Homework 9

[Please note that there is no Homework 8!]

Exercise 1. [Barker's algorithm]

Let $\pi = (\pi_i,\, i \in S)$ be a distribution on a finite state space S such that $\pi_i > 0$ for all $i \in S$ and let us consider the base chain with transition probabilities ψ_{ij}, which is assumed to be irreducible, aperiodic and such that $\psi_{ij} > 0$ if and only if $\psi_{ji} > 0$. Define the following acceptance probabilities:

$$a_{ij} = \frac{\pi_j\, \psi_{ji}}{\pi_i\, \psi_{ij} + \pi_j\, \psi_{ji}}$$

as well as a new chain with transition probabilities $p_{ij} = \psi_{ij}\, a_{ij}$ if $j \neq i$. Show that this new chain is ergodic and that it satisfies the detailed balance equation:

$$\pi_i\, p_{ij} = \pi_j\, p_{ji}, \quad \forall i, j \in S$$

Exercise 2. [Sampling a binomial distribution using the Metropolis-Hastings algorithm]

Suppose you want to get samples from the binomial distribution with parameters n and p, with $0 < p < 1$. We recall that a random variable $X \sim \text{Bin}(n,p)$ has probability mass function as follows:

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad \text{for } k \in \{0, 1, \ldots, n\}$$

To perform this sampling, you decide to use the Metropolis-Hastings algorithm with a base chain given by the simple symmetric random walk on $\{0, 1, 2, \ldots, n\}$: $\psi_{ij} = 1/2$ for $|i-j| = 1$, $\psi_{00} = 1/2$, $\psi_{nn} = 1/2$, and $\psi_{ij} = 0$ otherwise.

a) Why is the simple symmetric random walk ψ an appropriate base chain?

b) What are the acceptance probabilities a_{ij}? Compute them exactly. (*Note*: For your computations here, use that $\frac{0}{0} = 0$.)

c) Construct the matrix P as seen in class. What can you say about its stationary distribution?

d) What would be the procedure to get the actual samples from the binomial distribution? What specific conditions need to be ensured in order for the sampling to work as expected? Is it the case here?

Exercise 3. [Sampling a Gaussian distribution using the Metropolis-Hastings algorithm]

a) Preliminary question. Consider the random walk on $\mathbb{Z}$ with transition probabilities $\psi_{i,i\pm 1} = 1/2$. Does this chain admit a stationary distribution ? Is it positive-recurrent?

In this problem, we use the Metropolis-Hastings rule to bias the simple random walk so that the stationary distribution of the new walk $(X_n, n \geq 0)$ equals

$$\pi_i = \frac{e^{-ai^2}}{\sum_{i=-\infty}^{+\infty} e^{-ai^2}}$$

where $a > 0$ is a parameter. Moves $i \to j = i \pm 1$ are proposed with probability ψ_{ij} and accepted with probability $a_{ij} = \min\left(1, \frac{\pi_j \psi_{ji}}{\pi_i \psi_{ij}}\right)$.

b) Give the transition probabilities $p_{ij} = \mathbb{P}(X_1 = j \mid X_0 = i)$ of the final chain for *all* i and j.

c) Show that for this chain:

$$\lim_{n\to+\infty} \mathbb{P}(X_n = j \mid X_0 = i) = \pi_j, \quad \forall i \in \mathbb{Z}.$$

For the next question, we consider two coupled walks $(X_n, n \geq 0)$ and $(Y_n, n \geq 0)$ on $\mathbb{Z}$. The walks are coupled in the following way:

- At each time step $n \geq 0$, we draw a common uniform random variable $\xi_n \in \{+1, -1\}$ and for each walk we propose the moves $X_n \to X_n + \xi_n$ and $Y_n \to Y_n + \xi_n$.
- Each move is accepted or rejected according to the Metropolis-Hastings rule of question b).

We define the coalescence time (a random variable)

$$T = \inf\{n : X_n = Y_n \text{ given that } X_0 = z, Y_0 = z + d\}$$

where z and d are strictly *positive* integers.

d) What is the smallest possible coalescence time ? Compute the probability that the coalescence time takes this smallest possible value.