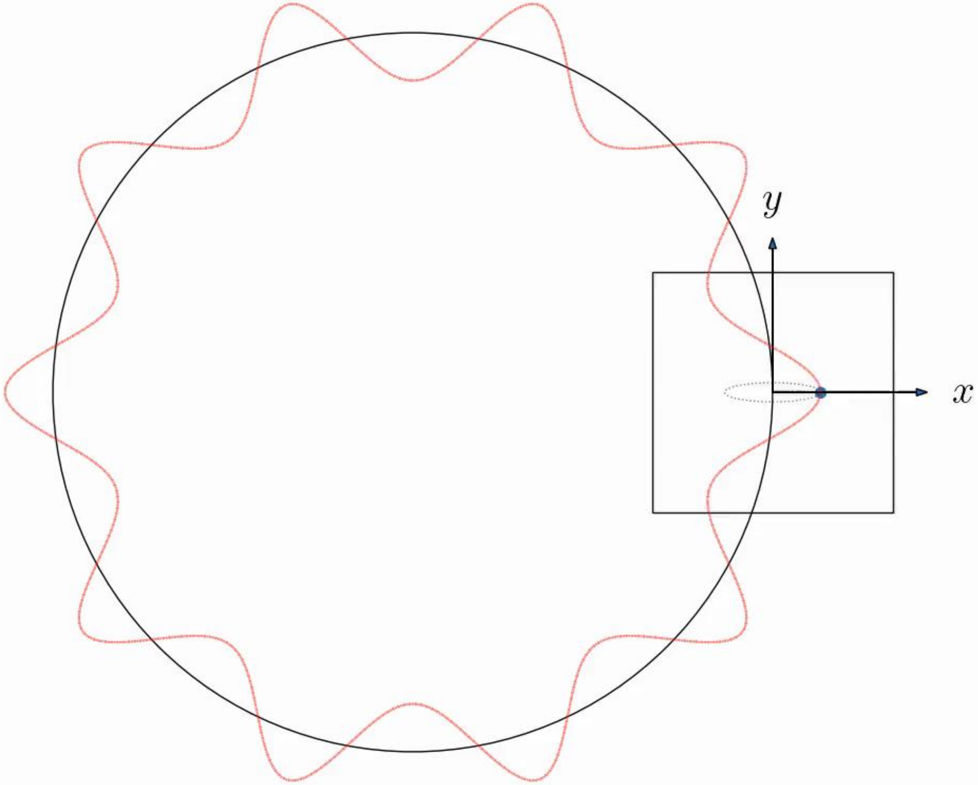
$$\kappa/\Omega = 1.5$$



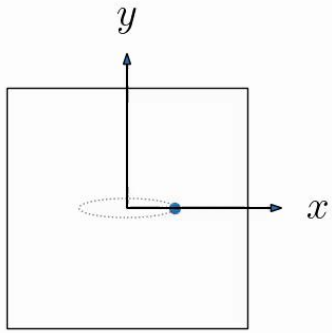


$$\kappa/\Omega = 1.0$$





$$\kappa/\Omega = 10.0$$



Stellar orbits

Motions of stars in the Sun neighbourhood

The Oort constants

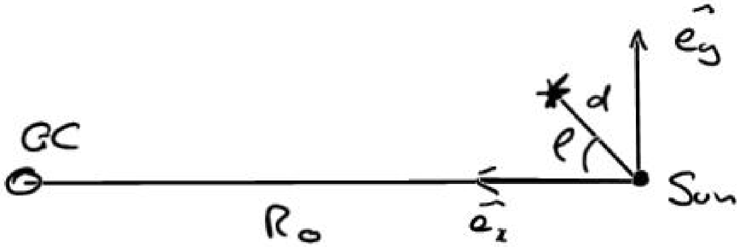
Motions of stars in the neighbourhood of the Sun

- How can we learn about the global motions of stars in the Milky Way ?
- Problem : we are living around the Sun, which is also moving around the Milky Way center ...
- Solution :
  - describe in a general framework the motions of nearby stars
  - deduce from observations of nearby stars global motions of the Milky Way

Motions of nearby stars

$\vec{x}$: position of a nearby star

d : distance to the star



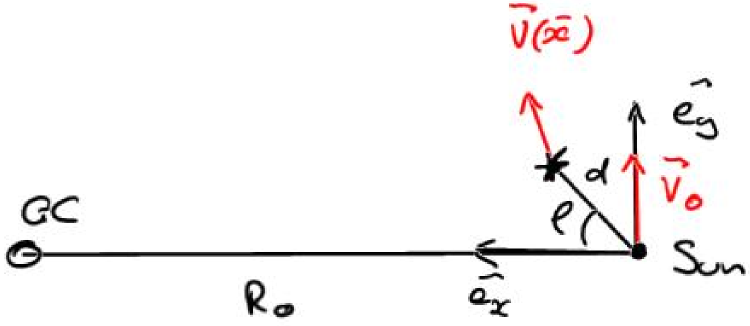
$$\vec{x} = \begin{cases} x = d\cos\ell \\ y = d\sin\ell \end{cases}$$

R_0 : distance to the GC

Velocities of stars

we assume a continuous velocity field $\vec{V}(\vec{x})$

⚠ no stars at the same position with different velocities



Sun velocity

$$\vec{V}_0 = \vec{V}(\vec{x}_0)$$

Taylor expansion of the velocity field around the Sun

$$\vec{V}(\vec{x}) = \vec{V}_\odot + \overset{\text{Jacobian matrix}}{\begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} \end{pmatrix}_{\vec{x}_0}} (\vec{x} - \underbrace{\vec{x}_0}_{=0 \text{ at the origin}})$$

Relative velocity field $\delta\vec{V}(\vec{x}) = \vec{V}(\vec{x}) - \vec{V}_\odot$

$$\begin{cases} \delta V_x = \frac{\partial V_x}{\partial x} x + \frac{\partial V_x}{\partial y} y \\ \delta V_y = \frac{\partial V_y}{\partial x} x + \frac{\partial V_y}{\partial y} y \end{cases}$$

$$\begin{pmatrix} K + C & A - B \\ A + B & K - C \end{pmatrix} := \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} \end{pmatrix}$$

A, B, C, K

The Oort constants describe the local velocity field

$$\begin{cases} A = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & K = \frac{1}{2}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) \\ B = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) & C = \frac{1}{2}\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right) \end{cases}$$

Interpretation

- "A" : radial shear $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ • GC 

- "B" : vorticity $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ • GC 

- "C" : azimuthal shear $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ • GC 

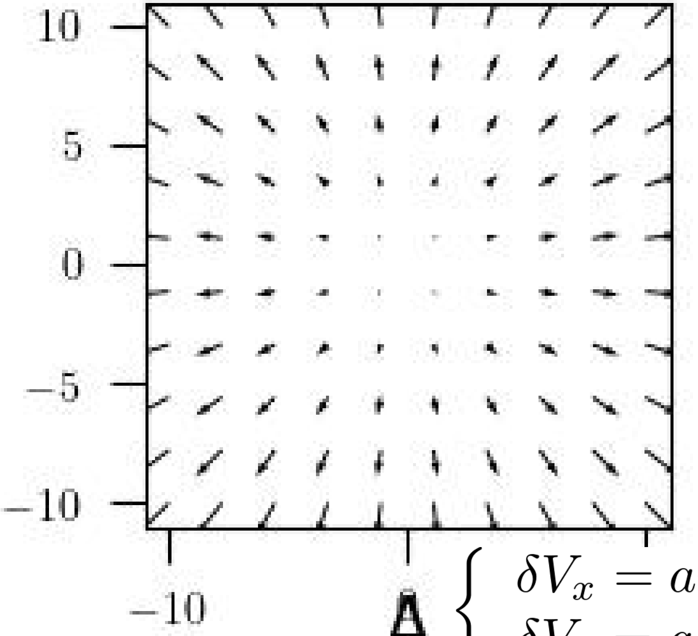
- "K" : divergeance $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ • GC 

The local velocity field (Jacobian matrix) may be decomposed on those basis.

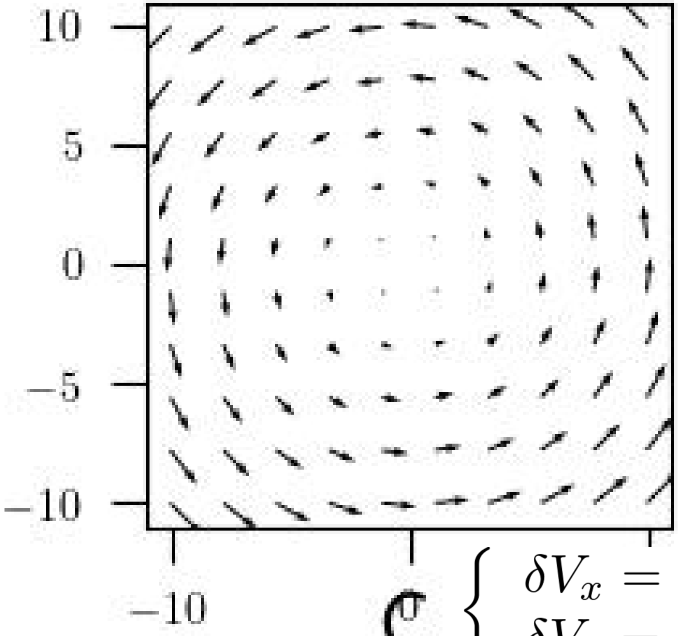
$$A \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + C \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + K \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note: we could do all this in 3D

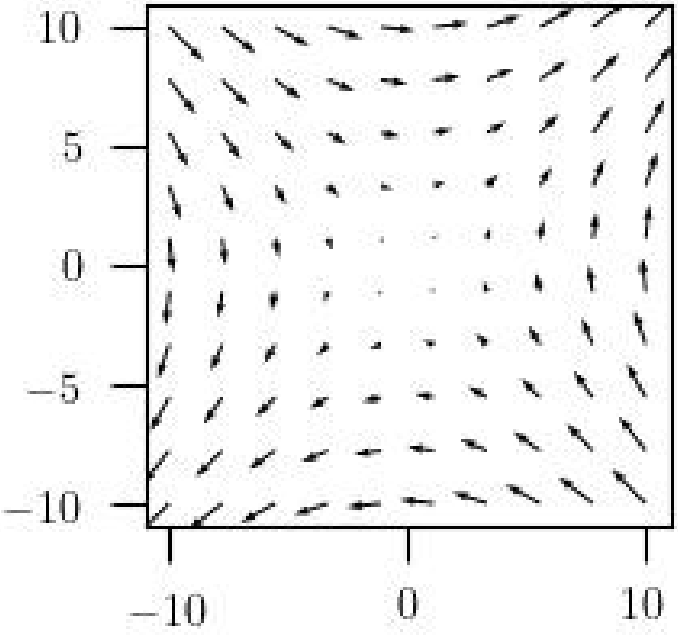
K $\begin{cases} \delta V_x = kx \\ \delta V_y = ky \end{cases}$



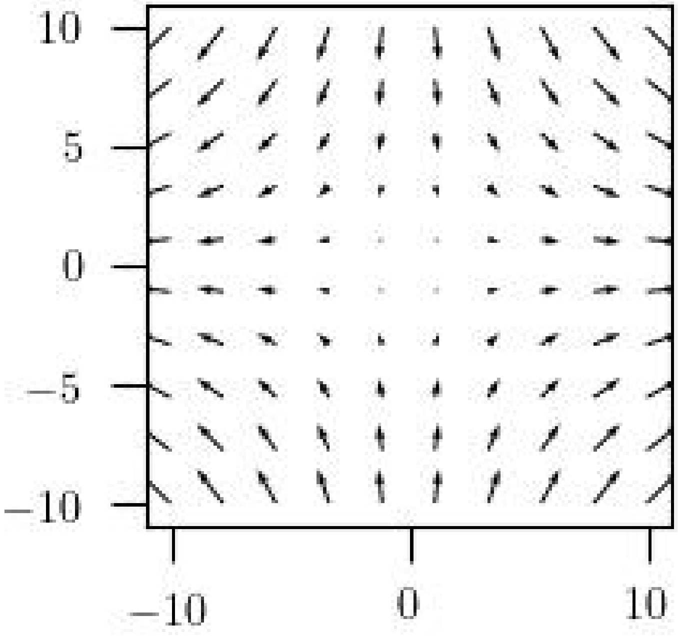
B $\begin{cases} \delta V_x = -by \\ \delta V_y = bx \end{cases}$

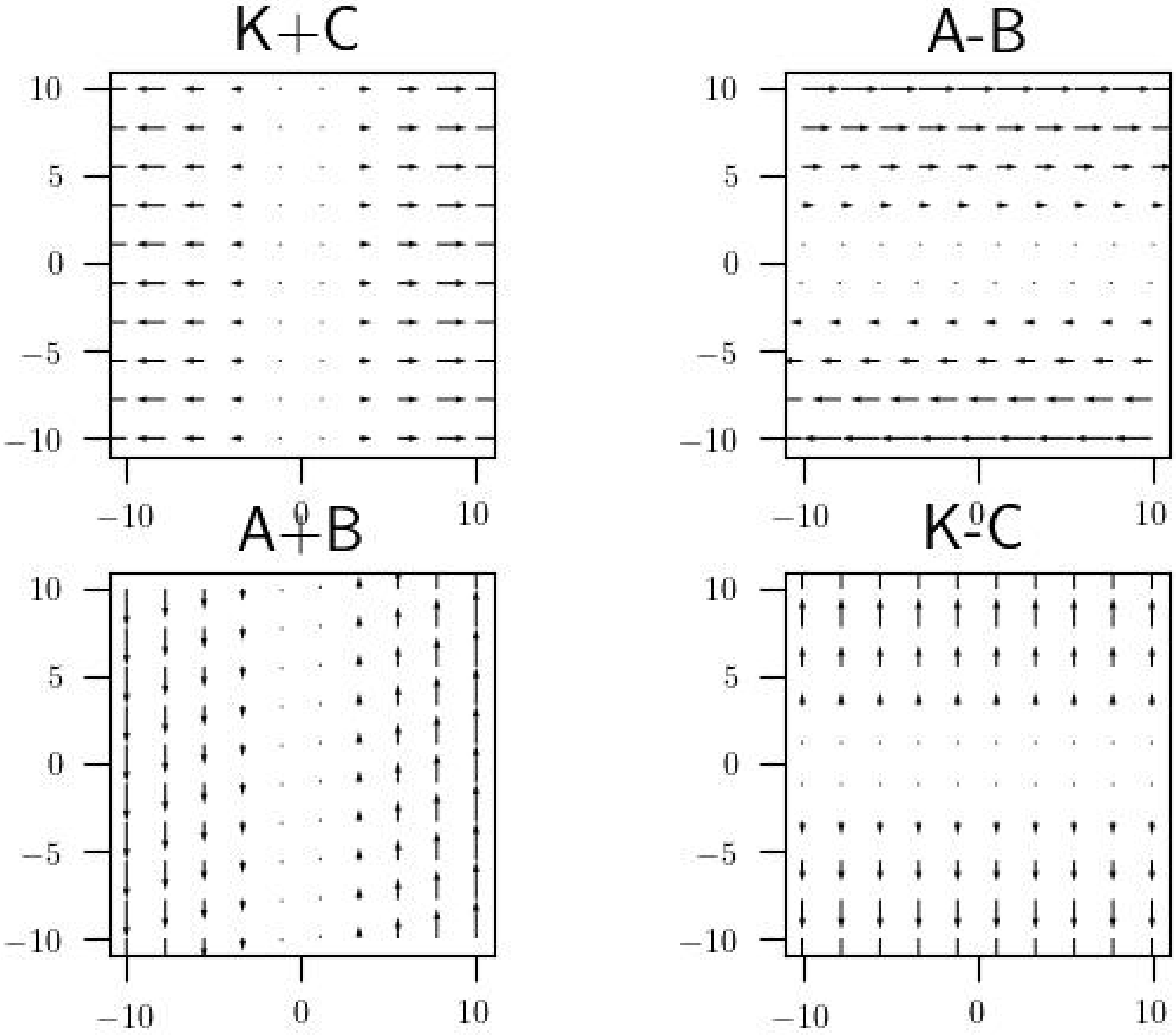


A $\begin{cases} \delta V_x = ay \\ \delta V_y = ax \end{cases}$



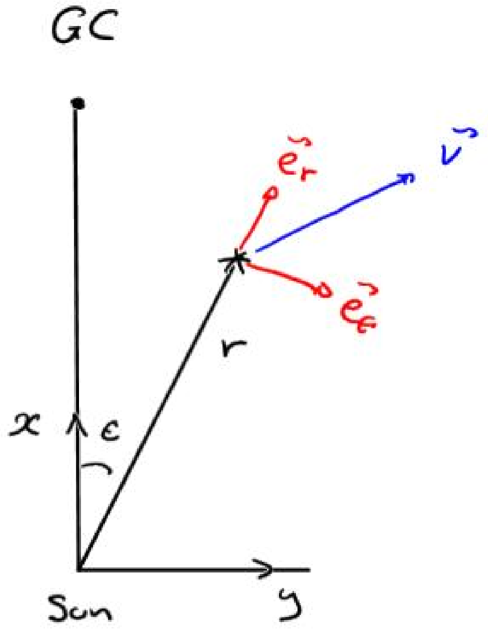
C $\begin{cases} \delta V_x = cx \\ \delta V_y = -cy \end{cases}$




K+C
A-B
A+B
K-C
10
5
0
−5
−10
−10
0
10

In term of radial and tangential velocities V_r , V_θ

polar coordinates $x = r\cos\theta$ $y = r\sin\theta$



$$V_r = \vec{v}\cdot\vec{e_r} = \vec{v}\cdot\frac{\vec{x}}{r} = \frac{1}{r}\left(x V_x + y V_y\right)$$

$$= \cos\theta\, V_x + \sin\theta\, V_y$$

$$V_\theta = \left|\vec{v}\times\frac{\vec{x}}{r}\right| = \frac{1}{r}\left(x V_y - y V_x\right)$$

$$= \cos\theta\, V_y - \sin\theta\, V_x$$

$$\boxed{\begin{aligned}
A &= \frac{1}{2}\left(-\frac{1}{r}\frac{\partial V_r}{\partial\theta} + \frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)\\
B &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)\\
C &= \frac{1}{2}\left(-\frac{1}{r}\frac{\partial V_\theta}{\partial\theta} - \frac{V_r}{r} + \frac{\partial V_r}{\partial r}\right)\\
K &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r} + \frac{\partial V_r}{\partial r}\right)
\end{aligned}}$$

Purely axisymmetric disk

$$A = \frac{1}{2}\left(-\frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)$$

$$B = \frac{1}{2}\left(\frac{1}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)$$

$$C = \frac{1}{2}\left(-\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} - \frac{V_r}{r} + \frac{\partial V_r}{\partial r}\right)$$

$$K = \frac{1}{2}\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r} + \frac{\partial V_r}{\partial r}\right)$$

$$\frac{\partial}{\partial \theta} = 0 \qquad V_r = 0 \qquad \frac{\partial V_r}{\partial r} = 0$$

$$A = \frac{1}{2}\left(\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)$$

$$B = \frac{1}{2}\left(-\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)$$

$$C = 0$$

$$K = 0$$

With $V_\Theta \equiv V_c$ (circular velocity)

$$A(R) := \frac{1}{2}\left[\frac{V_c(R)}{R} - \frac{d}{dR}V_c(R)\right] \equiv -\frac{1}{2}R\frac{d\Omega(R)}{dR}$$

$$B(R) := -\frac{1}{2}\left[\frac{V_c(R)}{R} + \frac{d}{dR}V_c(R)\right] \equiv -\left(\Omega(R) + \frac{1}{2}R\frac{d\Omega(R)}{dR}\right)$$

We can express Ω and $\varkappa$ from the Oort constants

$$\Omega = A - B$$

$$\varkappa^2 = -4B(A-B) = -4B\Omega$$

Rigid rotation $\Omega = cte$

$\frac{\partial}{\partial\theta} = 0$ $v_r = 0$ $\frac{\partial V_r}{\partial r} = 0$ $\frac{\partial \Omega}{\partial r} = 0$

$$A = 0 \qquad C = 0$$

$$B = -\Omega \qquad K = 0$$

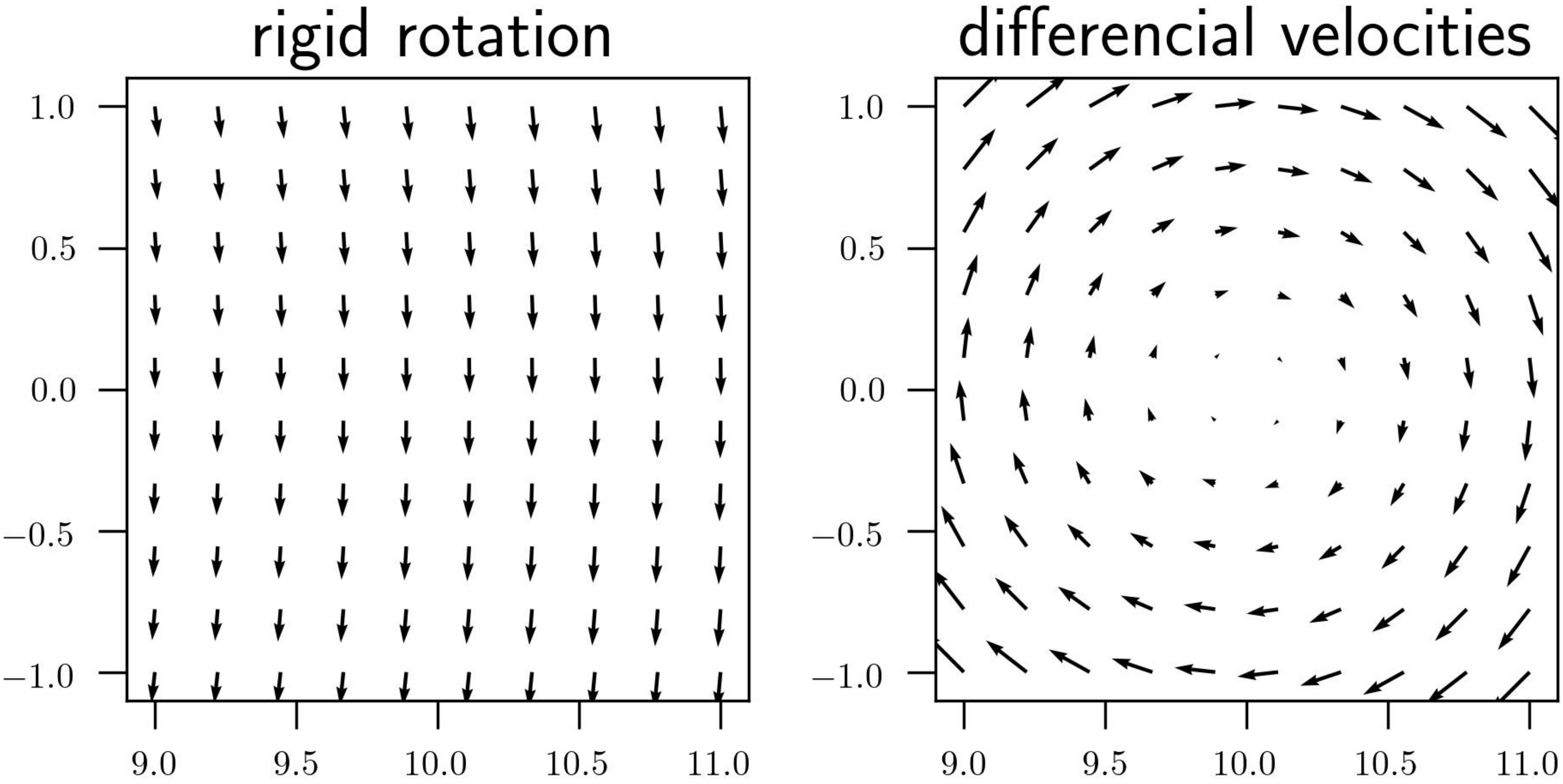
$$A = \frac{1}{2}\left(\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$B = \frac{1}{2}\left(-\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$C = 0$$

$$K = 0$$

$$J = \begin{pmatrix} 0 & A-B \\ A+B & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}$$



rigid rotation

differencial velocities

$$A = \frac{1}{2}\left(\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$B = \frac{1}{2}\left(- \frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$C = 0$$

$$K = 0$$

Rigid rotation $\Omega = \text{cte}$

$$\frac{\partial}{\partial \theta} = 0 \qquad V_r = 0 \qquad \frac{\partial V_r}{\partial r} = 0 \qquad \frac{\partial \Omega}{\partial r} = 0$$

$$A = 0 \qquad C = 0$$

$$B = -\Omega \qquad K = 0$$

$$J = \begin{pmatrix} 0 & A-B \\ A+B & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}$$

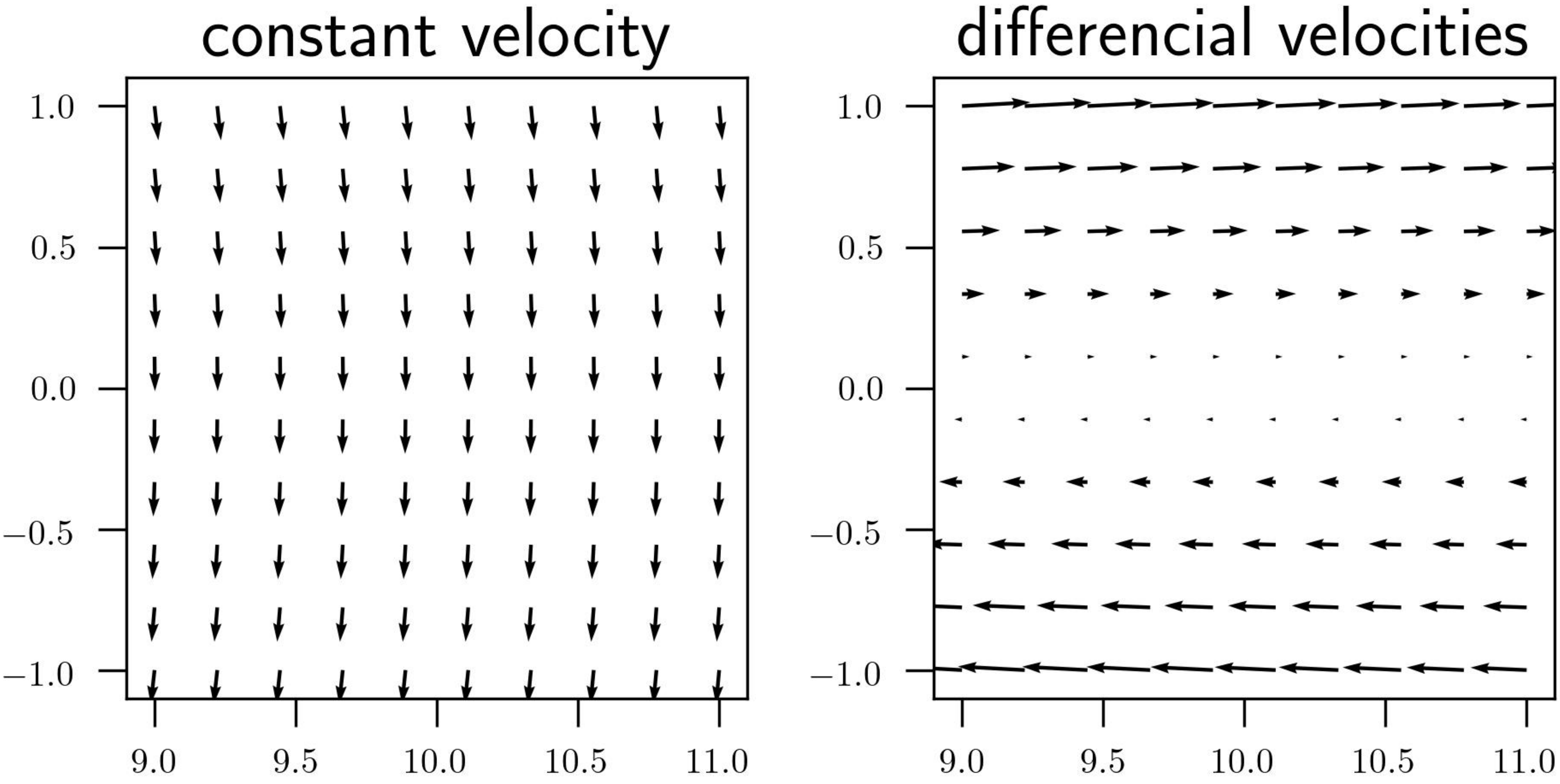
Constant rotation curve

$$\frac{\partial}{\partial \theta} = 0 \qquad V_r = 0 \qquad \frac{\partial V_r}{\partial r} = 0 \qquad \frac{\partial V_\theta}{\partial r} = 0$$

$$A = \frac{1}{2}\Omega \qquad C = 0$$

$$B = -\frac{1}{2}\Omega \qquad K = 0$$

$$J = \begin{pmatrix} 0 & A-B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ 0 & 0 \end{pmatrix}$$


constant velocity
differencial velocities
1.0
0.5
0.0
−0.5
−1.0
9.0
9.5
10.0
10.5
11.0

Keplerian decrease

$\frac{\partial}{\partial \theta} = 0 \qquad v_r = 0 \qquad \frac{\partial v_r}{\partial r} = 0 \qquad V_\theta \sim r^{-1/2}$

$$A = \frac{1}{2}\left(\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$B = \frac{1}{2}\left(-\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

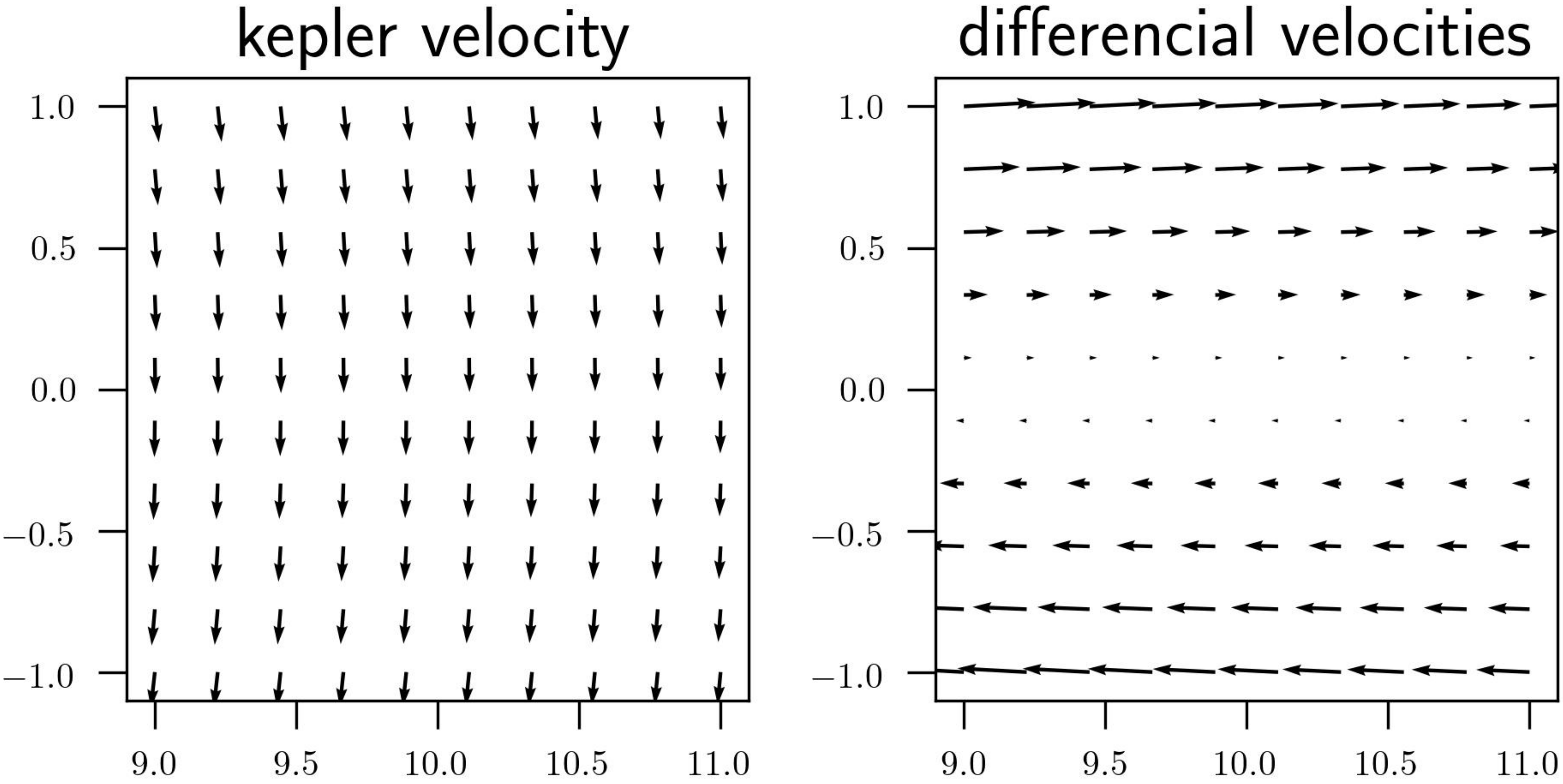
$$C = 0$$

$$K = 0$$

$A = \frac{3}{4}\Omega \qquad C = 0$

$B = -\frac{1}{4}\Omega \qquad K = 0$

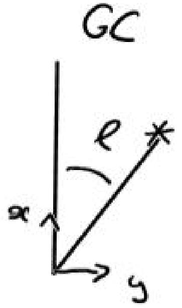
$$J = \begin{pmatrix} 0 & A-B \\ A+B & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \Omega \\ \frac{1}{2}\Omega & 0 \end{pmatrix}$$


kepler velocity
differencial velocities
1.0
0.5
0.0
−0.5
−1.0
9.0
9.5
10.0
10.5
11.0

Can we measure the Oort constants (at the Sun location) ?

In Galacto Centric coordinates

$$(\ell, b, \underset{\underset{r}{\shortparallel}}{d})$$



with

$$\begin{cases} x = d\cos\ell \\ y = d\sin\ell \end{cases} \qquad \begin{cases} V_x = (K+C)\,x + (A-B)\,y \\ V_y = (A+B)\,x + (K-C)\,y \end{cases}$$

- $$V_r = \vec{v}\cdot\frac{\vec{x}}{r} = \frac{1}{r}\left(x V_x + y V_y\right) \equiv V_x\cos\ell + V_y\sin\ell$$

$$\boxed{V_r = d\left[K + C\cos(2\ell) + A\sin(2\ell)\right]}$$

- $$V_t = \left|\vec{v}\times\frac{\vec{x}}{r}\right| = \frac{1}{r}\left(x V_y - y V_x\right) \equiv -V_x\sin\ell + V_y\cos\ell$$

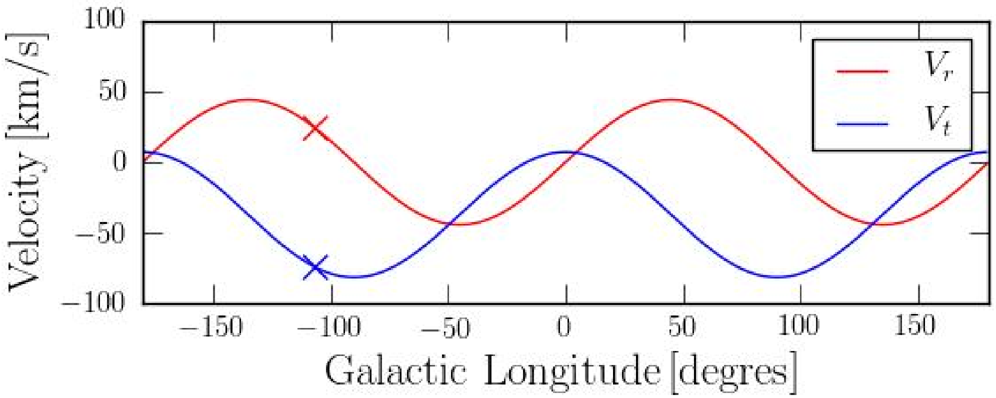
$$\boxed{V_t = d\left[B + A\cos(2\ell) - C\sin(2\ell)\right]}$$

In the axisymmetric case $C = K = 0$

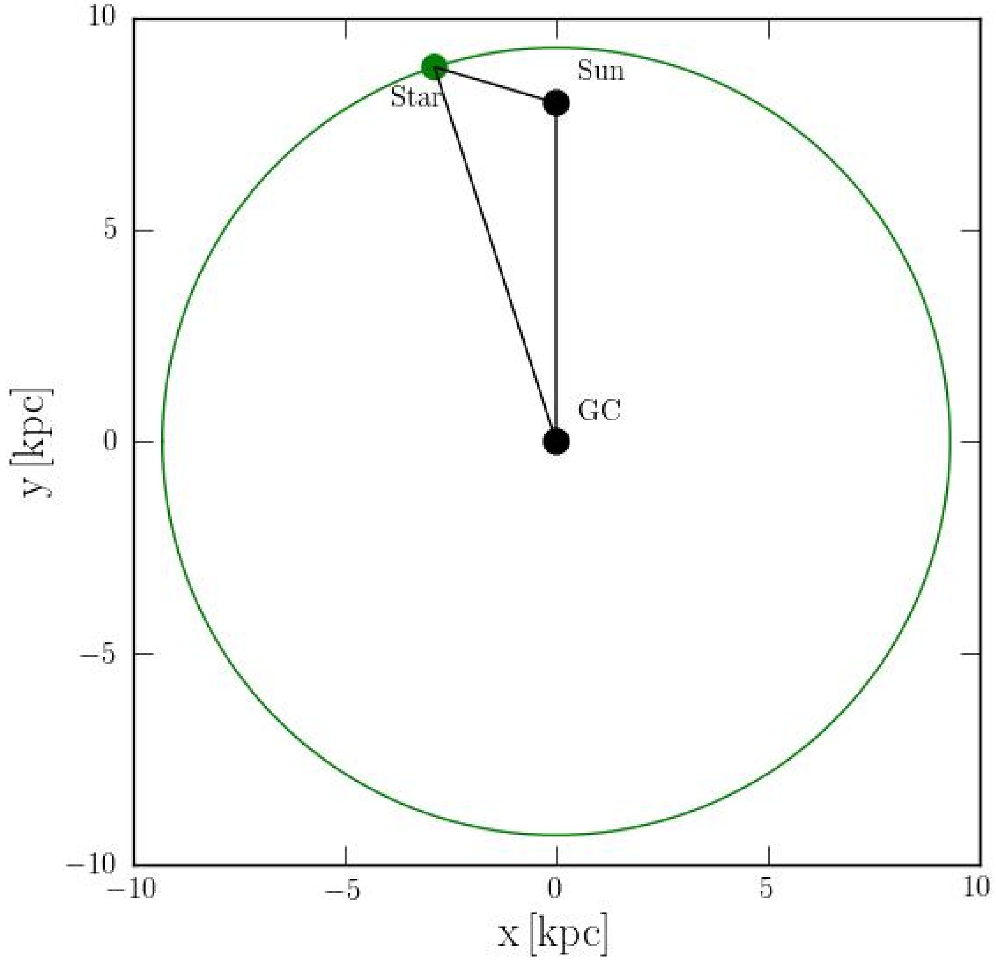
(purely circular orbits)

$$\boxed{\begin{cases} V_r = A d \sin(2\ell) \\ V_t = A d \cos(2\ell) + B d \end{cases}}$$

The Oort constants

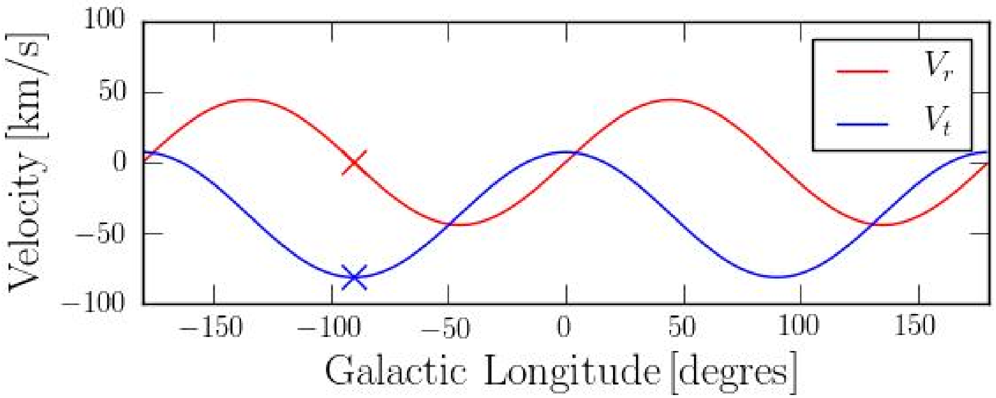


$$\begin{cases} V_r = Ad\sin(2l) \\ V_t = Ad\cos(2l) + Bd \end{cases}$$

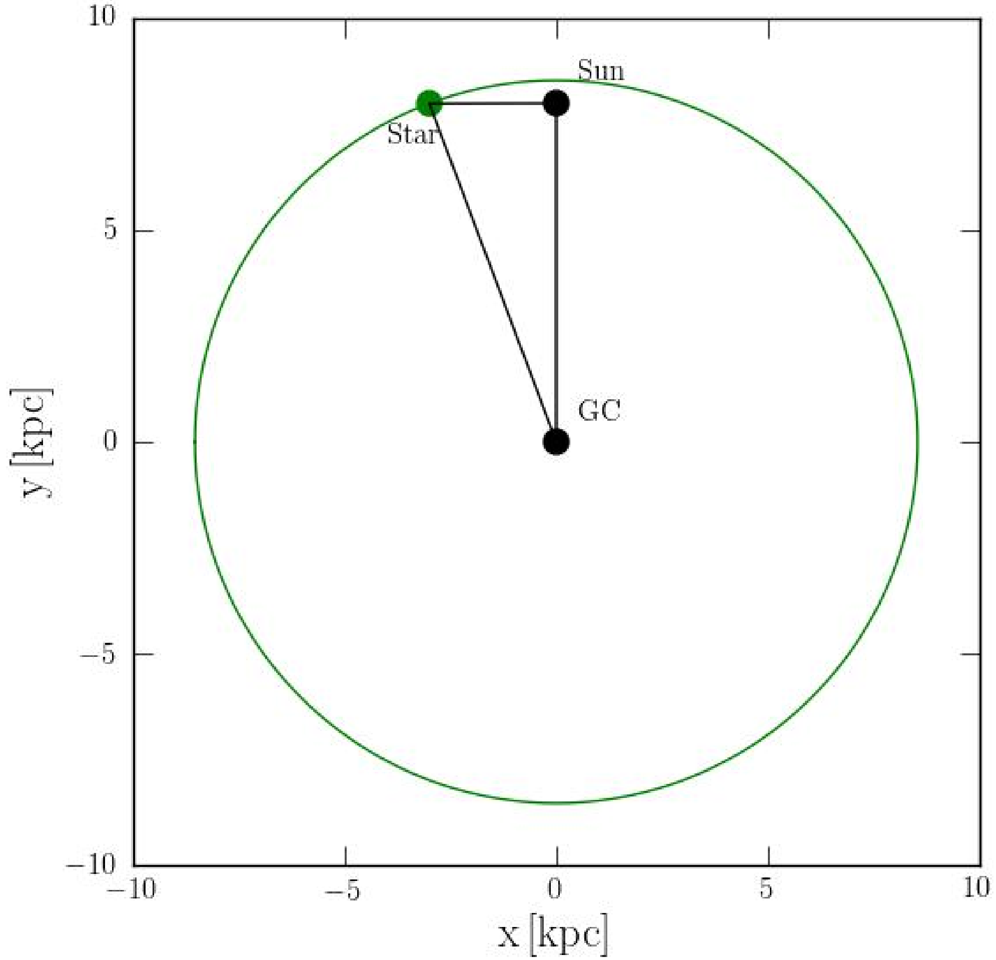


The Oort constants

$$l = -90$$



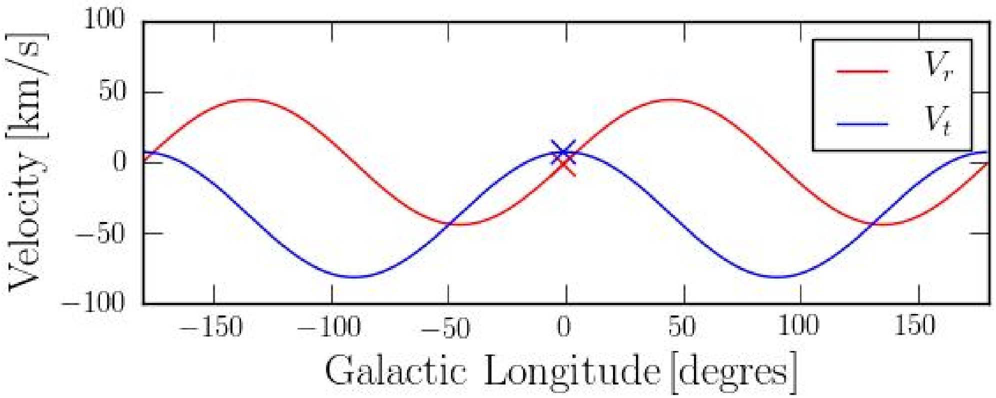
$$\begin{cases} V_r = Ad\sin(2l) \\ V_t = Ad\cos(2l) + Bd \end{cases}$$



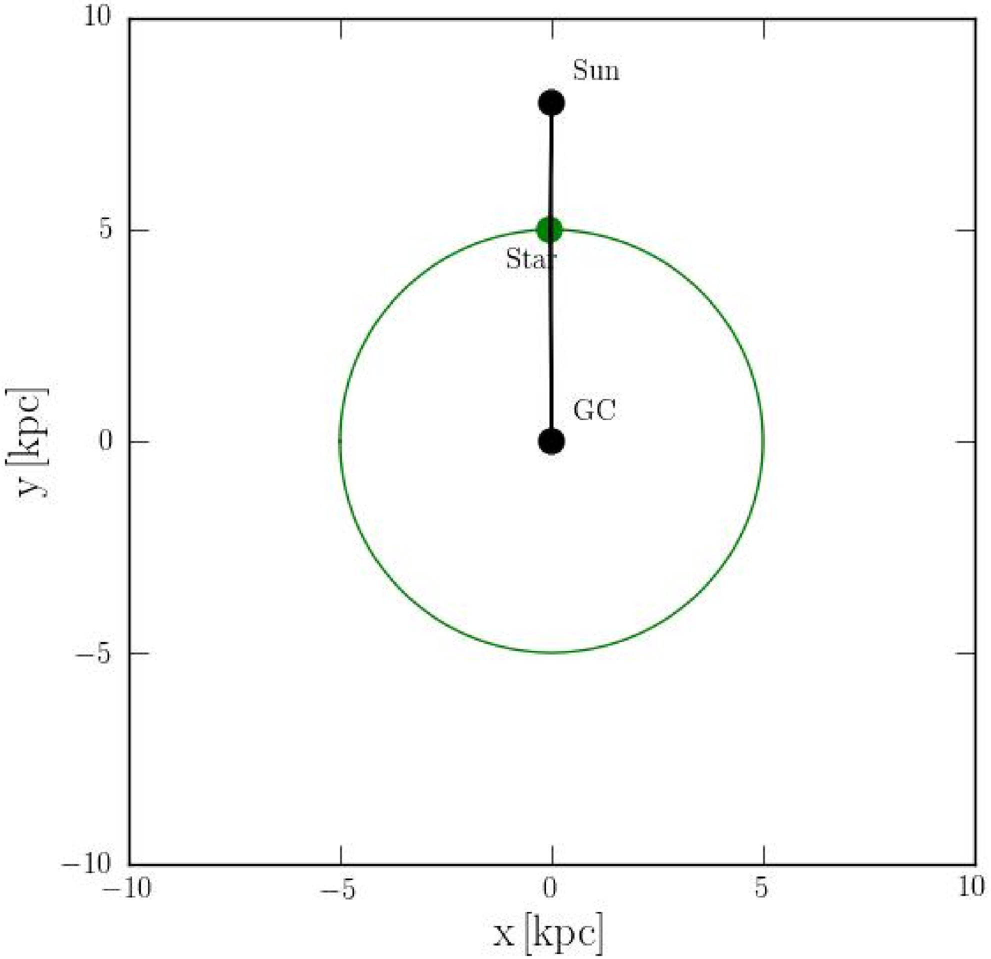
$$\begin{cases} V_r = 0 \\ V_t = (A + B)d \end{cases}$$

The Oort constants

$$l = 0$$



$$\begin{cases} V_r = Ad\sin(2l) \\ V_t = Ad\cos(2l) + Bd \end{cases}$$



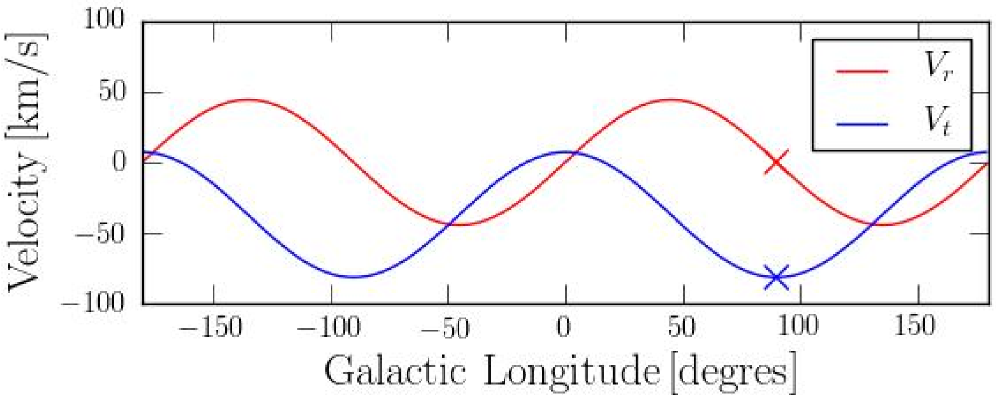
$$\begin{cases} V_r = 0 \\ V_t = (A + B)d \end{cases}$$

$$\Omega = \text{cte}$$

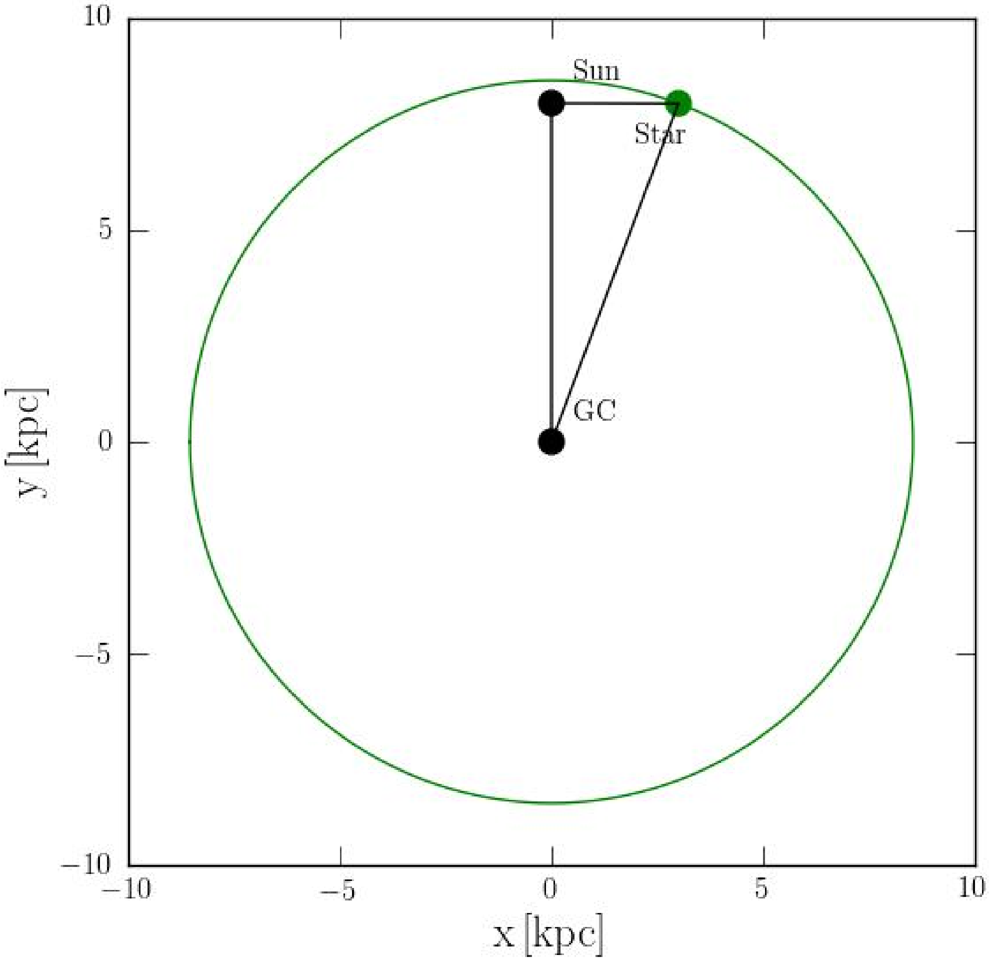
$$\begin{cases} V_r = 0 \\ V_t = -\Omega d \end{cases}$$

The Oort constants

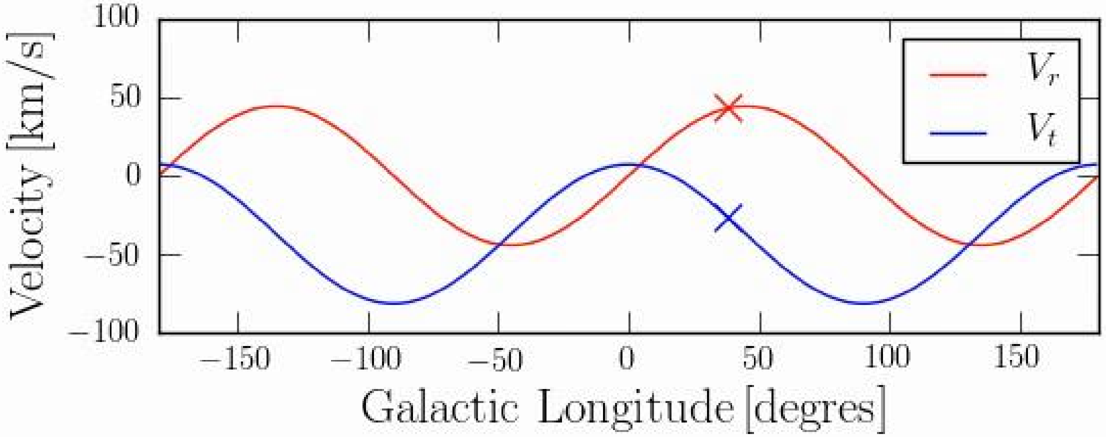
$$l = 90$$

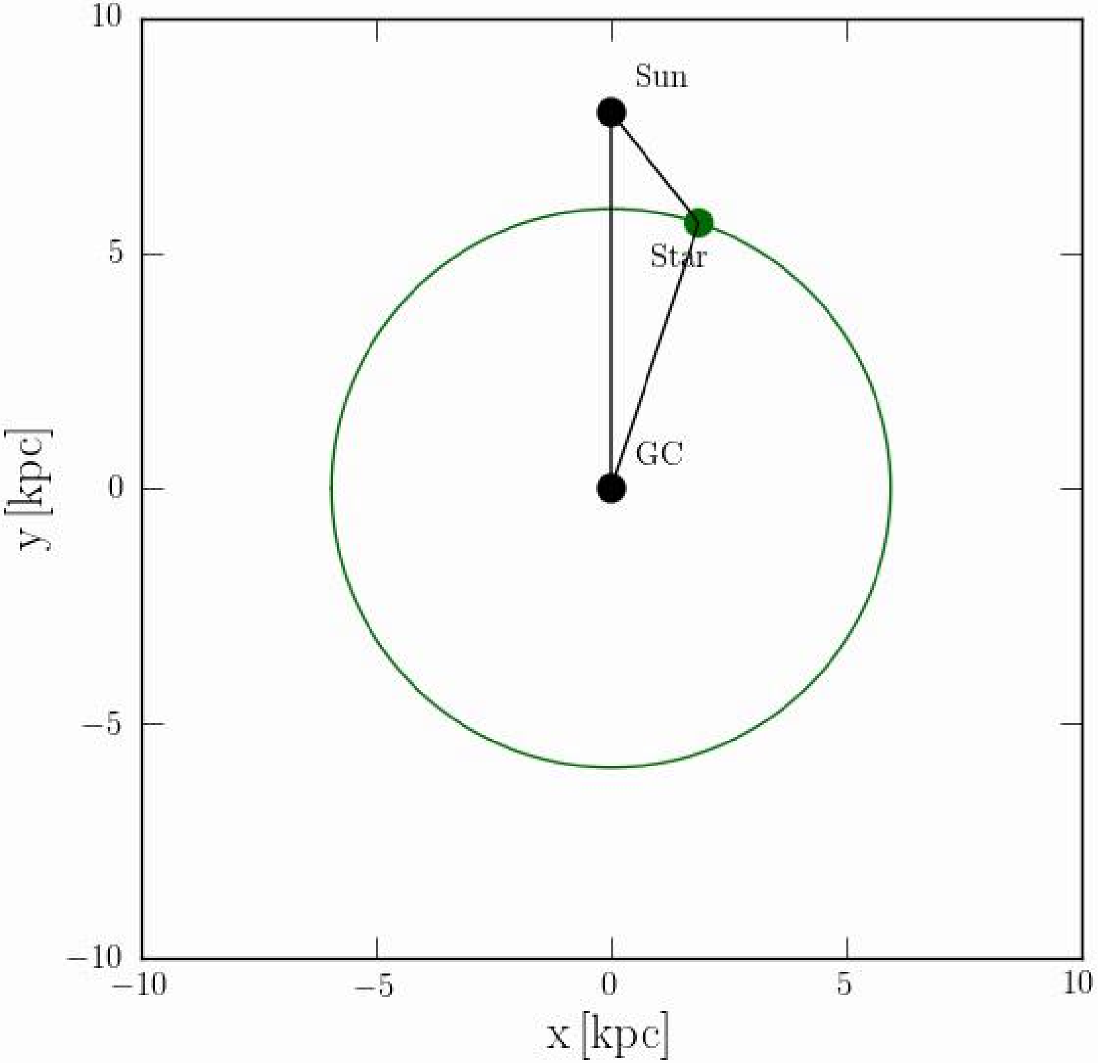


$$\begin{cases} V_r = Ad\sin(2l) \\ V_t = Ad\cos(2l) + Bd \end{cases}$$

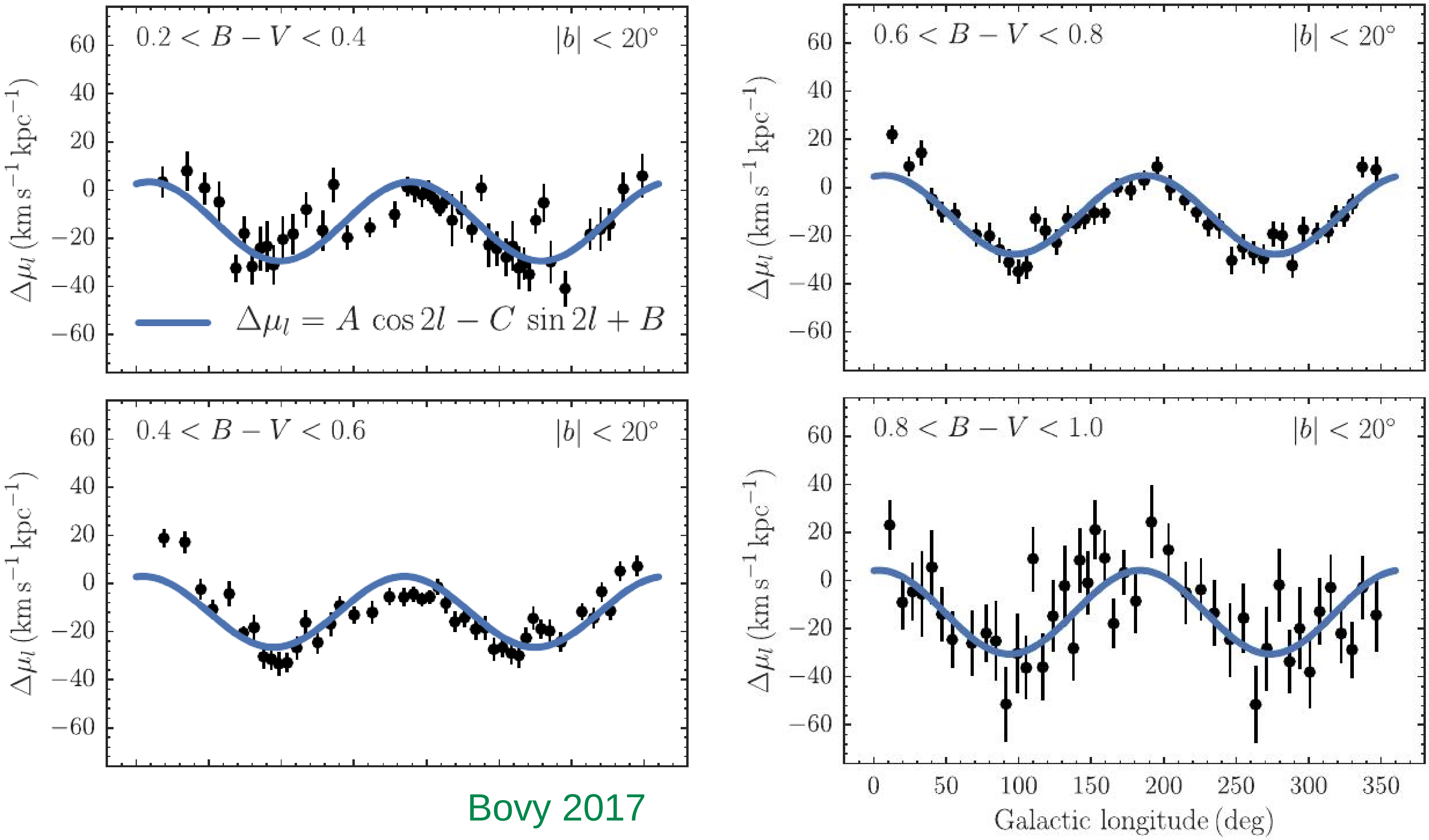


$$\begin{cases} V_r = 0 \\ V_t = (A + B)d \end{cases}$$


Velocity [km/s]
100
50
0
−50
−100
−150
−100
−50
0
50
100
150
Galactic Longitude [degres]
V_r
V_t


Sun
Star
GC
y [kpc]
10
5
0
−5
−10
−10
−5
0
5
10
x [kpc]

Proper motions measurements with GAIA



Bovy 2017

Figure 2. Comparison between the observed mean proper motion in Galactic longitude corrected for the solar motion (see equation 3) as a function of l and the best-fitting model for the four main colour bins used in the analysis. The data clearly display the expected signatures due to the differential rotation of the Galactic disc. The agreement between the model and the data is good.

The Oort constants

Monthly Notices
of the
ROYAL ASTRONOMICAL SOCIETY

MNRAS **468,** L63–L67 (2017)
Advance Access publication 2017 February 11

doi:10.1093/mnrasl/slx027

Galactic rotation in *Gaia* DR1

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$$A = 15.3 \pm 0.4\,\mathrm{km\,s^{-1}\,kpc^{-1}} \qquad B = -11.9 \pm 0.4\,\mathrm{km\,s^{-1}\,kpc^{-1}}$$

$$C = -3.2 \pm 0.4\,\mathrm{km\,s^{-1}\,kpc^{-1}} \qquad K = -3.3 \pm 0.6\,\mathrm{km\,s^{-1}\,kpc^{-1}}$$

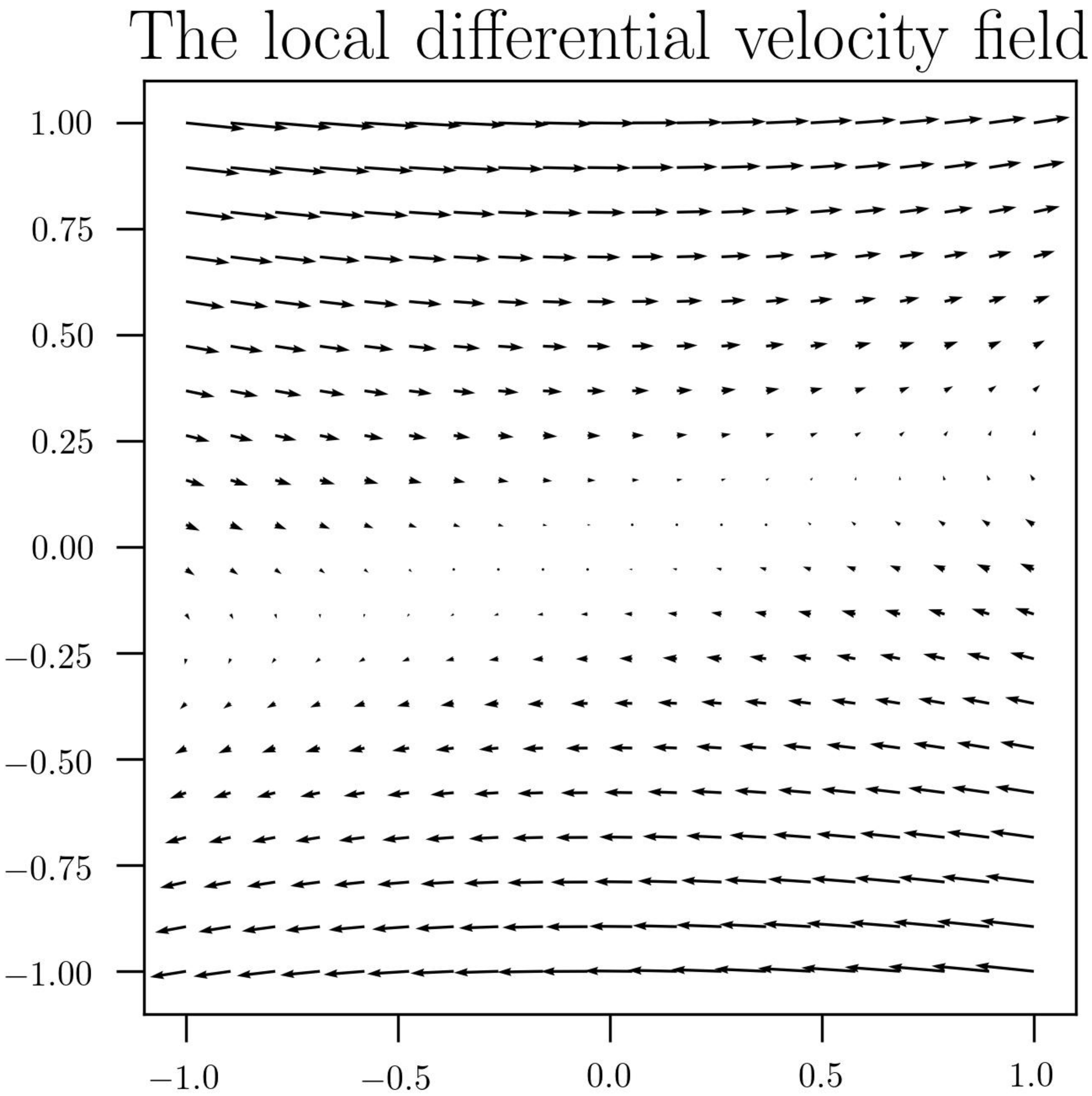
using

$$\begin{cases} \Omega = A - B \\ \kappa^2 = -4B\,(A - B) = -4\,B\,\Omega \end{cases}$$

$$\begin{cases} \Omega_0 = 27 \pm 0.8\,\mathrm{km\,s^{-1}\,kpc^{-1}} \\ \kappa_0 = 35 \pm 0.2\,\mathrm{km\,s^{-1}\,kpc^{-1}} \end{cases} \qquad \begin{cases} T_\phi \cong 227\,\mathrm{Myr} \\ T_R \cong 175\,\mathrm{Myr} \end{cases}$$

$$\frac{\kappa_0}{\Omega_0} = 2\,\frac{-B}{A - B} \cong 1.29 \qquad \kappa_0 > \Omega_0 \quad \text{as expected}$$

Shear (A+B) + compression (K+C)



Stellar Orbits

$$\nu$$

and the density relation

Galaxy properties from the vertical frequency ν

What can we learn from κ, Ω, ν ratios ?

Poisson equation in cylindrical coordinates

$$\nabla^2 \phi(R,z) = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \phi}{\partial R}\right) + \frac{\partial^2 \phi}{\partial z^2} = 4\pi G \rho(R,z)$$

$z = 0$

$$\nabla^2 \phi(R,z) = \frac{1}{R}\frac{\partial}{\partial R}\left(V_c^2\right) + \nu^2 = 4\pi G \rho(R,z=0)$$

① if $\rho(R,z) \rightarrow \delta(z)\Sigma(R)$ $\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \phi}{\partial R}\right) \ll \frac{\partial^2 \phi}{\partial z^2}$

② if $V_c = \text{cte}$, $\frac{1}{R}\frac{\partial}{\partial R}\left(V_c^2\right) = 0$

so, $\boxed{\nu^2 = 4\pi G \rho(R, z=0)}$ ①

Expected relation between $\varkappa$ and ν ?

In spherical systems : $\Omega^2 = \frac{GM(r)}{r^3} = \frac{4}{3}\pi G\bar{\rho}$ $\qquad \bar{\rho} = \frac{M(r)}{\frac{4}{3}\pi r^3}$

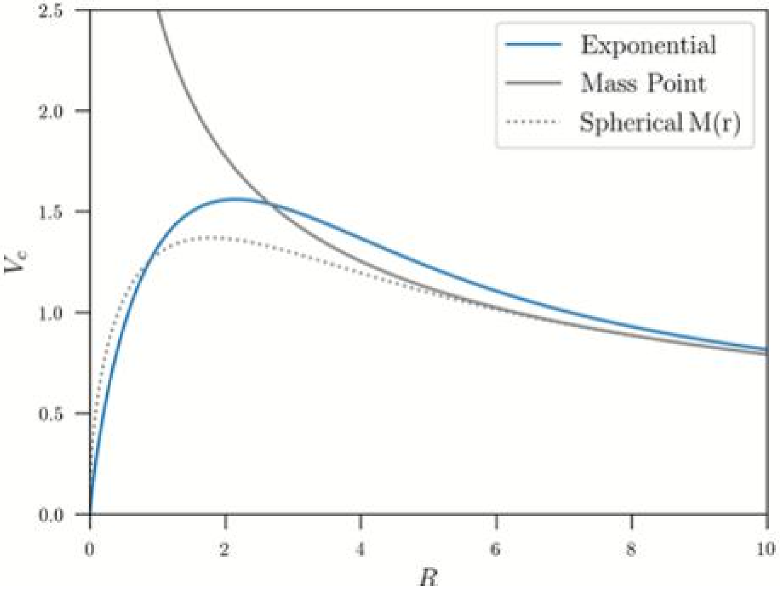
As V_c for a cylindrical distribution is not so different than a spherical one if $M(r) = M(R)$

and $\Omega \sim \frac{1}{V_c}$

For an axisymmetric disk,

we can estimate

$$\boxed{\Omega^2 = \frac{GM(R)}{R^3} = \frac{4}{3}\pi G\bar{\rho}}$$ ②



For the flat rotation curve part, we have

$$\varkappa^2 = \frac{1}{R}\frac{\partial}{\partial R}(V_c^2) + 2\Omega^2 \Rightarrow \boxed{\varkappa^2 = 2\Omega^2} \quad ③$$

Combining ① + ② + ③

$$\boxed{\frac{\nu^2}{\varkappa^2} = \frac{3}{2}\frac{\rho}{\bar{\rho}}}$$

← density in the plane

← mean density computed inside the radius

Estimation of the vertical frequency

1. using

$$\nu^2 = 4\pi G \rho_\mathrm{d}$$

$$\rho_\mathrm{d} \cong 0.1 \frac{\mathrm{M}_\odot}{\mathrm{pc}^3}$$

$$\boxed{T_z = \frac{2\pi}{\nu} = 87\,\mathrm{Myr}}$$

$$\boxed{\nu \cong 2\kappa}$$

2. using

$$\frac{\nu^2}{\kappa^2} = \frac{3}{2}\frac{\rho_\mathrm{d}}{\bar\rho}$$

$$\frac{4}{3}\pi G \bar\rho = \Omega_\odot^2 = \frac{V_{\mathrm{c},\odot}}{R_\odot^2} \qquad \left\{ \begin{array}{l} V_{\mathrm{c},\odot} \cong 200\,\mathrm{km/s} \\ R_\odot^2 \cong 8\,\mathrm{kpc} \end{array} \right.$$

$$\Rightarrow \bar\rho \cong 0.039 \frac{\mathrm{M}_\odot}{\mathrm{pc}^3}$$

$$\boxed{\frac{\nu}{\kappa} \cong 2}$$

Stellar Orbits

Integral of motion and Surfaces of section

Integrals of motion

A stellar orbit defines a path in the 6-D phase space ($x, y, z, \dot{x}, \dot{y}, \dot{z}$ in cartesian coordinates)

Definition :

An integral of motion $I\left[\mathbf{x}, \mathbf{v}\right]$ is any function of the phase-space co-ordinates alone that is constant along an orbit:

$$I\left[\mathbf{x}(t_1), \mathbf{v}(t_1)\right] = I\left[\mathbf{x}(t_2), \mathbf{v}(t_2)\right]$$

Examples :

- Hamiltonian	$H(x, y, z, \dot{x}, \dot{y}, \dot{z}) = E$
- Total angular momentum	$\vec{L}: \; L_x = \mathrm{cte}, \; \mathrm{L_y} = \mathrm{cte}, \; \mathrm{L_z} = \mathrm{cte}$
- z-component of the angular momentum	$L_z = \mathrm{cte}$

Remarks :

- Orbits may have between 0 to 5 integrals of motion.
- Integrals of motion may exist without an analytical form.

Integrals of motion

Interest of integral of motion :

Restrict the study of an orbit to a subset of the phase space

Example I :

Orbit in spherical potentials

- 6-D — 6 indep. variables $(x, y, z, \dot{x}, \dot{y}, \dot{z})$
- Angular momentum conservation
  3 integrals, 2 among the three

  $\vec{n} = \vec{L}/|\vec{L}|$ defines a plane → 4-D — 4 indep. variables $(r, \phi, \dot{r}, \dot{\phi})$
- Angular momentum conservation + energy — 2 indep. variables (r, ϕ)

  $L = |\vec{L}| \quad E$ → 2-D

$$\dot{\phi} = \frac{L}{r^2}$$

$$\dot{r} = \pm\sqrt{2(E - \Phi(r)) - L^2/r^2}$$

defines a 2-D surface

Given $E,\ \vec{L}$ the position and velocities of a star (i.e. the position in the phase space) is fully determined by providing two additional quantities, ex: $r,\ \phi$

Integrals of motion

Is there a fifth integral ?

Example of the Keplerian potential :

We showed that:

$$r(\phi) = \frac{1}{C\,\cos(\phi - \phi_0) + \frac{GM}{L^2}}$$

with:

$$E = \frac{1}{2}\,(C\,L)^2 - \frac{1}{2}\left(\frac{GM}{L}\right)^2$$

we have then the new integral of motion:

$$\phi_0 = \phi - \arccos\left[\frac{1}{C}\left(\frac{1}{r} - \frac{GM}{L^2}\right)\right]$$

→ 1-D 1 indep. variable (r)

a curve

Given $E,\ \vec{L},\ \phi_0$ the position and velocities of a star is fully determined by providing only one additional quantities, ex: r

Integrals of motion

Example II :

Orbit in axi-symmetric potentials

- 6-D

6 indep. variables $(x, y, z, \dot{x}, \dot{y}, \dot{z})$

- z-component angular momentum conservation
  1 integral $\dot{\theta} = \frac{L_z}{R^2}$

5 indep. variables $(R, z, \dot{R}, \dot{z}, \theta)$

- Initial azimuth $\theta(t) = L_z \int_{t_0}^{t} \frac{1}{R^2(t')} dt' + \theta_0$

not an integral,
a constant !

4 indep. variables $(R, z, \dot{R}, \dot{z})$
4-D (meridional plane)

- Energy E

3 indep. variables $(R, z, \dot{R})$
3-D

Given $E,\ L_z,\ \theta_0$ and t — the position and velocities of a star (in the phase space) is fully determined by providing three additional quantities, ex: $R, z, \dot{R}$

Given E — the position and velocities of a star (in the phase space of the meridional plane) is fully determined by providing three additional quantities, ex: $R, z, \dot{R}$

Is there a third integral ?

Surfaces of section

Can we visualize the phase phase and check if an additional integral of motion exists ?

<u>Idea</u> :

We study the orbits in the meridional plane

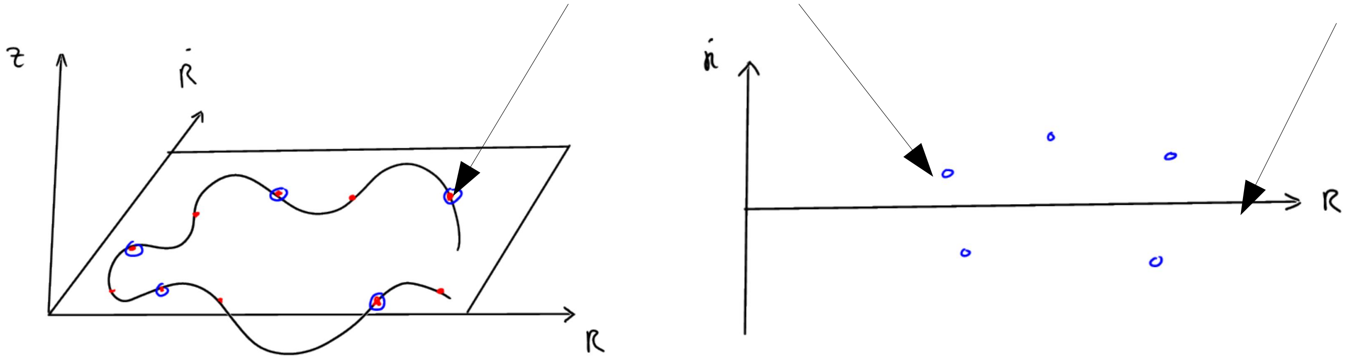
- 4-D — 4 indep. variables $(R, z, \dot{R}, \dot{z})$
- Energy E

→ 3-D — 3 indep. variables $(R, z, \dot{R})$

- Drawing a 3-D phase space is still not easy. Instead, we draw slices of the phase space. We plot only phase space points that:
  - cross the $z = 0$ plane
  - have $\dot{z} > 0$

consequents

Surface of section or Poincaré maps



Surfaces of section

- A point in the surface of section (for a given E and L_z) defines an orbit as the three independent variables $(R, \dot{R}, z=0)$ are defined.

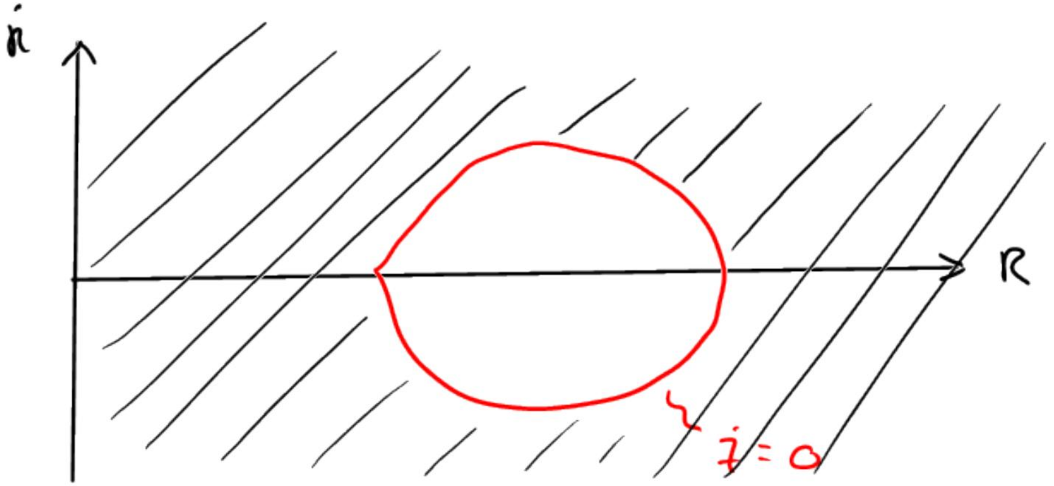
- Even if orbits have the same energy, they will never intersect in the plane (EoM are first order diff. equations).

- Zero velocity curve : curve defined by $\dot{z} = 0$

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \Phi_{\rm eff}(R, z=0) \qquad \Rightarrow \qquad \dot{R} \leqslant \pm\sqrt{2\left[E - \Phi_{\rm eff}(R, z=0)\right]}$$

$$\dot{R}(R) = \pm\sqrt{2\left[E - \Phi_{\rm eff}(R, z=0)\right]}$$

defines the accessible region of the phase space

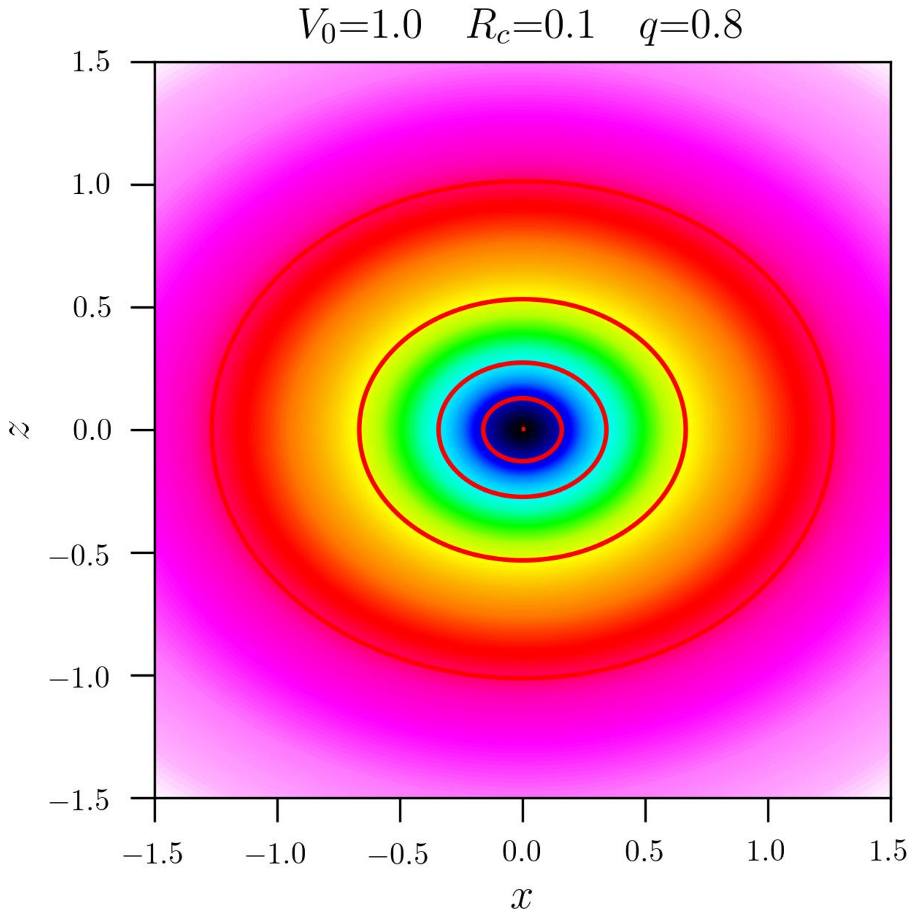


Surfaces of section

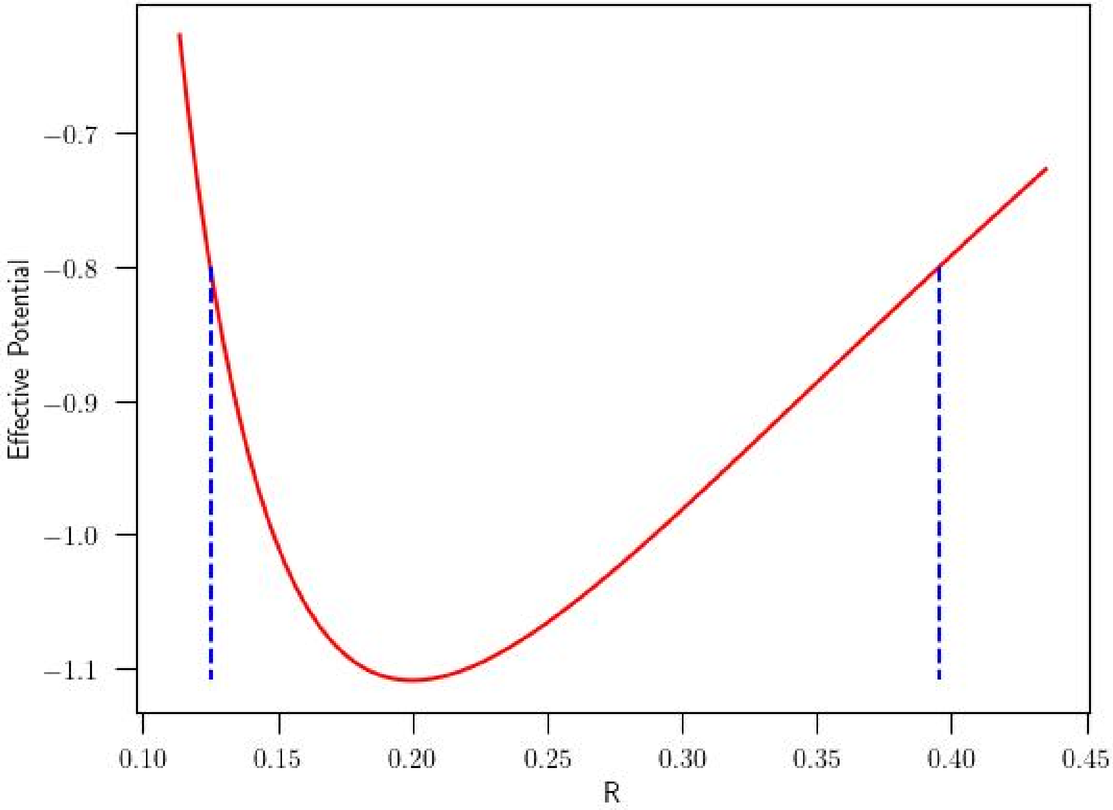
Examples

Logarithmic potential

$$\Phi_{\log}(R, z) = \frac{1}{2} V_0^2 \ln\left(R_c^2 + R^2 + \frac{z^2}{p^2}\right)$$

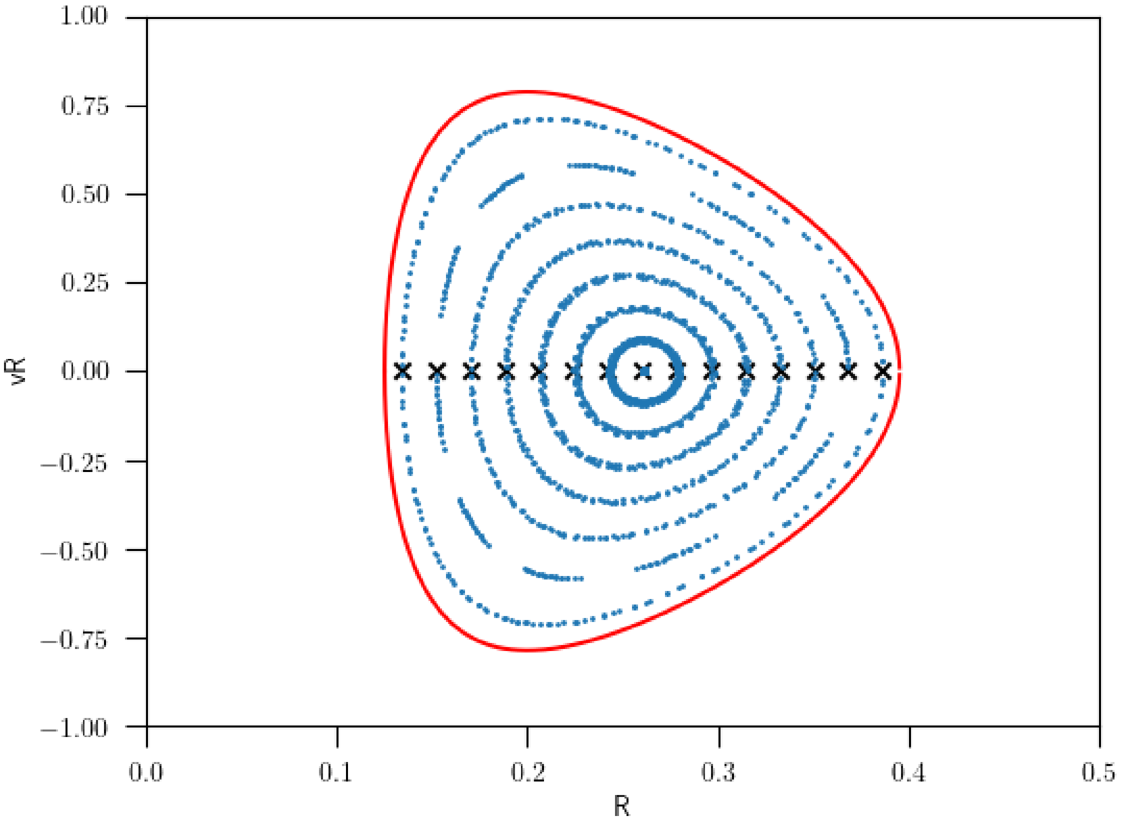


Effective Potential



```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --plotpotential
```

Invariant curves : Third Integral



```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 15 --nlaps 100
```

The Third Integral I (I is in general non analytical)

Spherical systems : $|\tilde{L}| \equiv L$ is conserved

Nearly spherical potential : L is nearly an integral $= I$?

What is the curve in the Poincaré map that satisfies $L =$ cte ?

in cylindrical coordinates

$$L^2 = \dot{z}^2 R^2 + L_z^2 \qquad (z=0)$$

$$\dot{z}^2 = \frac{1}{R^2}\left(L^2 - L_z^2\right)$$

Energy conservation

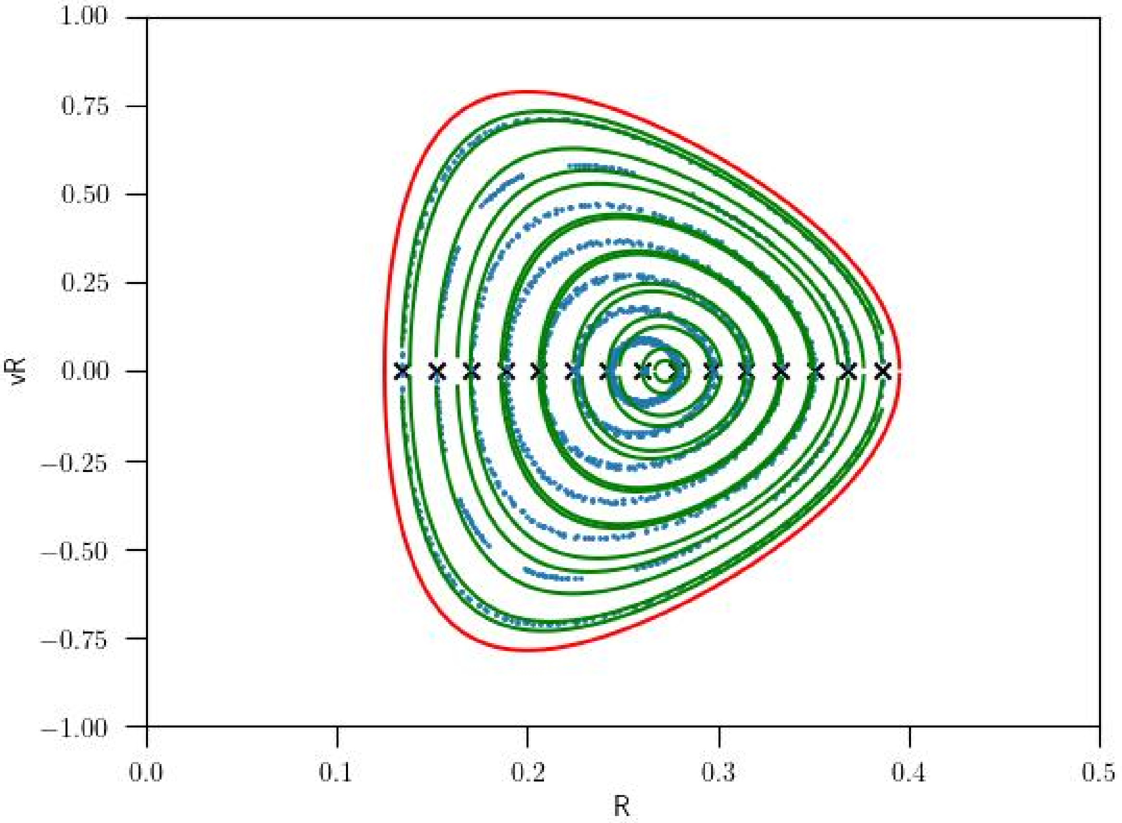
$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \Phi_{eff}(R,0)$$

$$= \frac{1}{2}\dot{R}^2 + \frac{1}{2R^2}\left(L^2 - L_z^2\right) + \Phi_{eff}(R,0)$$

$$\boxed{\dot{R} = \pm\sqrt{2(E - \Phi_{eff}(R,0) - \frac{1}{2R^2}\left(L^2 - L_z^2\right)}}$$

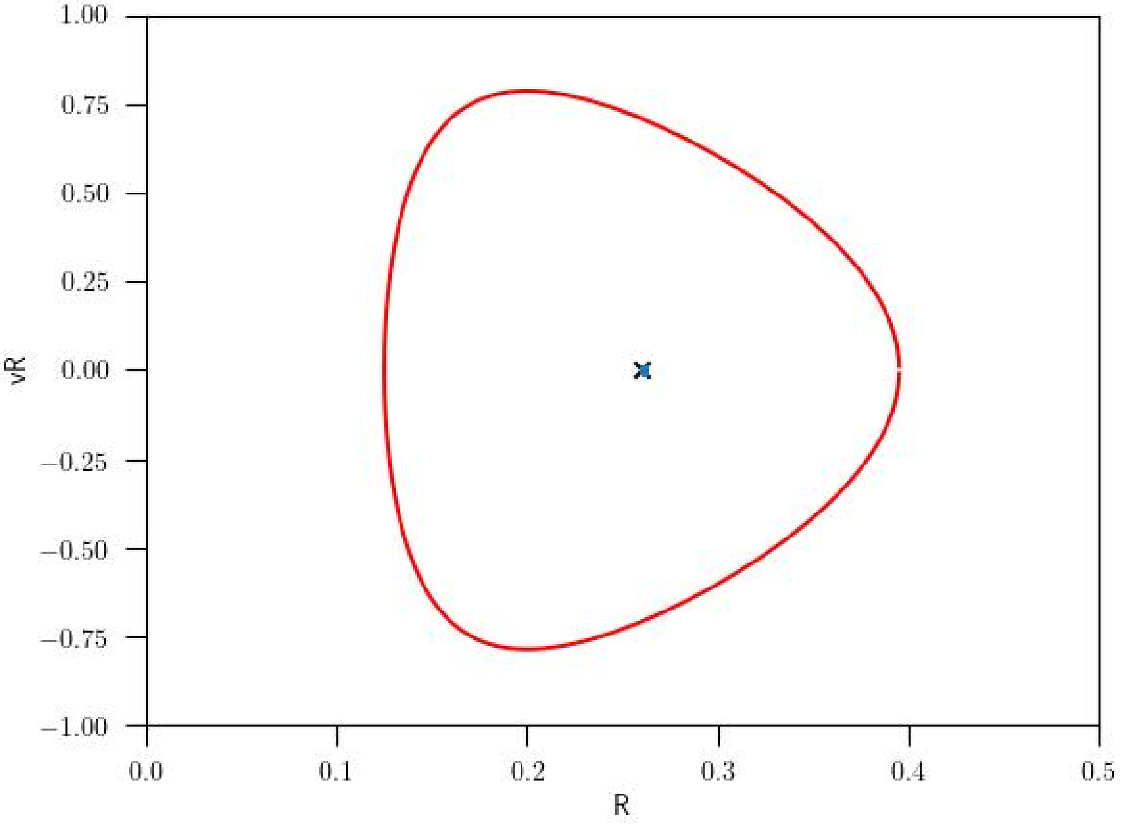
Invariant curves : Third Integral

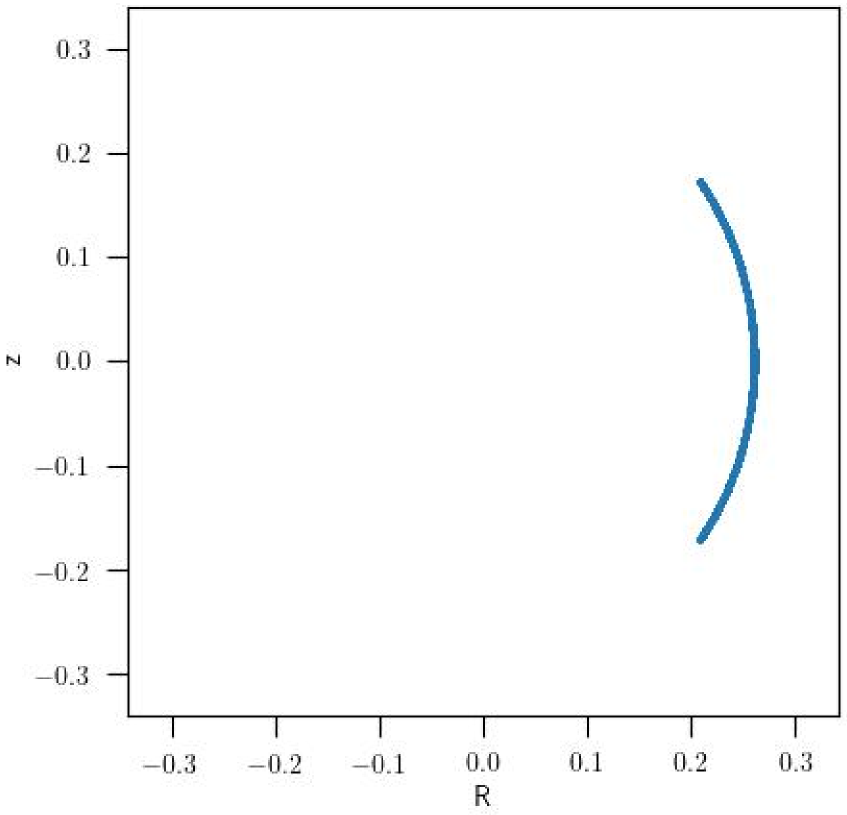
green : contours of constant total
angular momentum



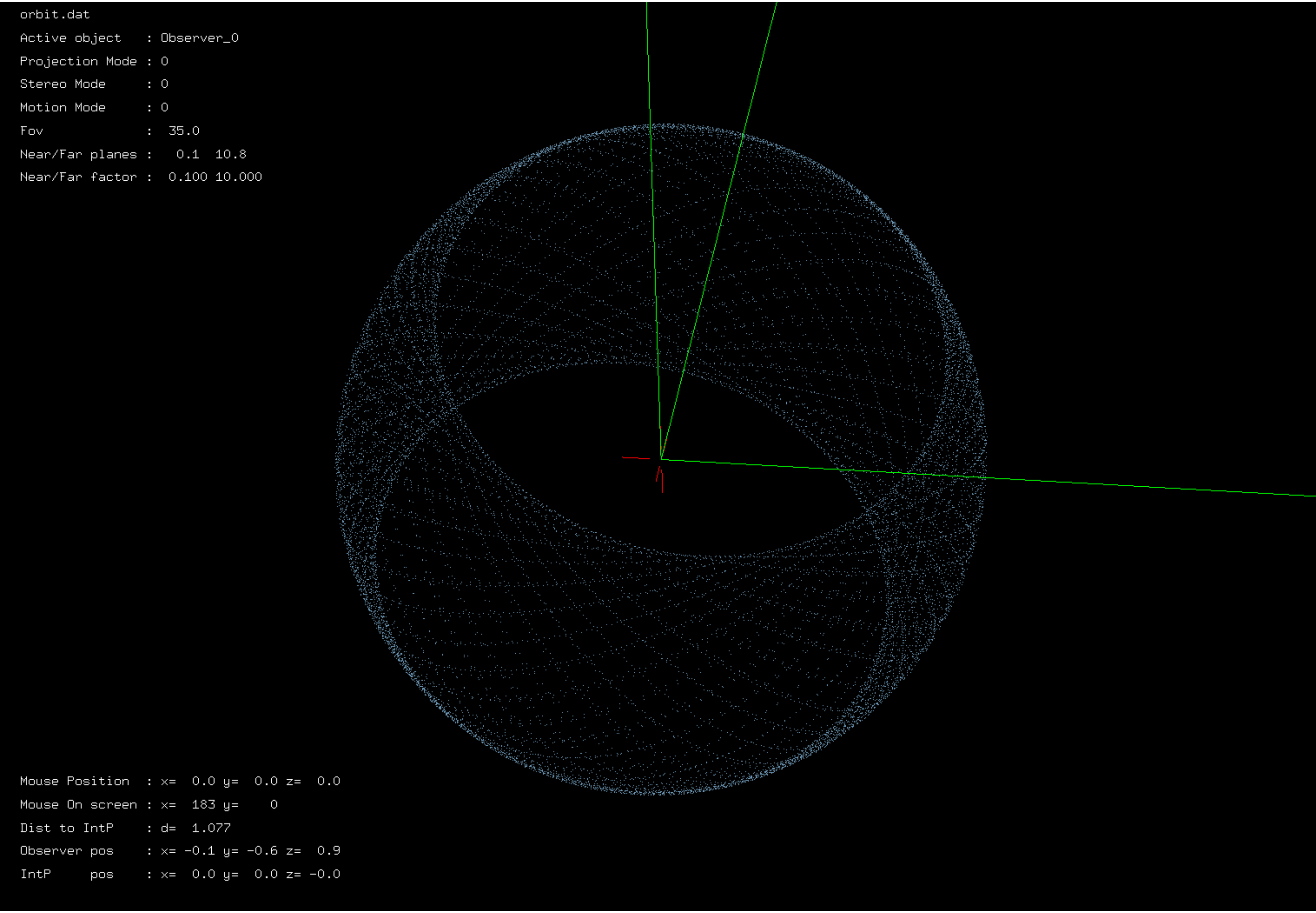
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 15 --nlaps 100 --add_IL
```

shell orbit





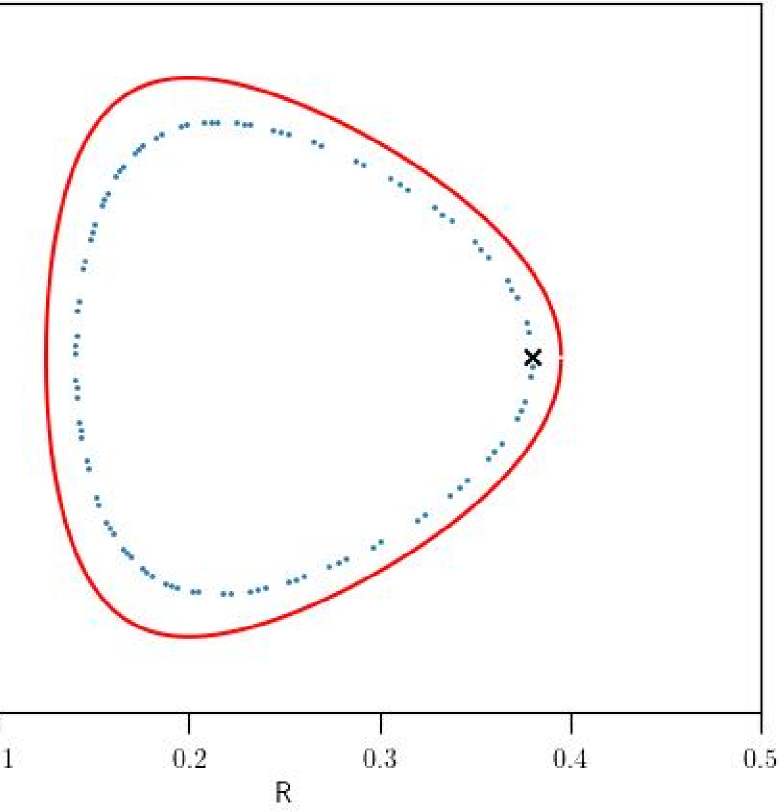
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8  --nlaps 100 --R 0.26
```

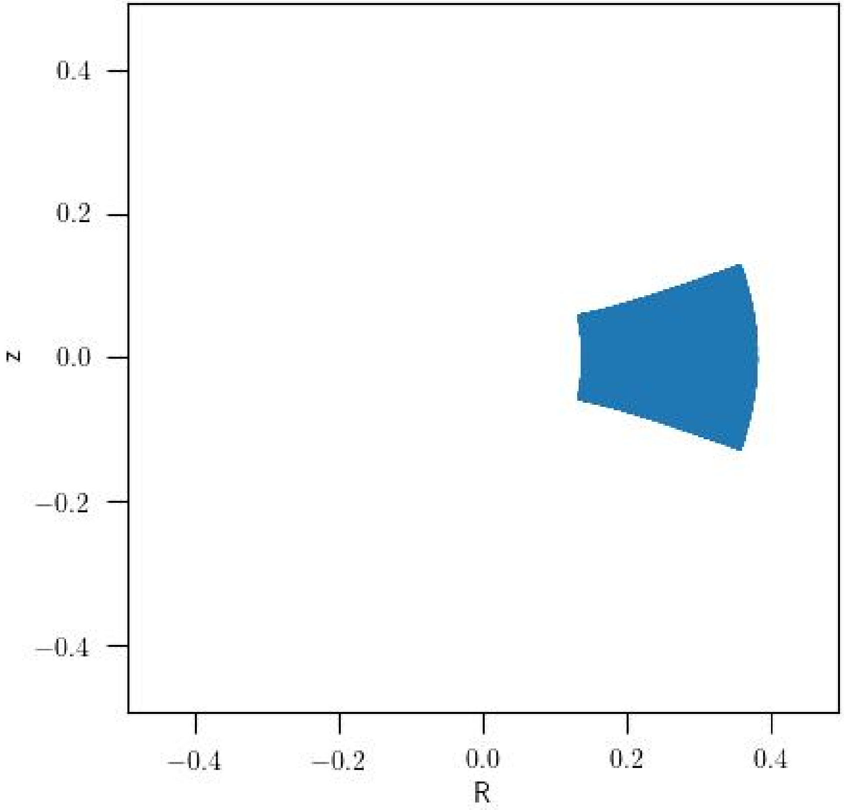


```
orbit.dat
Active object   : Observer_0
Projection Mode : 0
Stereo Mode     : 0
Motion Mode     : 0
Fov             :  35.0
Near/Far planes :   0.1  10.8
Near/Far factor :  0.100 10.000
```

```
Mouse Position  : x=   0.0 y=   0.0 z=   0.0
Mouse On screen : x=   183 y=     0
Dist to IntP    : d=   1.077
Observer pos    : x= -0.1 y= -0.6 z=   0.9
IntP     pos    : x=   0.0 y=   0.0 z= -0.0
```

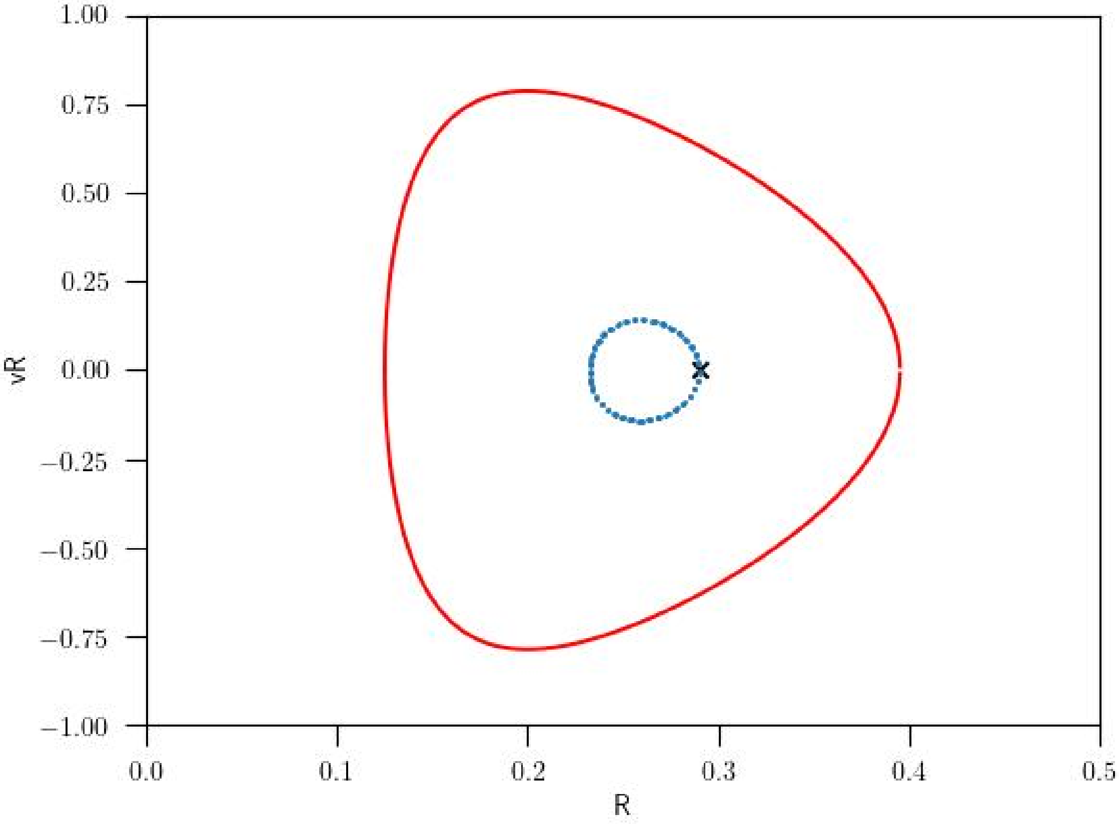
Large radius

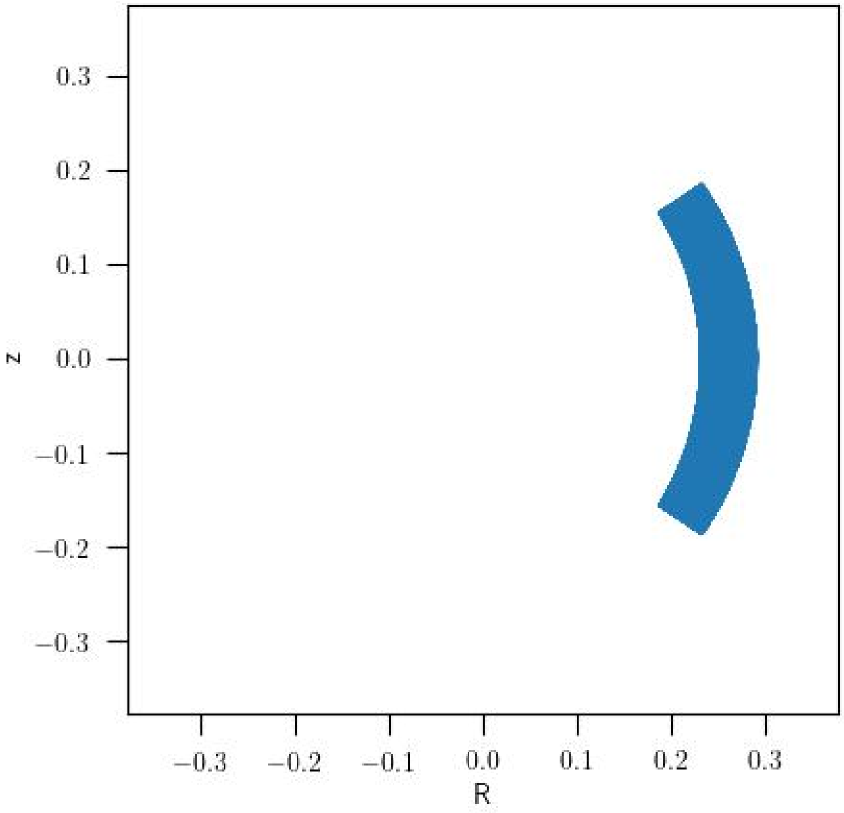




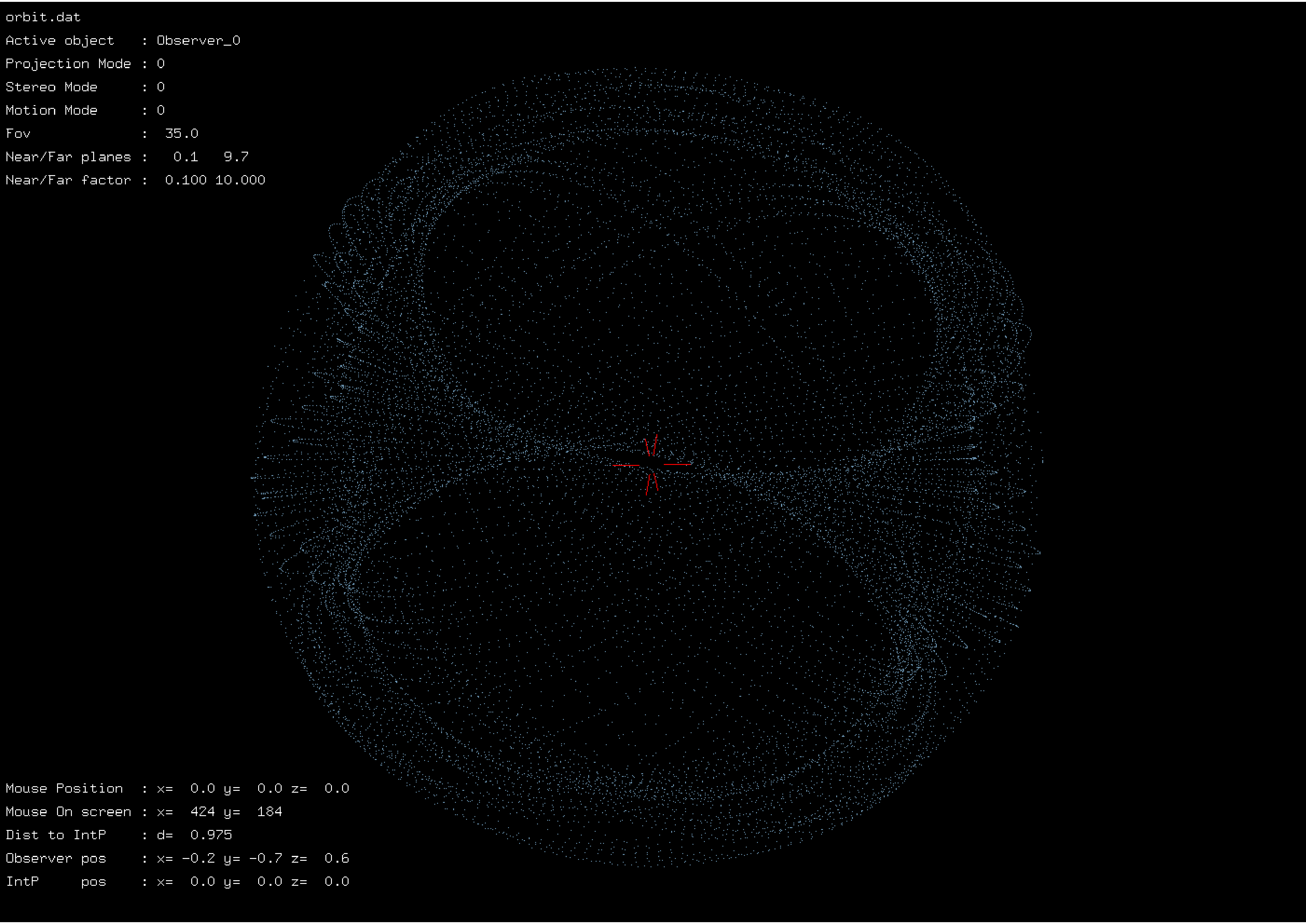
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 15 --nlaps 100 --R 0.38
```

Smaller radius





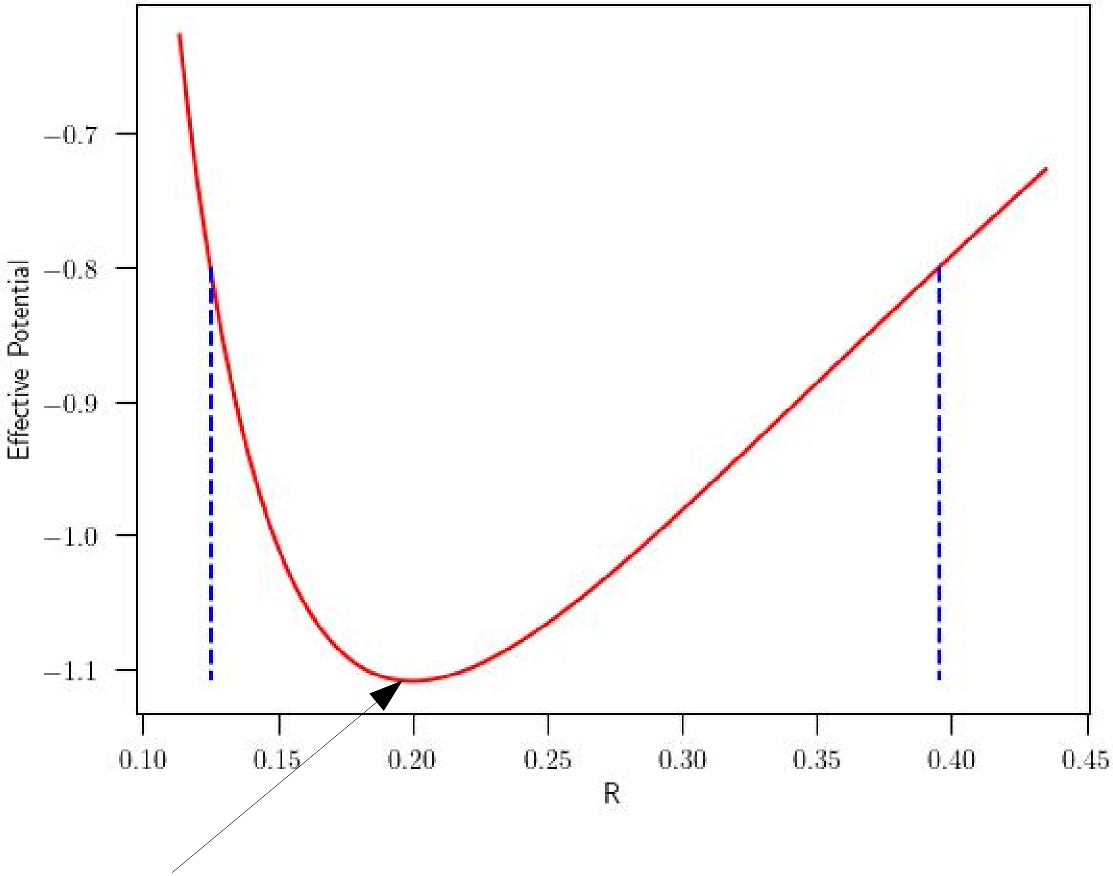
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 15 --nlaps 100   --R 0.29
```



```
orbit.dat
Active object   : Observer_0
Projection Mode : 0
Stereo Mode     : 0
Motion Mode     : 0
Fov             :  35.0
Near/Far planes :   0.1   9.7
Near/Far factor :  0.100 10.000
```

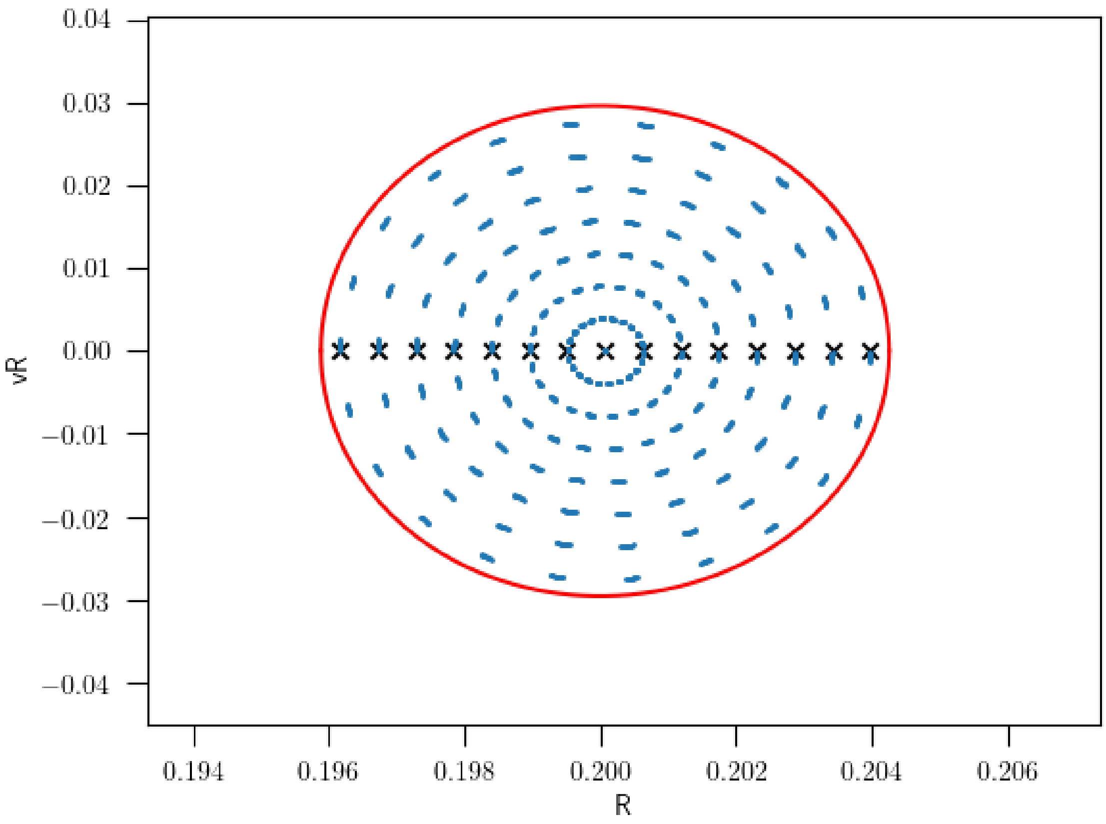
```
Mouse Position   : x=  0.0 y=  0.0 z=  0.0
Mouse On screen  : x=  424 y=  184
Dist to IntP     : d=  0.975
Observer pos     : x= -0.2 y= -0.7 z=  0.6
IntP     pos     : x=  0.0 y=  0.0 z=  0.0
```

Exploring orbits at lower energy



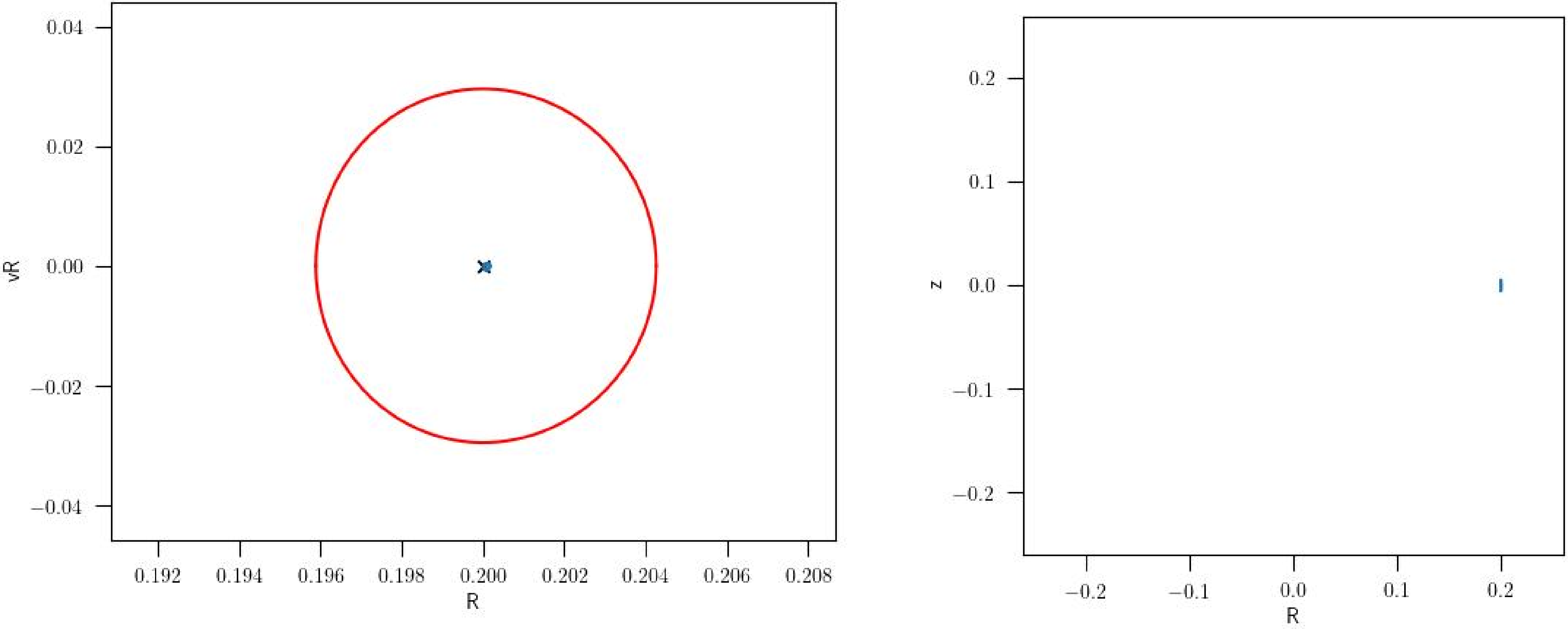
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.109 --plotpotential
```

Orbits near the circular orbit energy

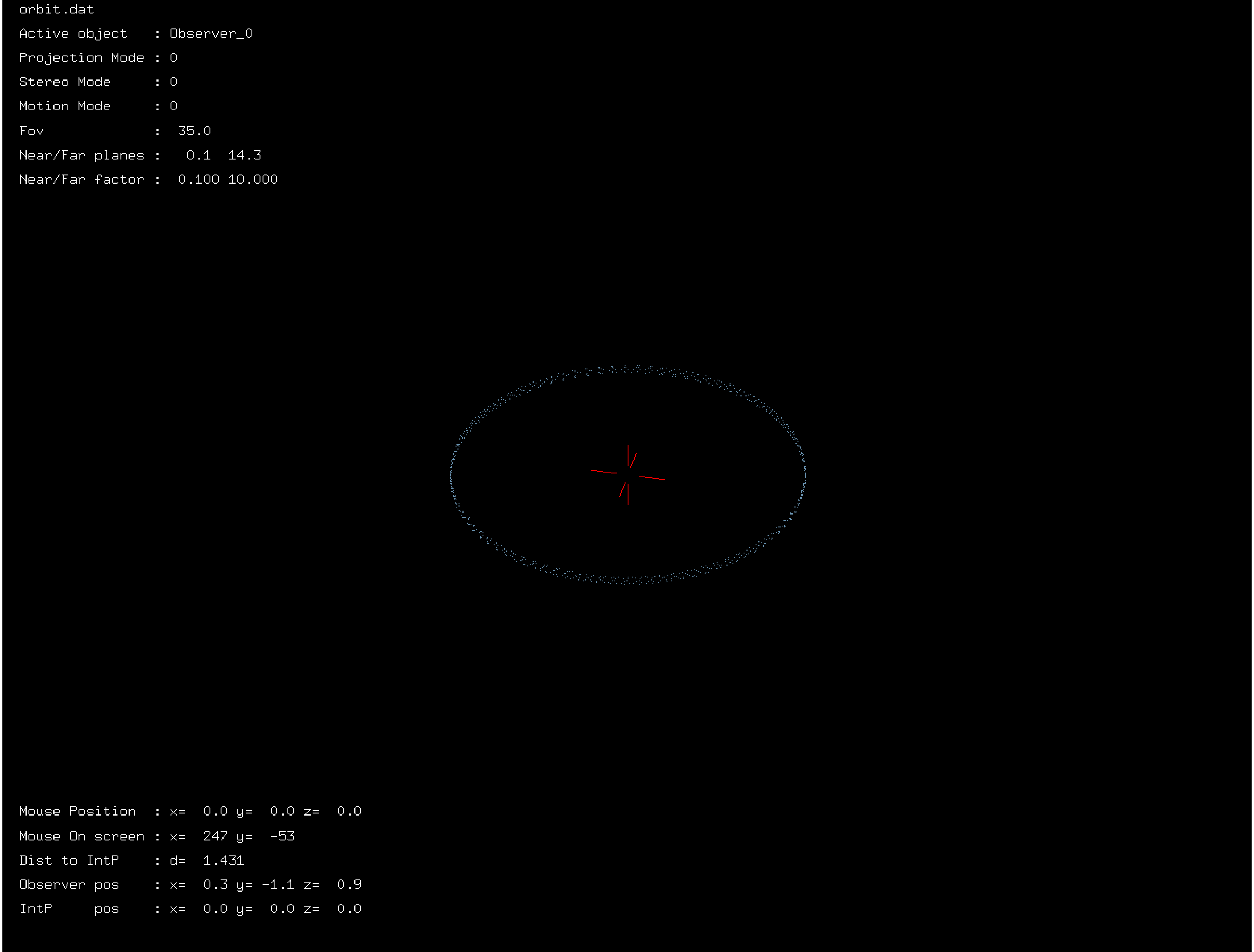


```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.109 --norbits 15 --nlaps 100
```

Circular orbit



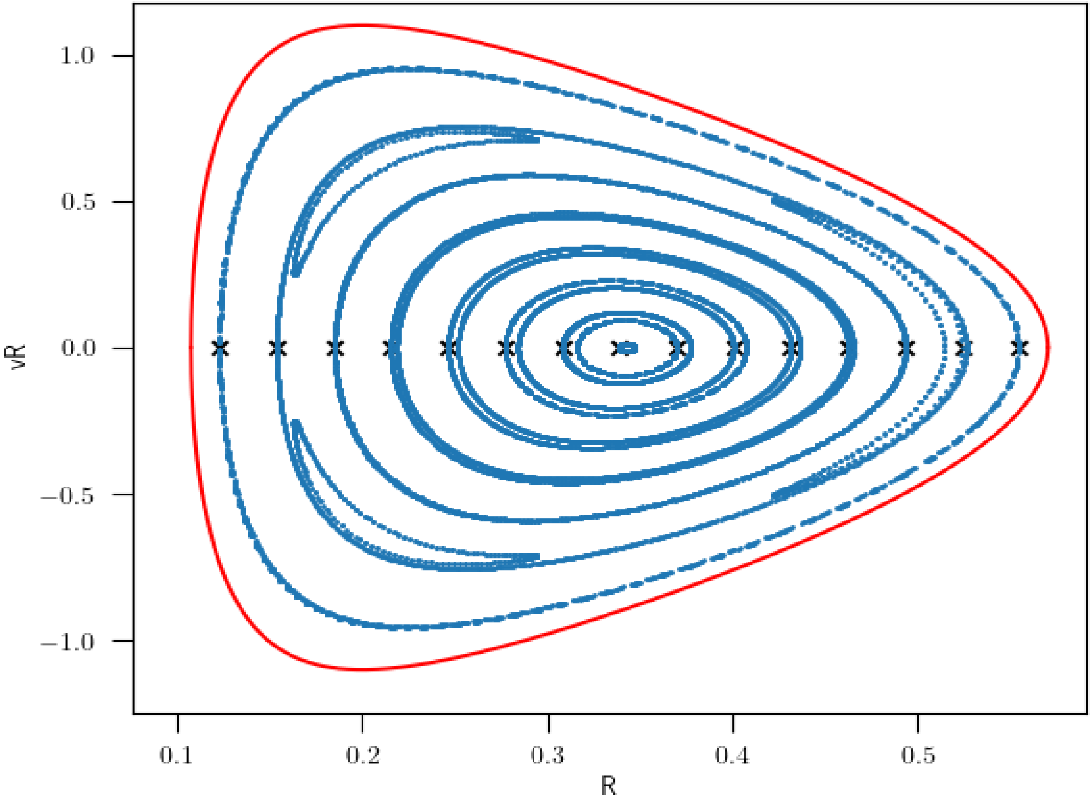
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.109 --vR 0 --R 0.2   --nlaps 10
```



```
orbit.dat
Active object   : Observer_0
Projection Mode : 0
Stereo Mode     : 0
Motion Mode     : 0
Fov             :  35.0
Near/Far planes :   0.1  14.3
Near/Far factor :  0.100 10.000
```

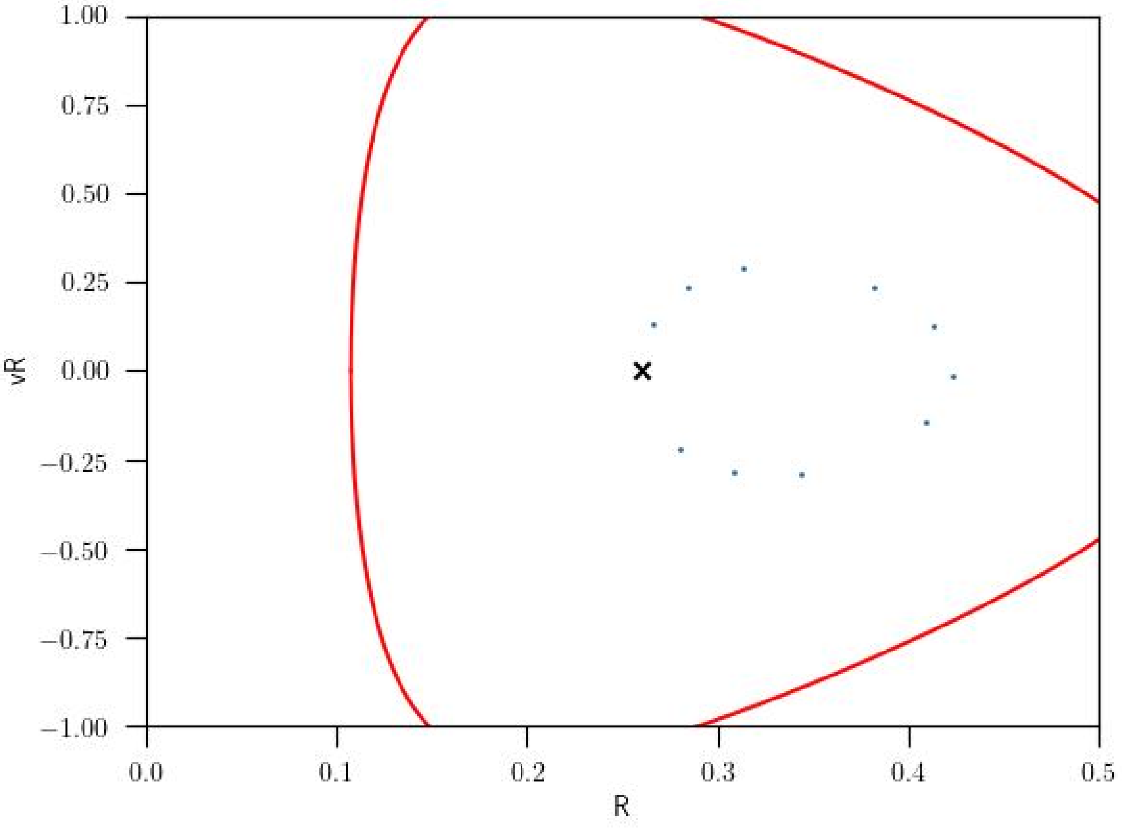
```
Mouse Position   : x=  0.0 y=  0.0 z=  0.0
Mouse On screen  : x=  247 y=  -53
Dist to IntP     : d=  1.431
Observer pos     : x=  0.3 y= -1.1 z=  0.9
IntP     pos     : x=  0.0 y=  0.0 z=  0.0
```

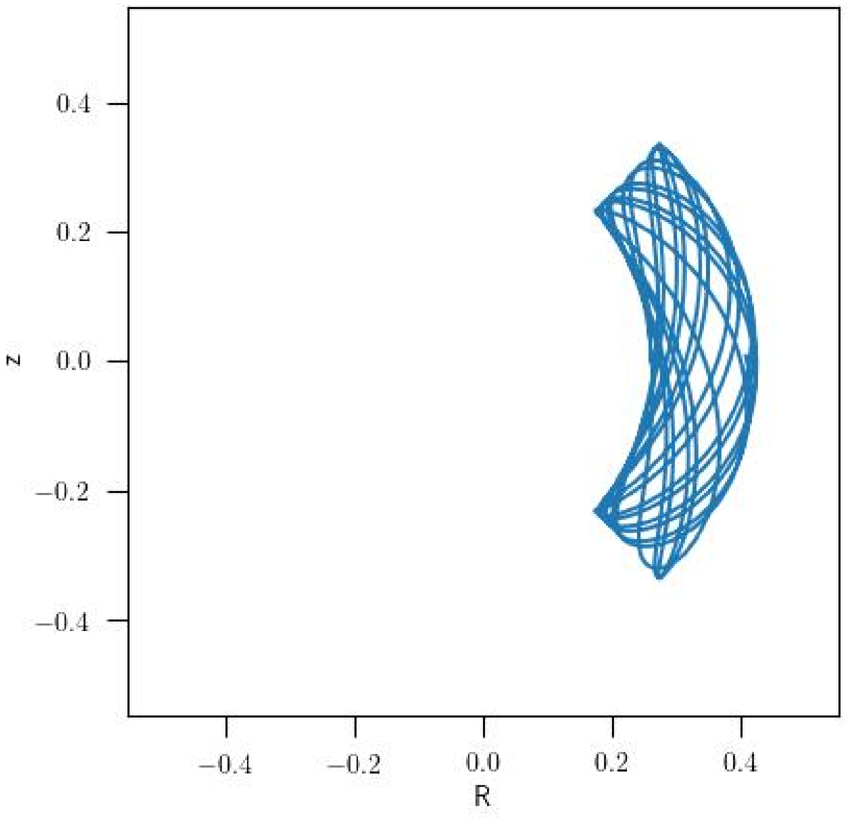
At higher energy



```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --norbits 15 --nlaps 1000
```

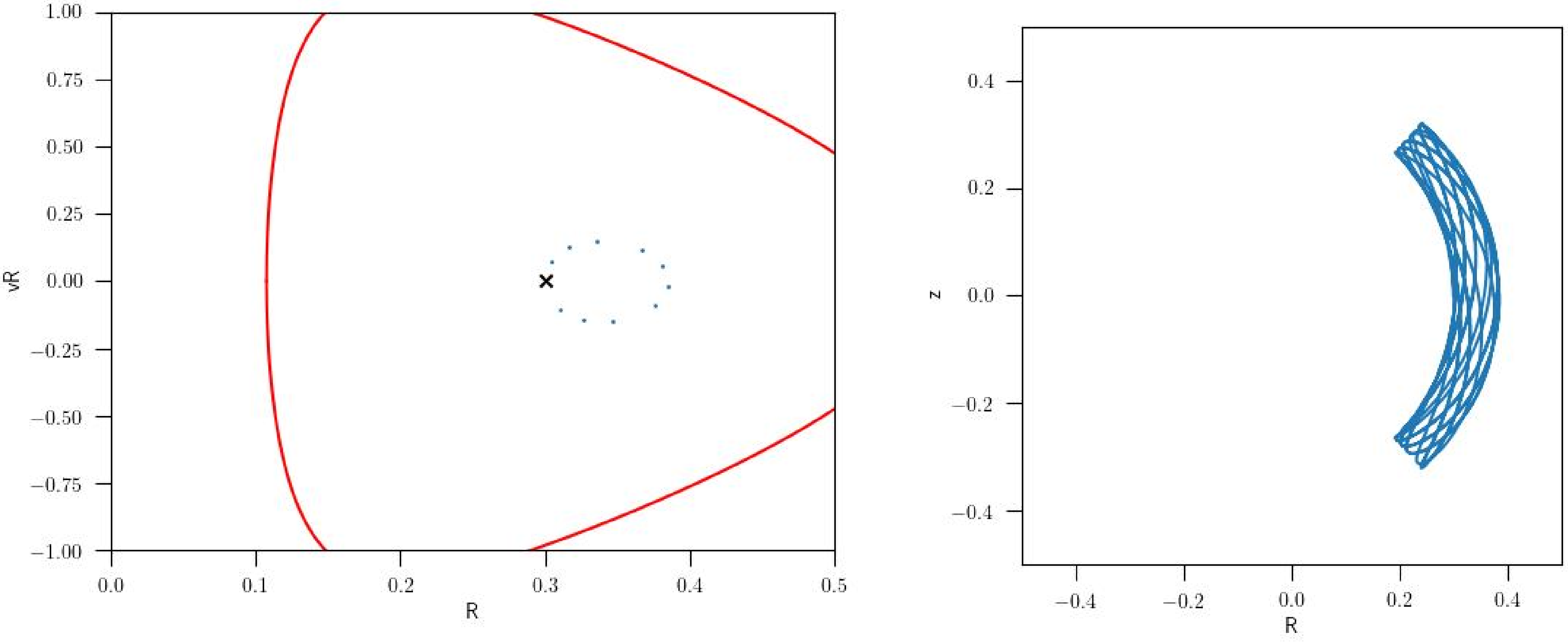
At higher energy





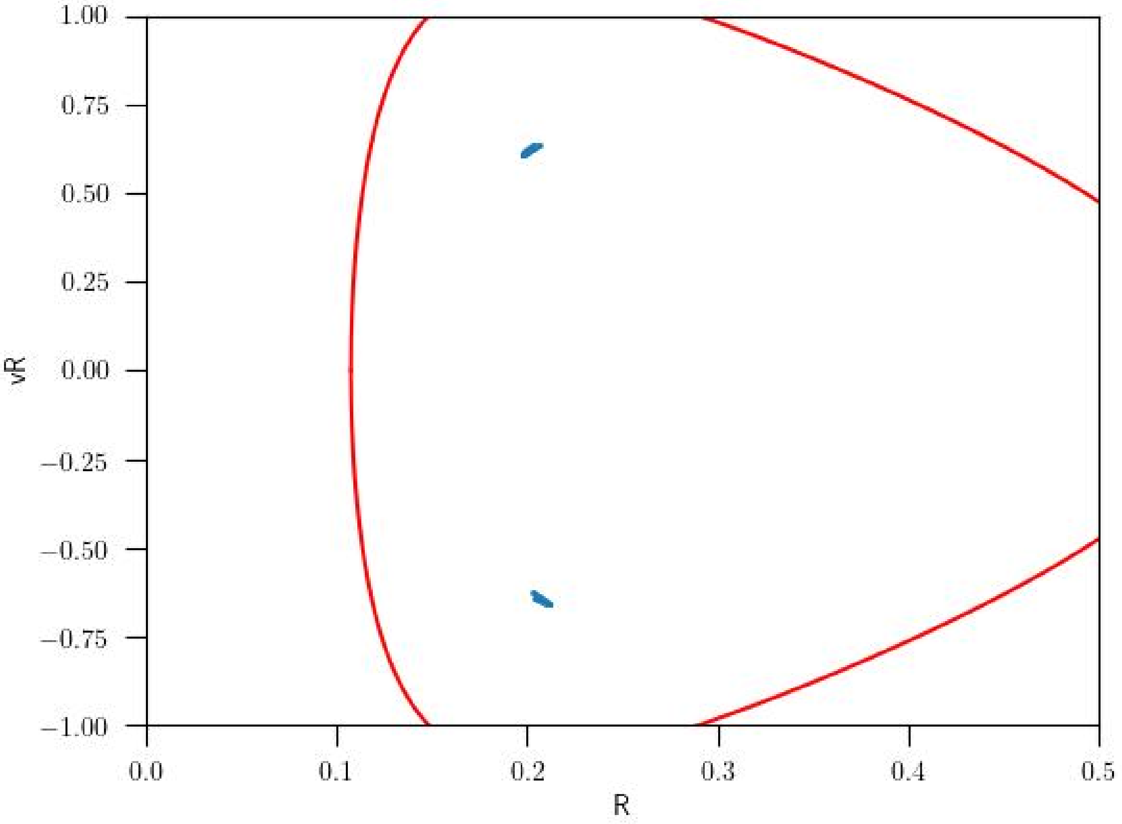
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --R 0.26   --nlaps 10
```

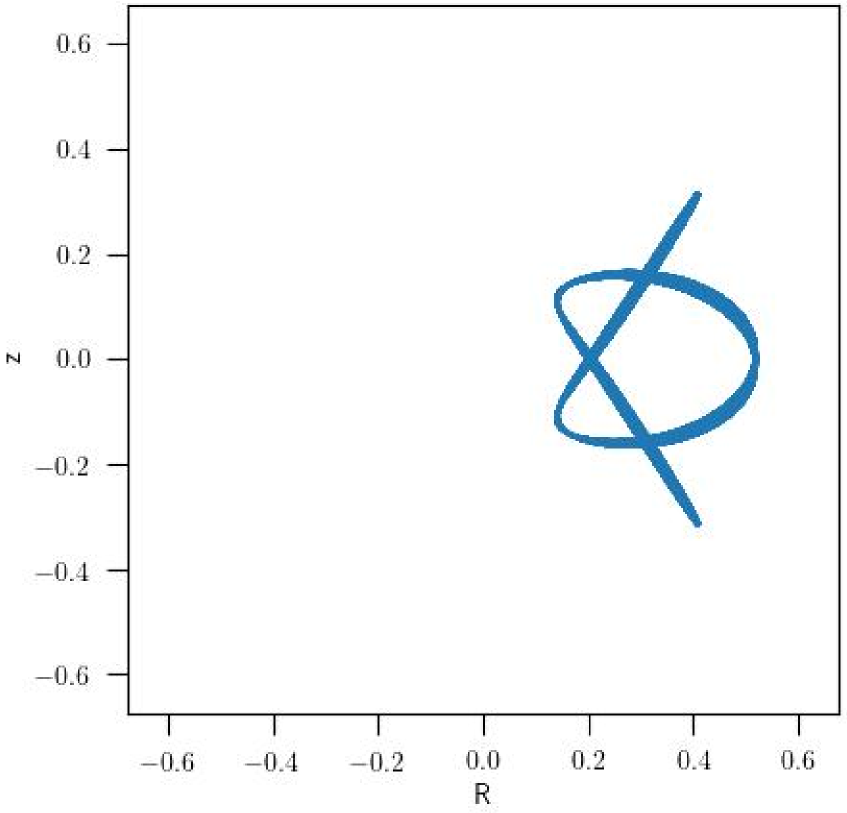
At higher energy



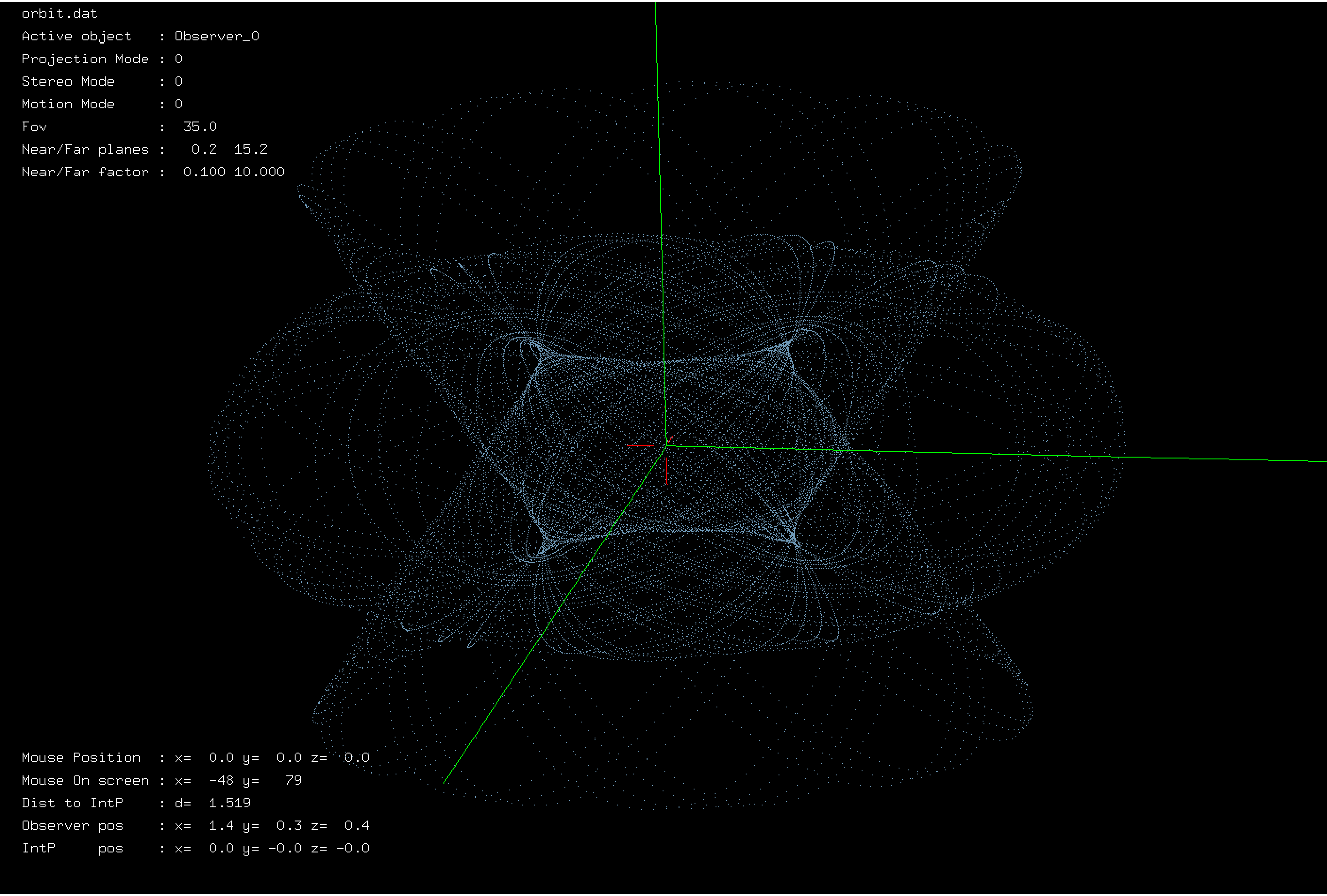
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --R 0.30   --nlaps 10
```

Bifurcation (resonance) : new orbit family





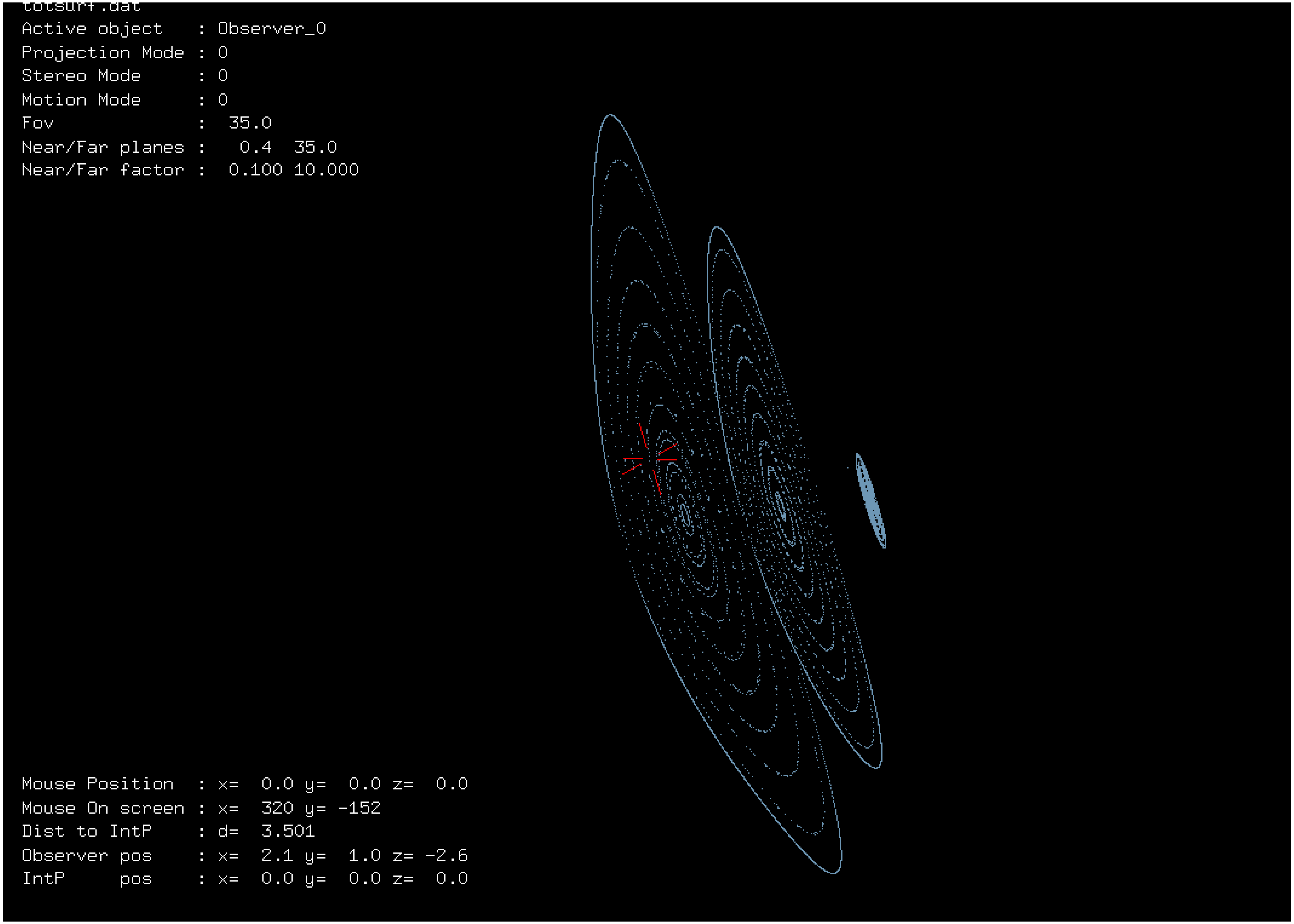
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --R 0.52   --nlaps 100
```



```
orbit.dat
Active object    : Observer_0
Projection Mode : 0
Stereo Mode      : 0
Motion Mode      : 0
Fov              :  35.0
Near/Far planes :   0.2  15.2
Near/Far factor :  0.100 10.000
```

```
Mouse Position   : x=   0.0 y=   0.0 z=   0.0
Mouse On screen : x=  -48 y=    79
Dist to IntP     : d=  1.519
Observer pos     : x=   1.4 y=   0.3 z=   0.4
IntP     pos     : x=   0.0 y= -0.0 z= -0.0
```

Slices of different energies



```
totsurf.dat
Active object   : Observer_0
Projection Mode : 0
Stereo Mode     : 0
Motion Mode     : 0
Fov             :  35.0
Near/Far planes :   0.4  35.0
Near/Far factor :  0.100 10.000
```

```
Mouse Position  : x=  0.0 y=  0.0 z=  0.0
Mouse On screen : x=  320 y= -152
Dist to IntP    : d=  3.501
Observer pos    : x=  2.1 y=  1.0 z= -2.6
IntP     pos    : x=  0.0 y=  0.0 z=  0.0
```

```
rm surf-*.dat
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.1 --vR 0 --norbits 18
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --vR 0 --norbits 18
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --norbits 18
./concatenate.py surf-0*
glups --fullscreen -p glparameters  totsurf.dat
```

The End