Introduction to Differentiable Manifolds	
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Solutions Series 7 - Vector bundles	**2021–11–20**

Exercise 7.1 (Trivial vector bundles). .

(a) (⇒) Show that a vector bundle is trivial if and only if it has a global frame.

Solution. Let E be a trivial vector bundle with projection $\pi : E \to M$. By definition, there exists an isomorphism of smooth vector bundles $\phi : E \to M \times \mathbb{R}^k$, where $M \times \mathbb{R}^k$ is the trivial bundle. For the latter we can construct smooth sections $\sigma'_i : M \to M \times \mathbb{R}^k$ by setting $\sigma'_i(p) := (p, e_i)$, for $i = 1, \dots, k$, where $(e_1, \dots, e_k)$ is the standard basis of $\mathbb{R}^k$. Then $\sigma' = (\sigma'_1, \dots, \sigma'_k)$ is a global frame for $M \times \mathbb{R}^k$. Let's define $\sigma_i : M \to E : p \to \phi^{-1}(\sigma'_i(p))$. These are smooth sections because ϕ is an isomorphism of smooth vector bundles. Moreover, for any $p \in M$, we know $\phi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a bijective linear map, hence we can conclude that $\{\sigma_i(p)\}_i$ forms a basis for E_p as $\{\sigma'_i(p)\}_i$ forms a basis for $\{p\} \times \mathbb{R}^k$. Thus $\sigma = (\sigma_1, \dots, \sigma_k)$ is a global frame for (E, π).

(⇐) Conversely, suppose we have a global frame $\sigma = (\sigma_1, \dots, \sigma_k)$ for a smooth vector bundle (E, π) where $\pi : E \to M$. We know that $\forall p \in M$, the vectors $\sigma_i(p)$ form a basis of E_p. We define a map $\Psi : M \times \mathbb{R}^k \to E : (p, v) \mapsto X = \sum_{i=1}^k v^i \sigma_i(p)$. Note that $\pi' = \pi \circ \Psi$ where $\pi' : M \times \mathbb{R}^k \to M : (p, v) \mapsto p$ is the standard projection.

We claim that Ψ is a vector bundle isomorphism.

Note that for each point $p \in M$, the function $\Psi_p : \mathbb{R}^k \to E_p$ sends $v \mapsto \sum_i \sigma_i(p)$ is a linear isomorphism because the vectors $\sigma_i(p)$ form a basis of T_pM. Therefore Ψ is bijective. The inverse map Ψ^{-1} restricts to a linear isomorphism $\Psi_p^{-1} : E_p \to \mathbb{R}^k$. Thus it suffices to prove that both Ψ and Ψ^{-1} are smooth to conclude that Ψ is an isomorphism of vector bundles.

In fact, since as we said Ψ is bijective, it suffices to prove that Ψ is a local diffeo (this will imply that it is open, hence a homeo, and clearly a homeo + local diffeo is a diffeo).

Thus to finish we will prove the following:

Claim: For any local trivialization $\phi : \pi^{-1}U \to U \times \mathbb{R}^k$ (where $U \subseteq M$ is an open set), the restricted bijection

$$\Phi_U := \Phi|_{U \times \mathbb{R}^k} : U \times \mathbb{R}^k \to \pi^{-1}U$$

is a diffeo.

Proof of claim: Since ϕ is a diffeo, it suffices to show that the composite map $\phi \circ \Psi_U : U \times \mathbb{R}^k \to U \times \mathbb{R}^k$ is a diffeo. This map $\phi \circ \Psi_U$ sends

$$(p, v) \mapsto \phi(\sum_i v^i \sigma_i(p)) = (p, \sum_i v^i \widetilde{\sigma}_i(p)),$$

where $\widetilde{\sigma}_i = \pi_1 \circ \phi \circ \sigma_i : U \to \mathbb{R}^k$, where in turn $\pi_1 : U \times \mathbb{R}^k \to \mathbb{R}^k : (x, v) \mapsto v$ is the projection. This expression shows that $\phi \circ \Phi_U$ is smooth and that its differential is

$$\mathrm{D}_{p,v}(\phi \circ \Phi_U)(a, b) = (a, \sum_i b^i \widetilde{\sigma}_i + \mathrm{D}_p \widetilde{\sigma}_i(b).$$

This linear transformation is represented by a matrix $\begin{pmatrix} \mathrm{id}_{T_pU} & 0 \\ * & A_p \end{pmatrix}$ where A_p is a matrix whose columns are the vectors $\widetilde{\sigma}_i(p)$. This matrix is invertible, hence the bijection $\phi \circ \Phi_U$ is a local diffeo, hence it is a diffeo. □

(b) Show that the vector bundle $\mathrm{T}\mathbb{S}^1$ is trivial.

Solution. Since $\mathbb{S}^1$ is diffeomorphic to $\mathbb{T}^1$, it suffices to show that the tangent bundle of the n-torus $\mathbb{T}^n$ is trivial.

We denote $\kappa : \mathbb{R}^n \to \mathbb{T}^n$ the quotient map, since the letter π is now used for the projection $\pi : \mathrm{T}\mathbb{T}^n \to \mathbb{T}^n$.

Recall that there is an inverse atlas of $\mathbb{T}^n$ consisting of the parametrizations $\phi = \kappa|_{\widetilde{U}} : \widetilde{U} \to U \subseteq \mathbb{T}^n$, where $\widetilde{U} \subseteq \mathbb{R}^n$ is any open set where κ is injective and $U = \kappa(\widetilde{U})$.

Each such parametrization ϕ of $\mathbb{T}^n$ induces a parametrization $\Phi : \widetilde{U} \times \mathbb{R}^n \to \pi^{-1}U$ of $\mathrm{T}\mathbb{T}^n$ that sends $(x, v) \mapsto (\phi_{\widetilde{U}}(x), \sum_i v^i \left.\frac{\partial}{\partial(\phi^{-1})^i}\right|_p)$. Note here that ϕ^{-1} is a chart of $\mathbb{T}^n$. The parametrizations Φ of this kind form an atlas of $\mathrm{T}\mathbb{T}^n$, which defines the smooth structure on $\mathrm{T}\mathbb{T}^n$.

We define a frame of $\mathrm{T}\mathbb{T}^n$ consisting of n vector fields E^i defined as follows. For each parametrization $\phi : \widetilde{U} \to U$ as above, we let

$$E^i(p) = \Phi(\phi^{-1}(p), e_i) \quad \text{for all } p \in U.$$

This formula defines $E^i|_U$. Let us check that E^i is well defined (i.e. that the formula agrees on an intersection $U \cap V$ of images of two parametrizations $\phi : \widetilde{U} \to U$, $\psi : \widetilde{V} \to V$. For this, recall that the transition map $\psi^{-1} \circ \phi$ between the parametrizations ϕ, ψ of $\mathbb{T}^n$ is locally a translation. Therefore the transition map between the parametrizations Φ, Ψ of $\mathrm{T}\mathbb{T}^n$ is

$$\Psi^{-1} \circ \Phi(x, v) = (\psi^{-1} \circ \phi(x), \mathrm{D}_{\phi(x)}(\psi^{-1} \circ \phi)(v)) = (\psi^{-1} \circ \phi(x), v)$$

since the differential of a translation is the identity map. Equivalently, we have $\Phi(x, v) = \Psi(y, v)$ if $\phi(x) = \psi(y)$. In particular, for a point $x \in U \cap V$, putting $x = \phi^{-1}(p)$ and $y = \psi^{-1}(p)$, we have

$$\Phi(\phi^{-1}(p), e_i) = \Phi(x, y) = \Psi(y, e_i) = \Psi(\psi^{-1}(p), e_i),$$

as needed to show that E_i is well defined.

The vector fields E^i are clearly smooth because they are smooth on each open set U as above, since the maps Φ and ϕ^{-1} are smooth. The vector fields E^i are also linearly independent at each point $p = \phi(x) \in \mathbb{T}^n$, since the vectors e_i are linearly independent. Therefore the vectors E_i constitute a frame of TTT^n, defined globally (i.e. on the whole torus $\mathbb{T}^n$). We conclude that that the tangent bundle $\mathrm{T}\mathbb{T}^n$ is trivial.

□

Exercise 7.2 (Properties of smooth vector fields)**.** Let M be a smooth manifold and let $X : M \to TM$ be a vector field. Show that the following are equivalent:

(a) X is a smooth vector field.
(b) The component functions of X are smooth with respect to all charts of one particular smooth atlas of M.
(c) For any smooth function $f : U \to \mathbb{R}$ on an open set $U \subset M$, the function $Xf : U \to \mathbb{R}$ defined by $Xf(p) := X_p(f)$ is smooth.

Solution. Let $(M, \mathcal{A})$ be a smooth manifold and X a vector field. Recall that we say that X is a smooth vector field if the component functions of X are smooth for any chart $(U, \varphi) \in \mathcal{A}$. The component functions w.r.t (U, φ) were defined as the functions $X^i : U \to \mathbb{R}$ such that

$$X_p = \sum_i X^i(p) \left.\frac{\partial}{\partial \varphi^i}\right|_p, \quad p \in U.$$

$(a) \Rightarrow (b)$ is clear.

(b) $\Rightarrow$ *(a)* Let $\mathcal{A}' \subset \mathcal{A}$ and suppose the component functions are smooth wrt all $(U, \varphi) \in \mathcal{A}'$. Let $(V, \psi) \in \mathcal{A}$. We write

$$X_p = \sum_i \widetilde{X}^i(p) \left.\frac{\partial}{\partial \psi^i}\right|_p, \quad p \in V$$

where $\widetilde{X}^i$ are the component functions of X wrt (V, ψ). To show that the $\widetilde{X}^i$ are smooth on V it suffices to show that they are smooth in a neighborhood of every point of V. So let $p \in V$, let $(U, \varphi) \in \mathcal{A}'$ be a chart containing p and let X^i be the component functions of X wrt (U, φ). Then from the change of coordinates formula it follows that (Exercise 3.iii from last week)

$$\widetilde{X}^i(q) = \sum_j \left(\left.\frac{\partial}{\partial \varphi^j}\right|_q \psi^i \right) X^j(q), \quad q \in U \cap V$$

and we conclude that $\widetilde{X}^i$ is smooth on $U \cap V$, $i = 1, \ldots, n$.

(c) $\Rightarrow$ *(a)* Let $(U, \varphi) \in \mathcal{A}$. Applying X to one of the components of φ yields $X\varphi^i = X^i$, which is smooth by hypothesis, i.e. the component functions of X wrt (U, φ) are smooth.

(a) $\Rightarrow$ *(c)* Conversely, suppose X is a smooth vector field, let $f \in \mathcal{C}^\infty(U)$ for an open set $U \subset M$. To check that Xf is smooth, it suffices to check that it is smooth in a neighborhood of every point of U. Given $p \in U$, let (W, φ) be a smooth chart containing p and satisfying $W \subset U$. Then on W we can write

$$Xf(q) = \sum_i X^i(q) \left.\frac{\partial}{\partial \varphi^i}\right|_q f$$

Then Xf is smooth on W since the component function of X are smooth by hypothesis and f is smooth (so in particular $\left.\frac{\partial}{\partial \varphi^i}\right|_q f = \left.\frac{\partial (f \circ \varphi^{-1})}{\partial x^i}\right|_{\varphi(q)}$ is smooth as a function of $q \in W$). □

Exercise 7.3 (Vector field on S^2)**.** Show that there is a smooth vector field on S^2 which vanishes at exactly one point.

Hint: Try using stereographic projection and consider one of the coordinate vector fields.

Solution. Recall that

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

Let us denote (u, v) the stereographic coordinates relative to the projection from the north pole $N = (0, 0, 1)$, that is, the map

$$\begin{aligned} \phi : \mathbb{S}^2 \setminus \{N\} &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (u, v) = \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right). \end{aligned}$$

(Note that we use the letters u, v to denote real numbers but also to denote the component functions ϕ^0, ϕ^1 of the chart ϕ, which are functions $\mathbb{S}^2 \to \mathbb{R}$.)

Similarly, denote $(\overline{u}, \overline{v})$ the stereographic coordinates relative to the projection from the south pole $S = (0, 0, -1)$, which is the map

$$\begin{aligned} \psi : \mathbb{S}^2 \setminus \{S\} &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (\bar{u}, \bar{v}) = \left(\frac{x}{1 + z}, \frac{y}{1 + z} \right) \end{aligned}$$

The transition function $\psi \circ \phi^{-1}(u, v)$ is obtained after some computation:

$$(\overline{u}, \overline{v}) = \left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2} \right)$$

For this we use the inverse of the north stereographic projection which is

$$x = \frac{2u}{1+u^2+v^2} \qquad y = \frac{2v}{1+u^2+v^2} \qquad z = \frac{-1+u^2+v^2}{1+u^2+v^2}.$$

Let $X = \frac{\partial}{\partial \phi^0} = \frac{\partial}{\partial u}$ be the first coordinate vector field of the chart ϕ. This vector field X is a non-vanishing smooth vector field defined on $\mathbb{S}^2 \setminus \{N\}$. (Its component functions w.r.t. ϕ are just the constant functions 1 and 0; therefore X is smooth.) The important step is to show that X extends to a smooth vector field defined on the whole sphere.

For this we compute the component functions w.r.t. ψ on the intersection of the two charts, i.e. on $\mathbb{S}^2 \setminus \{N, S\}$:

$$\begin{aligned} X &= \frac{\partial \psi^0}{\partial \phi^0} \frac{\partial}{\partial \psi^1} + \frac{\partial \psi^1}{\partial \phi^0} \frac{\partial}{\partial \psi^1} \\ &= \frac{\partial \overline{u}}{\partial u} \frac{\partial}{\partial \overline{u}} + \frac{\partial \overline{v}}{\partial u} \frac{\partial}{\partial \overline{v}} \\ &= \frac{v^2-u^2}{(u^2+v^2)^2} \frac{\partial}{\partial \overline{u}} + \frac{-2uv}{(u^2+v^2)^2} \frac{\partial}{\partial \overline{v}} \\ &= (\overline{v}^2 - \overline{u}^2) \frac{\partial}{\partial \overline{u}} - 2\overline{uv} \frac{\partial}{\partial \overline{v}} \end{aligned}$$

From this we see that X can be extended to a smooth vector field X on the whole sphere by setting its value on the north pole to zero., i.e.

$$X|_p = \begin{cases} \frac{\partial}{\partial u}\big|_p & \text{if } p \in \mathbb{S}^2 \setminus \{N\} \\ 0 & \text{if } p = N. \end{cases}$$

□