Introduction to Differentiable Manifolds
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Exercise series 5, with solutions **2021–11–03**

Exercise 5.1. Consider the map

$$f : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (2 + \tanh t) \cdot (\cos t, \sin t).$$

Show that f is an injective immersion. Is it a smooth embedding?

Solution. First notice that f is an immersion since $f'(t) \neq 0$ for every $t \in \mathbb{R}$. To see this observe that

$$f_*\big|_t(\frac{\partial}{\partial t}\big|_t) = \sum_{0\leq j<2} \frac{\partial}{\partial t}\big|_t(x^j \circ f)\frac{\partial}{\partial x^j}\big|_{f(t)} = f_0'(t)\frac{\partial}{\partial x^0}\big|_{f(t)} + f_1'(t)\frac{\partial}{\partial x^1}\big|_{f(t)}$$

Hence if $f'(t) \neq 0$ then we have $\operatorname{Ker} f_*\big|_t = \{0\}$ which is equivalent to $f_*\big|_t$ injective for every $t \in \mathbb{R}$. Thus it suffices to compute

$$f_0'(t) = \left(\frac{1}{\cosh^2 t}\right)\cos t - (2 + \tanh t)\sin t$$

and

$$f_1'(t) = \left(\frac{1}{\cosh^2 t}\right)\sin t - (2 + \tanh t)\cos t$$

To see that $f'(t) \neq 0$ notice that

$$\|f'(t)\|^2 = \left(\frac{1}{\cosh^2 t}\right)^2 + (2 + \tanh t)^2 > 0$$

where $\|\cdot\|$ denotes the euclidean norm. This proves that f is an immersion. Furthermore the function f is an injection since the function $r(t) = \|f(t)\| = 2 + \tanh t$ is strictly increasing.

Note that f is an injective immersion. Let us prove that it is a smooth embedding. Consider the open set $U = \{x \in \mathbb{R}^2 : 1 < \|x\| < 3\}$. We will show that $f|^U : \mathbb{R} \to U$ is a proper map (hence a closed map; see e.g. Thm. 4.95 of Lee's book on topological manifolds). It follows that f is an embedding, since its the composite $f = \iota_U \circ f|^U$ of a closed embedding $f|^U$ and the inclusion map $\iota_U : U \to M$, which is an open embedding.

To see that $f|^U$ is proper we let $K \subseteq U$ be a compact set and verify that $f^{-1}(K) \subseteq \mathbb{R}$ is compact as well. Since K is closed (because it is a compact subset of a Hausdorff space) and f is continuous, the preimage $f^{-1}(K)$ is closed. Finally, we have to check that $f^{-1}(K)$ is bounded. Let a (resp b) be the minimum (resp. maximum) norm of a point $x \in X$. Note that $[a, b] \subseteq (1, 3)$. It follows that $f^{-1}(K) \subseteq [a', b']$, where a', b' are the preimages of a, b by the monotonic map $t \mapsto 2 + \tanh t$. □

Exercise 5.2. Consider the following subsets of $\mathbb{R}^2$. Which is an embedded submanifold ? Which is the image of an immersion ?

(a) The "cross" $S := \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$.

Solution. The cross S is not an embedded submanifold, because it is the union of the lines $y = 0$ and $x = 0$, and is therefore not locally Euclidean at the origin (exercise of series 1).

On the other hand, S is the disjoint union of two embedded submanifolds: S_0 = the horizontal axis, and S_1 = the vertical axis minus the origin. Let M be the 1-manifold obtained as disjoint union of S_0 and S_1. The inclusion map of M into $\mathbb{R}^2$ is an injective immersion and has S as image. □

(b) The "corner" $C := \{(x, y) \in \mathbb{R}^2 \mid xy = 0, x \geq 0, y \geq 0\}$

Solution. We will show that C is not even an immersed submanifold of $\mathbb{R}^2$, so in particular it cannot be an embedded submanifold.

We proceed by contradiction. Suppose that C is an immersed submanifold, i.e. it has a topology τ and smooth structure such that the canonical inclusion $\iota : C \hookrightarrow \mathbb{R}^2$ is an immersion. Let (U, φ) be a smooth chart s.t. $(0,0) \in U$, $\varphi(0,0) = 0$ where $U \subset (C, \tau)$ is open [1]. By making the image $\varphi(U)$ smaller if necessary we can suppose that it is an open interval containing 0, $\varphi(U) = J \subset \mathbb{R}$.

Since ι is an immersion, then

$$f := \iota \circ \varphi^{-1} : J \to \mathbb{R}^2$$

is a smooth map with non-zero derivatives everywhere. Here we emphasize that on J and $\mathbb{R}^2$ we have the standard Euclidean topology and smooth structure. In particular, we find that $f'(0) \neq (0,0)$. Hence either $f_1'(0) \neq 0$ or $f_2'(0) \neq 0$. If $f_1'(0) \neq 0$ then for any neighborhood of $0 \in J$, we can find points $t_1, t_2 \in J$ s.t. $f_1(t_1) < 0$ and $f_1(t_2) > 0$. It contradict the fact that $f_1 \geq 0$. Similarly we arrive at a contradiction if $f_2'(0) \neq 0$. □

Exercise 5.3. Let N be a $\mathcal{C}^k$-embedded n-submanifold of some m-manifold M, with $k \geq 1$. Show that there exists an open set $U \subseteq M$ that contains N as a closed subset.

Solution. Consider a family of charts $\varphi_i : W_i \to V_i$ that cover N and are slice charts for N, meaning that $\varphi_i(x) \in \mathbb{R}^n \times \{0\}$ iff $x \in N$, or equivalently, that $N \cap W_i = \varphi_i^{-1}(\mathbb{R}^n \times \{0\})$. Therefore $N \cap W_i$ is a closed subset of W_i for all i. We conclude that N is closed in $W = \bigcup_i W_i$, which is an open subset of M. □

Exercise 5.4. Let $f : M \to N$ be an injective immersion of $\mathcal{C}^k$ manifolds. Show that there exists a closed embedding $M \to N \times \mathbb{R}$.

Hint: Recall that there exists a proper map $g : M \to \mathbb{R}$.

Solution. The map $h : M \to N \times \mathbb{R} : x \mapsto (f(x), g(x))$ is an immersion and is proper, hence it is a closed embedding.

Proof that h is proper: Let $K \subseteq N \times \mathbb{R}$ a compact set. Note that K is closed in N since it's a compact subset of a Hausdorff space. It follows that $h^{-1}(K)$ is closed. In addition $h^{-1}(K)$ is contained in the compact set $g^{-1}(\pi_1(K))$, where $\pi_1 : N \times \mathbb{R} \to \mathbb{R}$ is the projection. Therefore $h^{-1}(K)$ is compact. This proves that h is proper, hence closed. Since in addition it is injective, it's a closed topological embedding.

Proof that h is an immersion: for each nonzero vector $v \in \mathrm{T}_p M$, the vector $\mathrm{T}_p h(v) = (\mathrm{T}_p f(v), \mathrm{T}_p g(v))$ is nonzero because its first component $\mathrm{T}_p f(v)$ is nonzero. □

Exercise 5.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + y^3 + 1$.

(a) What are the regular values of f? For which $c \in \mathbb{R}$ is the level set $f^{-1}(\{c\})$ an embedded submanifold of $\mathbb{R}^2$?

Solution. The gradient of f,

$$\nabla f(x,y) = (3x^2, 3y^2),$$

vanishes precisely at the origin $(x,y) = (0,0)$. Thus $\mathrm{T}_p f : \mathrm{T}_p \mathbb{R}^2 \to T_{f(p)}\mathbb{R}$ has rank 0 if and only if $p = (x,y) = (0,0)$. Thus every $c \in \mathbb{R}$ is a regular value except $c = 1$.

By the regular preimage theorem, each level set $f^{-1}(\{c\})$ with $c \neq 1$ is a smooth embedded submanifold in $\mathbb{R}^2$. As for the level set $f^{-1}(\{1\})$ we

[1] Note that in the case of an embedded manifold we could assume that $U = V \cap C$ for some $V \subset \mathbb{R}^2$ open, but here a-priori we do not know the topology τ.

have to argue differently. The theorem does not say that $f^{-1}(\{1\})$ is not a smooth submanifold. We have to study this case separately. Observe that in this case one has

$$f^{-1}(\{1\}) = \{x^3 + y^3 = 0\} = \{x = -y\}$$

i.e., $f^{-1}(\{1\})$ is a line going through the origin. Thus, also $f^{-1}(\{1\})$ is a smooth submanifold of $\mathbb{R}^2$. Summing up, all level sets of this function are smooth submanifolds. □

(b) In the case where $S = f^{-1}(\{c\})$ is an embedded submanifold, $p \in S$, write down an equation for the tangent space $\iota_*(\mathrm{T}_p S) \subset \mathrm{T}_p \mathbb{R}^2$ where as usual we identify $T_p\mathbb{R}^2 \cong \mathbb{R}^2$ (i.e. you are expected to write down the equation for a line in $\mathbb{R}^2$).

Solution. By the regular preimage theorem, if $c \neq 1$ we have $\mathrm{T}_p S = \operatorname{Ker} \mathrm{T}_p f$ for all $p \in S = f^{-1}(c)$.

Let us compute $\mathrm{T}_p f$. If $V = (V_x, V_y) \in \mathrm{T}_p \mathbb{R}^2 \equiv \mathbb{R}^2$, then $\mathrm{T}_p f(V) = 3\, p_x^2\, V_x + 3\, p_y^2\, V_y$, where $p = (p_x, p_y)$. Hence

$$\operatorname{Ker} \mathrm{T}_p f = \{V \in \mathrm{T}_p \mathbb{R}^2 : p_x^2\, V_x + p_y^2\, V_y = 0\}.$$

When $c = 1$ we notice that $S = \{x = -y\}$, thus $\mathrm{T}_p S = \{V \in \mathrm{T}_p \mathbb{R}^2 : V_x = -V_y\}$. □

Exercise 5.6. Consider the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the projection map.

(a) Give $\mathbb{T}^n$ a natural smooth structure so that π is a local diffeomorphism.

Solution. We have already seen in a previous exercise that π is locally injective. This means that $\mathbb{R}^2$ is covered by open sets U such that the restriction $\pi|_U : U \to \mathbb{T}^n$ is injective. We take these maps $\phi = \pi|_U$ as local parametrizations of $\mathbb{T}^n$. Their inverses form a smooth atlas for $\mathbb{T}^n$. (The transition maps are locally translations, hence smooth.) □

(b) Show that a map $f : \mathbb{T}^n \to M$ (where M is a $\mathcal{C}^k$ manifold) is $\mathcal{C}^k$ if and only if the composite $f \circ \pi$ is $\mathcal{C}^k$.

Solution. If f is $\mathcal{C}^k$, it is clear that $f \circ \pi$ is $\mathcal{C}^k$.

Now suppose $f \circ \pi$ is $\mathcal{C}^k$. To show that f is $\mathcal{C}^k$, it suffices to show that $f \circ \phi$ is $\mathcal{C}^k$ for all parametrizations $\phi = \pi|_U$ as above. And indeed, by decomposing $\phi = \pi \circ \iota_U$, where $\iota_U : U \to \mathbb{R}^n$ is the inclusion map, we see that the map $f \circ \phi$ is $\mathcal{C}^k$ because $f \circ \phi = f \circ \pi \circ \iota_U$ and both $f \circ \pi$ and ι_U are $\mathcal{C}^k$. □

(c) Show that $\mathbb{T}^n$ is diffeomorphic to the product of n copies of the circle $\mathbb{S}^1$.

Solution. Recall the homeomorpism $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1 \subseteq \mathbb{R}^2$ that sends $[t] \mapsto (\cos(2\pi t), \sin(2\pi t))$. We will construct an n-dimensional version of it.

For this exercise it is convenient to define the torus as $\mathbb{T}^n := \mathbb{R}^n/2\pi\mathbb{Z}^n$. We define a map $f : \mathbb{R}^n \to (\mathbb{S}^1)^n \subseteq \mathbb{R}^{2n}$ that sends

$$(t^i)_{0 \leq i < n} \mapsto (\cos t^0, \sin t^0, \cos t^1, \sin t^1, \dots).$$

Since the map f is $2\pi\mathbb{Z}^n$-periodic, by the previous part of the exercise it passes to the quotient giving a smooth map $\overline{f} : \mathbb{T}^n \to (\mathbb{S}^1)^n$ that satisfies $f = \overline{f} \circ \pi$.

Note that map $\overline{f}$ is an immersion. To prove this, since π is a surjective, it suffices to check that the map $\iota \circ f = \iota \circ \overline{f} \circ \pi : \mathbb{R}^n \to \mathbb{R}^{2n}$ is an immersion, where ι is the inclusion map $(\mathbb{S}^1)^n \to \mathbb{R}^{2n}$. To see that $\iota \circ \overline{f}$ is an immersion we note that the n vectors

$$\mathrm{T}_p(\iota \circ f)(e_i) = (0, \dots, 0, -\sin t^i, \cos t^i, 0, \dots, 0)$$

are linearly independent, since they are nonzero and contained in different coordinate planes.

Since $\overline{f} : \mathbb{T}^n \to (\mathbb{S}^1)^n$ is an immersion between n-dimensional manifolds, it follows that $\overline{f}$ is a local diffeomorphism, and in particular it is an open map. Since in addition $\overline{f}$ is bijective, it is a diffeomorphism. □

Exercise 5.7. Show that the map $g : \mathbb{T}^2 \to \mathbb{R}^3$ given by

$$g([s,t]) = ((2+\cos s)\cos t, (2+\cos s)\sin t, \sin s)$$

is a smooth embedding of the 2-torus in $\mathbb{R}^3$.
(In this case the torus is defined as $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$.)

Solution. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the quotient map. We define the composite map $f = g \circ \pi : \mathbb{R}^2 \to \mathbb{R}^3$. Note that

$$f(s,t) = ((2+\cos s)\cos t, (2+\cos s)\sin t, \sin s).$$

Clearly f is smooth, therefore (by the previous exercise) g is smooth.

Let us show that g is an embedding. We first show that f is an immersion. This follows because for any point $p = (s,t)$, the vectors

$$T_p f(e_0) = \left.\frac{\partial f(s,t)}{\partial s}\right|_p = (-\sin(s)\cos t, -\sin s \sin t, \cos s)$$
$$T_p f(e_1) = \left.\frac{\partial f(s,t)}{\partial t}\right|_p = (-(2+\cos s)\sin(t), (2+\cos s)\sin t, 0)$$

are linearly independent. Since π is a surjective local diffeomorphism, it follows that g is an immersion. (Indeed, each point $q \in \mathbb{T}^2$ is of the form $q = \pi(p)$, with $p \in \mathbb{R}^2$. Differentiating the composite map $f = g \circ \pi$ at p we get

$$\mathrm{T}_p\, f = \mathrm{T}_q\, g \circ \mathrm{T}_p\, \pi,$$

and since $\mathrm{T}_p\, f$ is injective and $\mathrm{T}_p\, \pi$ is an isomorphism, we conclude that $\mathrm{T}_q\, g$ is injective as well.)

Finally, $g : \mathbb{T}^2 \to \mathbb{R}^3$ is a closed map because its domain is compact and its codomain is Hausdorff. Since g is injective, we conclude that g is a a topological embedding. □

Exercise 5.8. Show that the following subgroups of $GL_n(\mathbb{R})$ are closed submanifolds. Compute their dimension and their tangent space at the identity.

(a) The *special linear group* $\mathrm{SL}_n(\mathbb{R})$, consisting of matrices with determinant equal to 1.

Solution. The determinant function $\det : M_n \to \mathbb{R}$ is continuous, which implies that the preimage of a closed (resp. open) set is a closed (resp. open) set. We have already used this to show that the *general linear group* $GL_n = \det^{-1}(\mathbb{R}_{\neq 0})$ is open in M_n. And now we can use it to show that the *special linear group* $SL_n = \det^{-1}(1)$ is a closed subset of M_n. (And since SL_n is contained in GL_n, it is also closed in GL_n).

To show that SL_n is a submanifold we use the regular preimage theorem. We apply the theorem to the determinant map $\det : M_n \to \mathbb{R}$, which is a smooth map (by a previous exercise).

To apply the theorem we have to show that 1 is a regular value of det. Thus we have to show that the linear transformation

$$\mathrm{D}_A \det : \mathrm{T}_A\, M_n \equiv \mathbb{R}^{n^2} \longrightarrow \mathrm{T}_{\det(A)}\, \mathbb{R} \equiv \mathbb{R}$$

is surjective for all points $A \in SL_n$. Since the codomain of this linear transformation has dimension 1, we have two possibilities: either the transformation is surjective (if it has rank 1) or it is null (if it has rank 0). Thus it suffices to show that the transformation $\mathrm{D}_A \det$ is not null. We have already computed the differential

$$\mathrm{D}_A \det(X) = \det(A)\,\mathrm{tr}(A^{-1}X)$$

Putting $X := A$ we get

$$\mathrm{D}_A \det(X) = \det(A)\,\mathrm{tr}(I_n) = n$$

This implies that $\mathrm{D}_A \det$ is surjective for every $A \in SL_n$. Therefore $SL_n = \det^{-1}(1)$ is an embedded submanifold of M_n of dimension

$$\dim(SL_n) = \dim(M_n) - \dim(\mathbb{R}) = n^2 - 1.$$

Finally, the regular preimage theorem also tells us that the tangent space of SL_n at any point $A \in SL_n$ is

$$\mathrm{T}_A(SL_n) = \mathrm{Ker}(\mathrm{D}_A \det) = \{X \in M_n \mid \mathrm{tr}(A^{-1}X) = 0\}$$

In particular,

$$\mathrm{T}_{I_n}(SL_n) = \{X \in M_n \mid \mathrm{tr}(X) = 0\}.$$

□

(b) The *orthogonal group* $O_n(\mathbb{R})$, consiting of the orthogonal matrices A (which satisfy $A^\top A = I_n$).

Hint: Consider the map $f : M_n \to M_n^{sym}$ that sends $A \mapsto A^\top A$, there M_n^{sym} is the vector space of *symmetric* $n \times n$ matrices.

Solution. Note that $f^{-1}(I_n) = O_n$. To apply the regular preimage theorem we have to verify that I_n is a regular value of f. Thus we have to show that for each point $A \in O_n$, the linear transformation

$$\mathrm{D}_A f : \mathrm{T}\, M_n \equiv M_n \longrightarrow \mathrm{T}\, M_n^{sym} \equiv M_n^{sym}$$

is surjective. Note that

$$\begin{aligned} \mathrm{D}_A f(X) &= A^\top X + X^\top A \\ &= A^\top X + (A^\top X)^\top. \end{aligned}$$

Let $Y \in M_n^{sym}$ be an antisymmetric matrix. Let us find some $X \in M_n$ such that $\mathrm{D}_A f(X) = Y$. We can write $Y = \frac{1}{2}Y + \frac{1}{2}Y^\top$, thus it suffices to find $X \in M_n$ such that $A^\top X = \frac{1}{2}Y$. We put simply $X = (A^\top)^{-1}\frac{1}{2}Y = \frac{1}{2}AY$. This finishes the proof that I_n is a regular value of f. Therefore, by the regular preimage theorem, the set $O_n = f^{-1}(I_n)$ is a closed embedded submanifold of M_n of dimension

$$\dim(O_n) = \dim(M_n) - \dim(M_n^{sym}) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Its tangent space at any point $A \in O_n$ is

$$\mathrm{T}_A\, O_n = \mathrm{Ker}\, \mathrm{D}_A f = \{X \in \mathrm{T}\, M_n \mid A^\top X + X^\top A = 0\}$$

In particular, its tangent space at the identity matrix is

$$\mathrm{T}_{I_n}(O_n) = \{X \in M_n \mid X + X^\top = 0\},$$

that is, the space of antisymmetric matrices. □