Introduction to Differentiable Manifolds

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Solutions Series 6 - Submanifolds (more!) 2021–11–20

Exercise 6.1. (a) If $f : M \to N$ is an immersion, show that a continuous map $h : L \to M$ is $\mathcal{C}^k$ if the composite $f \circ h$ is $\mathcal{C}^k$.

Solution. This is called the *initial property* of immersions; it is now included in Section 3.3 of the lecture notes. □

(b) If $f : M \to N$ is an embedding, show that a function $h : M \to L$ is $\mathcal{C}^k$ if the composite $f \circ h$ is $\mathcal{C}^k$.

Solution. This is the initial property of embeddings, also proved in Section 3.3 of the notes. □

(c) If $f_0 : M_0 \to N$ and $f_1 : M_1 \to N$ are $\mathcal{C}^k$ embeddings with the same image, show that there is a diffeomorphism $h : M_0 \to M_1$ such that $f_1 \circ h = f_0$.

Solution. Define the bijection $h = f_1^{-1} \circ f_0 : M_0 \to M_1$. The composite $f_1 \circ h = f_0$ is $\mathcal{C}^k$, and since the map f_1 has the initial property because it is an embedding, it follows that h is $\mathcal{C}^k$. The inverse of h is the map $h^{-1} = f_0^{-1} \circ f_1$ and it is $\mathcal{C}^k$ by the same argument. Therefore h is a diffeomorphism. □

Exercise 6.2. If S_0, S_1 are $\mathcal{C}^r$-embedded submanifolds of M_0, M_1 respectively, then $S_0 \times S_1$ is a $\mathcal{C}^r$-embedded submanifold of $M_0 \times M_1$.

Solution. By hypothesis there exists embeddings $f_0 : L_0 \to M_0$ and $f_1 : L_1 \to M_1$ whose images are S_0 and S_1 respectively. The set $S_0 \times S_1$ is the image of the $\mathcal{C}^r$ map $f_0 \times f_1 : L_0 \times L_1 \to M_0 \times M_1$ that sends $(p_0, p_1) \mapsto (f_0(p_0), f_1(p_1))$. Thus it suffices to prove that $f_0 \times f_1$ is a $\mathcal{C}^r$ embedding.

We check first that $f_0 \times f_1$ is an immersion. For this, note that for each point $p = (p_0, p_1)$ the tangent transformation $\mathrm{T}_p f_0 \times f_1$ sends $(v_0, v_1) \mapsto (\mathrm{T}_{p_0} f_0(v_0), \mathrm{T}_{p_1} f_1(v_1))$. (Here we are using the identification $T_{p_0,p_1}(L_0 \times L_1) \equiv \mathrm{T}_{p_0} L_0 \times \mathrm{T}_{p_1} L_1$.) This transformation is injective since both $\mathrm{T}_{p_0} f_0$ and $\mathrm{T}_{p_1} f_1$ are injective.

Finally, let us check that $f_0 \times f_1$ is a topological embedding. We know that each map $f_i|^{S_i}$ has a topological inverse $g_i : S_i \to L_i$. Thus the map $g_0 \times g_1 : S_0 \times S_1 \to L_0 \to L_1$ is an inverse of $(f_0 \times f_1)|^{S_0 \times S_1}$. This proves that $f_0 \times f_1$ is an homeomorphism onto its image $S_0 \times S_1$. We conclude that $f_0 \times f_1$ is a $\mathcal{C}^r$ embedding. □

Exercise 6.3. (a) Show that a subset $S \subseteq \mathbb{R}^n$ is a $\mathcal{C}^r$-embedded k-submanifold if each point $x \in S$ has an open neighborhood W such that the set $S \cap W$ is the graph of a $\mathcal{C}^r$ function that expresses some $n-k$ coordinates in terms of the remaining k coordinates. (More precisely, the function is of the form $f : U \subseteq \mathbb{R}^I \to \mathbb{R}^{I'}$, where I is a k-element subset of $n := \{0, \dots, n-1\}$, I' is its complement, and $U \subseteq \mathbb{R}^I$ is an open set.)

Solution. Let $x \in S$. By hypothesis there exists a k-element set $I \subseteq \{0, \dots, n-1\}$ (we assume w.l.o.g. $I = \{0, \dots, k-1\}$), an open set $W \subseteq \mathbb{R}^n$, an open set $U \subseteq \mathbb{R}^I$ and a $\mathcal{C}^r$ function $f : U \to \mathbb{R}^{I'}$ such that $S \cap W = \mathrm{Gra}_f$. Instead of the open set W, it is better to use the smaller open set $W' = W \cap (U \times \mathbb{R}^{I'})$. Note that this set contains the graph of f, therefore we still have $S \cap W' = \mathrm{Gra}_f$.

The set $S \cap W'$ is the image of the map $g : U \to W' : x \mapsto (x, f(x))$. This map is a $\mathcal{C}^r$ embedding because it is $\mathcal{C}^r$ and it admits a $\mathcal{C}^r$ retraction $W' \to U : (x, y) \mapsto x$. Therefore its image $\mathrm{Img}(g) = \mathrm{Gra}_f = S \cap W'$ is an embedded submanifold of W'. This proves that S fulfills the condition of being locally an embedded k-submanifold of $\mathbb{R}^n$. By a theorem of the course, we conclude that S is an embedded submanifold of $\mathbb{R}^n$. □

(b) Let S be the set of real $m \times n$ matrices of rank k. Show that S is a smooth submanifold of $\mathbb{R}^{m\times n}$. What is its dimension ?

Hint: A rank-k matrix $A \in \mathbb{R}^{m\times n}$ has an invertible $k \times k$ submatrix $A|_{I\times J}$ (where $I \subseteq m$, $J \subseteq n$ are k-element sets). Show that the coefficients $A_{i',j'}$ with $i' \notin I$ and $j' \notin J$ can be expressed as a smooth function of the other coefficients of A.

Solution. For any pair of k-element sets $I \subseteq m$, $J \subseteq n$ we define an open set $U_{I,J} \subseteq \mathbb{R}^{m\times n}$ by

$$U_{I,J} = \{A \in \mathbb{R}^{m\times n} \mid \text{the } k \times k \text{ matrix } A|_{I\times J} \text{ is invertible}\},$$

where $A|_{I\times J} = (a_{i,j})_{i\in I,\, j\in J}$. Note that the sets $U_{I,J}$ cover S because every matrix of rank k has an invertible $k \times k$ submatrix.

Let us show that $S_{I,J} = S \cap U_{I,J}$ is the graph of a smooth function. For a matrix $A \in S_{I,J}$ we will show that the part $A|_{I'\times J'}$ of the matrix can be expressed as a function of the remaining coefficients. (Recall that $I' \subseteq m$ and $J' \subseteq n$ are the complements of I and J).

Since the column space of A has dimension k, and the k columns $A_{*,j}$ with $j \in J$ are linearly independent (because the block $A|_{I\times J}$ is invertible), these columns form a base of the column space. Hence any other column $A_{*,j'}$, with $j' \in J'$, is a linear combination of the columns $A_{*,j}$ with $j \in J$. That is, we can write

$$A_{*,j'} = \sum_{j\in J} A_{*,j}\, x_{j,j'},$$

using some real coefficients $(x_{j,j'})_{j\in J, j'\in J'}$. Thus we have

$$A_{i,j'} = \sum_{j\in J} A_{i,j}\, x_{j,j'} \quad \text{for } i \in n. \tag{1}$$

Using these equations just for $i \in I$ we can find out the coefficients $x_{j,j'}$ because the matrix $A|_{I\times J}$ is invertible. Denote its inverse by $B = (B_{l,i})_{l\in J, i\in I}$. (Note that B depends smoothly on A, this can be seen using the formula for the inverse matrix in terms of cofactors.) Multiplying equations (1) for $i \in I$ by the matrix B (that is, multiplying by the coefficient $B_{l,i}$ and summing over $i \in I$), we get

$$\sum_{i\in I} B_{l,i}\, A_{i,j'} = \sum_{i\in I}\sum_{j\in J} B_{l,i}\, A_{i,j}\, x_{j,j'} = \sum_{j\in J} \delta_{l,j}\, x_{j,j'} = x_{l,j'} \quad \text{for } l \in J,$$

or, renaming,

$$\sum_{i\in I} B_{j,i}\, A_{i,j'} = x_{j,j'} \quad \text{for } j \in J,\ j \in J'$$

Now that we know the value of the coefficients $x_{j,j'}$ we can replace in equation (1), this time restricted to the remaining values of i, that is, for $i \in I'$. Renaming the index i by i', we get

$$A_{i',j'} = \sum_{j\in J} A_{i',j}\, x_{j,j'} = \sum_{j\in J} A_{i',j} \sum_{i\in I} B_{j,i}\, A_{i,j'} \quad \text{for } i' \in I',\ j' \in J'.$$

Since B depends smoothly on A, this last equation shows that the $(m-k)(n-k)$ coefficients $A_{i',j'}$ with $i' \in I'$, $j' \in J'$ can be expressed as a smooth function of the remaining $k^2 + (m-k)k + k(n-k) = k(m+n-k)$ coefficients. In summary, for each open set $U_{I,J}$, the set $S \cap U_{I,J}$ is the graph of the smooth function

$$\begin{array}{ccc} GL_k \times \mathbb{R}^{(m-k)\times k} \times \mathbb{R}^{k\times(n-k)} & \to & \mathbb{R}^{(m-k)\times(n-k)} \\ (A|_{I\times J}, A|_{I'\times J}, A|_{I\times J'}) & \mapsto & (\sum_{j\in J}\sum_{i\in I} A_{i',j}B_{j,i}\, A_{i,j'})_{i'\in I',\, j'\in J'} \end{array}$$

where B is the inverse of $A|_{I\times J}$. Therefore S is a smoothly embedded $k(m+n-k)$-submanifold of $\mathbb{R}^{m\times n}$. $\square$

Exercise 6.4. * If M is connected and $f : M \to M$ is an idempotent $\mathcal{C}^k$ map ("idempotent" means that $f \circ f = f$), then $f(M)$ is an embedded submanifold of M.
Hint: Show that f has constant rank. Use what you know about a linear projector $P : V \to V$ and the complementary projector $\mathrm{id}_V - P$.

Solution. Unfortunately the place where this exercise was taken from has an incomplete solution, thus we will not follow the hint. We will give a more complicated solution that is suggested in `https://mathoverflow.net/questions/162552/idempotents-split-in-category-of-smooth-manifolds/162556#162556`.

We first record some facts that do not involve differentiability.

Lemma. If X is a topological space and $f : X \to X$ is an idempotent continuous map, then:

(a) The image $f(X)$ is the set of fixed points $\mathrm{fix}(X) = \{x \in X : f(x) = x\}$.
(b) In consequence, the image $f(X)$ is a closed subset of X.
(c) If X is connected, then $f(X)$ is connected.
(d) Every open neighborhood U of a point $p \in f(X)$ contains a smaller open neighborhood U' of p that is invariant by f, i.e. $f(U') \subseteq U'$.

Proof. **??** If $y \in f(X)$, we can write $y = f(x)$ for some $x \in X$, therefore $f(y) = f(f(x)) = f(x) = y$, thus $y \in \mathrm{fix}(f)$. Reciprocally, if $x \in \mathrm{fix}(f)$, then $f(x) = x$ and it is clear that $x \in f(X)$.

?? follows from **??** since the equation $f(x) = x$ define a closed subset of M.

?? is a general property of continuous maps.

?? We define $U' = U \cap f^{-1}(U)$. We claim that U' is invariant by f. Indeed, take any point $x \in U'$. This means that both x and $f(x)$ are in U. Then the point $y = f(x)$ is in U' because both y and $f(y) = f(f(x)) = f(x) = y$ are in U. ☐

Now we solve the following local version of the problem.

Proposition. If $M \subseteq \mathbb{R}^n$ is an open set and $f : M \to M$ is an idempotent $\mathcal{C}^r$ map, then each point $p \in f(M)$ has an open neighborhood U such that $f(M) \cap U$ is a $\mathcal{C}^r$-embedded submanifold of U of dimension $k = \mathrm{rank}_p(f)$.

Proof. For a point $p \in f(M)$, the tangent operator $\mathrm{T}_p f$ is a linear endomorphism of $\mathrm{T}_p M = \mathbb{R}^n$ which satisfies

$$\mathrm{T}_p f = \mathrm{T}_p(f \circ f) = \mathrm{T}_p f \circ \mathrm{T}_p f.$$

Thus $\mathrm{T}_p f$ is a linear projector in $\mathbb{R}^n$, and its image and kernel are complementary subspaces of $\mathbb{R}^n$ of dimensions k and $k' = n - k$.

Let $\pi = \mathrm{id}_{\mathbb{R}}^n - \mathrm{T}_p f$ be the complementary projector of $\mathrm{T}_p f$. (Check that π is also a linear projector and has $\mathrm{Ker}(\pi) = \mathrm{Img}(\mathrm{T}_p f)$ and $\mathrm{Img}(\pi) = \mathrm{Ker}(\mathrm{T}_p f)$.)

We may assume w.l.o.g that $\mathrm{Ker}\,\mathrm{T}_p f = \mathbb{R}^{k'}$ and we consider π as a map $\mathbb{R}^n \to \mathbb{R}^{k'}$.

We define a map $g : M \to \mathbb{R}^{k'}$ that sends $x \mapsto \pi(x - f(x))$.

Note that $\mathrm{T}_p g = \pi \circ (\mathrm{T}_p f - \mathrm{id}_{\mathbb{R}^n}) = \pi \circ \pi = \pi$. Therefore g has rank k' and hence there is an open neighborhood W of p such that $g|_W : W \to \mathbb{R}^{k'}$ is a submersion. By the Lemma, we may assume that W is invariant by f.

By the regular preimage theorem, the set

$$S = \{q \in W : g(q) = 0\} = (g|_W)^{-1}(0),$$

is a k-submanifold of W.

Note that $f(M) \cap W = \mathrm{fix}(f|_W)$ is contained in S. However, it is not clear that all points of S are in $f(M)$.

Now, consider the $\mathcal{C}^r$ map $f|_W^S : W \to S$. Since f has rank k at p, and $\dim S = k$, we see that $f(W)$ contains an open neighborhood V' of p in S. We write $V' = V \cap S$, where V is an open set of M.

Let $U = W \cap V$. We claim that $f(M) \cap U$ is an embedded k-submanifold of U. In fact $f(M) \cap U = V'$. Indeed, if $x \in V' = V \cap S$, then $x \in f(W)$ (by definition of V') and it follows that $x \in W$, thus $x \in U = V \cap W$. We conclude that $x \in f(W) \cap U$. Reciprocally, if $x \in f(M) \cap U = f(M) \cap W \cap V$, we see that x is fixed by f, and also $x \in W$, so it follows that $x \in \operatorname{fix}(f|_W) \subseteq S$, thus $x \in S \cap V = V'$. This shows that $f(M) \cap U$ coincides with V', which is an open subset of S, which in turn is an embedded k-submanifold of W. Thus $f(M) \cap U$ is an embedded k-submanifold of U. □

Now we can solve the original problem. Let $f : M \to M$ be an idempotent $\mathcal{C}^r$ map, where M is a connected $\mathcal{C}^r$ manifold. We will show that $f(M) = \operatorname{fix}(f)$ is an embedded submanifold of M.

We first note that the Proposition holds for the manifold M even though M is not an open subset of $\mathbb{R}^n$.

Claim. Each point $p \in \operatorname{fix}(f) = f(M)$ has an open neighborhood U in M such that $\operatorname{fix}(f) \cap U$ is an embedded submanifold of U of dimension $k_p = \operatorname{rank}_p f$.

Proof. Proof: Take a chart (V, ϕ) that is defined at p. By the Lemma, we may assume that its domain V is f-invariant. Therefore the map $f|_V^V$ is an idempotent map $V \to V$. It follows that the local expression $\widetilde{f} = \phi \circ f \circ \phi^{-1}$ is an idempotent $\mathcal{C}^r$ map of the open set $\widetilde{V} = \phi(V) \subseteq \mathbb{R}^n$. In addition, the point $\widetilde{p} = \phi(p)$ is fixed by $\widetilde{f}$. By the Proposition, there is an open neighborhood $\widetilde{U}$ of $\widetilde{p}$ in $\widetilde{V}$ such that $\operatorname{fix}(\widetilde{f}) \cap \widetilde{U}$ is a k_p-submanifold of $\widetilde{U}$. Applying the diffeomorphism ϕ^{-1}, we get an open subset $U = \phi^{-1}(\widetilde{U})$ of M such that $\operatorname{fix}(f) \cap U$ is a k_p-submanifold of U. □

To finish showing that $\operatorname{fix}(f)$ is an embedded submanifold of M, we must show that the function $p \mapsto k_p = \operatorname{rank}_p f$ is constant throughout $f(M)$. Since $\operatorname{fix}(f) = f(M)$ is connected, it suffices to show that k_p is locally constant. But this follows from the claim. Indeed, if $p \in \operatorname{fix}(f)$ and U_p is an open neighborhood of p such that $\operatorname{fix}(f) \cap U_p$ is a k_p-submanifold of U_p, then for any point $q \in \operatorname{fix}(f) \cap U_p$ we have $k_q = k_p$, because applying the claim again we get an open set U_q such that $\operatorname{fix}(f) \cap U_q$ is a k_q submanifold, and then $\operatorname{fix}(f) \cap U_q \cap U_p$ is a submanifold of dimensions k_p and k_q at the same time. This manifold is nonempty because it contains the point q, therefore $k_p = k_q$.

□