Astrophysics III, Dr. Yves Revaz | EPFL

4th year physics | Exercises week 8
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Astrophysics III: Stellar and galactic dynamics

Solutions

Problem 1:

For a given potential in cylindrical coordinates $\Phi(R, \theta, z)$, the vertical and epicyclic frequencies read:

$$\begin{aligned}
\Omega^2(R_g) &= \frac{1}{R}\left(\frac{\partial \Phi}{\partial R}\right)_{(R_g, z=0)} &(1)\\
\nu^2(R_g) &= \left(\frac{\partial^2 \Phi}{\partial z^2}\right)_{(R_g, z=0)} &(2)\\
\kappa^2(R_g) &= \left(\frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2}\right)_{(R_g, z=0)} = \left(R\frac{\partial(\Omega^2)}{\partial R} + 4\Omega^2\right)_{(R_g, z=0)} &(3)
\end{aligned}$$

For the point mass, one obtains (dropping the g subscript for short):

$$\begin{aligned}
\Omega^2(R) &= \frac{GM}{R^3} &(4)\\
\nu^2(R) &= \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial z}\right)_{(R_g, z=0)} = \frac{\partial}{\partial z}\left(\frac{GMz}{(x^2+y^2+z^2)^{3/2}}\right)_{(R_g, z=0)} &(5)\\
&= \frac{GM}{R^3} - 3\left(\frac{GMz^2}{R^5}\right)_{z=0} = \frac{GM}{R^3} &(6)\\
\kappa^2(R) &= \left[\frac{\partial^2 \Phi}{\partial R^2} - \frac{1}{R}\frac{\partial \Phi}{\partial R} + \frac{4}{R}\frac{\partial \Phi}{\partial R}\right]_{(R_g, z=0)} = \frac{GM}{R^3} &(7)
\end{aligned}$$

For the Plummer-Schuster potential:

$$\begin{aligned}
\Omega^2(R) &= \frac{GM}{(e^2+R^2)^{3/2}} &(8)\\
\nu^2(R) &= = \frac{GM}{(e^2+R^2)^{3/2}} &(9)\\
\kappa^2(R) &= 4\frac{GM}{(e^2+R^2)^{3/2}} - 3R^2\frac{GM}{(e^2+R^2)^{5/2}} &(10)
\end{aligned}$$

For the Miyamoto-Nagai potential:

$$\Omega^2(R) = \frac{GM}{(R^2+(a+b)^2)^{3/2}} \tag{11}$$

$$\nu^2(R) = \frac{GM(a+b)}{b(R^2+(a+b)^2)^{3/2}} \tag{12}$$

$$\kappa^2(R) = 4\frac{GM}{(R^2+(a+b)^2)^{3/2}} - 3R^2\frac{GM}{(R^2+(a+b)^2)^{5/2}} \tag{13}$$

Problem 2:

The code to implement is given bellow:

For pm.py:

```
def Potential(x,y,z):
  R2 = x**2+y**2
  R  = sqrt(R2)
  return -GM/R

def Nu2(r):
  return GM/r**3


def Omega2(r):
  return GM/r**3


def Kappa2(r):
  return GM/r**3
```

For plummer.py:

```
def Potential(x,y,z):
  R2 = x**2+y**2
  R  = sqrt(R2)
  return -GM/sqrt( R2 + e**2 )

def Nu2(r):
  return GM/( e**2 + r**2 )**(3./2.)


def Omega2(r):
  return GM/( e**2 + r**2 )**(3./2.)


def Kappa2(r):
  return 4* GM/( e**2 + r**2 )**(3./2.) - 3* r**2 * GM/( e**2 + r**2 )
     **(5./2.)
```

For miyamoto.py:

```
def Potential(x,y,z):
  R2 = x**2+y**2
  R  = sqrt(R2)
  return -GM/sqrt( R2 + ( a + sqrt(z**2+b**2)  )**2 )

def Nu2(r):
  ab = a+b
```

```
    ab2 = ab**2
    return GM*ab / ( b* ( r**2 + ab2)**(3./2.) )


def Omega2(r):
    ab = a+b
    ab2 = ab**2
    return GM / ( r**2 + ab2 )**(3./2.)


def Kappa2(r):
    ab = a+b
    ab2 = ab**2
    return 4*GM / ( r**2 + ab2 )**(3./2.) - 3* r**2 * GM/( r**2 + ab2 )
        **(5./2.)
```

Problem 3:

Using the definition

$$v_c = R\Omega(R)$$

it follows that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}R} = \frac{1}{R}\frac{\mathrm{d}v_c}{\mathrm{d}R} - v_c\frac{1}{R^2}$$

Then

$$\begin{aligned}
A(R) &\equiv \frac{1}{2}\left(\frac{v_c}{R} - \frac{\mathrm{d}v_c}{\mathrm{d}R}\right) = \frac{1}{2}\left(-R\left(\frac{1}{R}\frac{\mathrm{d}v_c}{\mathrm{d}R} - \frac{v_c}{R^2}\right)\right) = -\frac{1}{2}R\frac{\mathrm{d}\Omega}{\mathrm{d}R}\\
B(R) &\equiv -\frac{1}{2}\left(\frac{v_c}{R} + \frac{\mathrm{d}v_c}{\mathrm{d}R}\right) = -\frac{1}{2}\left(\Omega + \left(\frac{1}{R}\frac{\mathrm{d}v_c}{\mathrm{d}R} - \frac{v_c}{R^2}\right) + \frac{v_c}{R}\right) = -\left(\Omega + \frac{1}{2}R\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)\\
\Omega &= A - B = \frac{1}{2}\left(\frac{v_c}{R} - \frac{\mathrm{d}v_c}{\mathrm{d}R}\right) + \frac{1}{2}\left(\frac{v_c}{R} + \frac{\mathrm{d}v_c}{\mathrm{d}R}\right) = \frac{v_c}{R} = \Omega\\
\kappa^2 &= \left(R\frac{\mathrm{d}(\Omega^2)}{\mathrm{d}R} + 4\Omega^2\right) = \left(2R\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}R} + 4\Omega^2\right) = 2\Omega\left(R\frac{\mathrm{d}\Omega}{\mathrm{d}R} + 2\Omega\right)\\
&= 2\Omega\left(-2B\right) = -4B(A-B)
\end{aligned}$$

Problem 4:

We have:

$$\begin{aligned}
R^2 &= x^2 + y^2\\
\vec{L} &= \vec{r}\times\vec{v} = \vec{x}\times\dot{\vec{x}} = \begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}\dot{x}\\\dot{y}\\\dot{z}\end{pmatrix} = \begin{pmatrix}y\dot{z}-z\dot{y}\\z\dot{x}-x\dot{z}\\x\dot{y}-y\dot{x}\end{pmatrix}
\end{aligned}$$

Since we're working in the $z = 0$ plane, and the z component of $\vec{L}$ is given by $L_z = x\dot{y} - y\dot{x}$, inserting this to compute L^2 gives

$$L^2 = (y\dot{z} - z\dot{y})^2 + (z\dot{x} - x\dot{z})^2 + (x\dot{y} - y\dot{x})^2 = (x^2 + y^2)\dot{z}^2 + L_z^2 = R^2\dot{z}^2 + L_z^2$$

Now we use the energy conservation:

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \Phi_{eff}(R, z)$$

Now eliminate $\dot{z}^2$ by using our expression for L^2:

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\frac{1}{R^2}(L^2 - L_z^2) + \Phi_{eff}(R, z)$$

Solving for $\dot{R}$ gives and using $\Phi_{eff} = \frac{1}{2}\frac{L_z^2}{R^2} + \Phi$:

$$\dot{R} = \pm\sqrt{2\left(E - \frac{1}{2}\frac{L^2 - L_z^2}{R^2} - \Phi_{eff}\right)} = \pm\sqrt{2\left(E - \frac{L^2}{2R^2} - \Phi\right)}$$