Equilibria of collisionless systems

2rd part

Outlines

The Jeans theorems

- Symmetry and integrals of motion

Connections between barotropic fluids and ergodic stellar systems

Self-consitent spherical models with Ergodic DF

- DFs from mass distribution
    - The Eddington formula
    - Examples
- Models defined from DFs
    - Polytropes and Plummer models

Quick summary of the last lecture

Distribution function (DF)

Definition (1) $f(\vec{x}, \vec{v}, t)$ or $f(\vec{w}, t)$ such that

$f(\vec{x}, \vec{v}, t)\, d^3\vec{x}\, d^3\vec{v}$ or $f(\vec{w}, t)\, d^3\vec{w}$

is the probability that at the time t,
a randomly chosen star "i" has its position $\vec{x}_i$
an velocity $\vec{v}_i$, or phase space coordinates $\vec{w}_i$
in the ranges

$$\vec{x}_i \in [\vec{x}, \vec{x} + d\vec{x}]$$
$$\vec{v}_i \in [\vec{v}, \vec{v} + d\vec{v}]$$
$$\equiv \quad \vec{w}_i \in [\vec{w}, \vec{w} + d\vec{w}]$$

obviously :

(normalisation)

$$\boxed{\begin{aligned} &\int f(\vec{x}, \vec{v}, t)\, d^3\vec{x}\, d^3 v = 1 \\ \equiv &\int f(\vec{w}, t)\, d^6\vec{w} = 1 \end{aligned}}$$

the particle is for sure somewhere in the phase space

$f(\vec{x}, \vec{v}, t)$ is the probability density of the phase space.

The collisionless Boltzmann equation

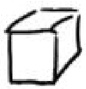
- What is the evolution of $f(\vec{w})$ over time ?

  As the mass, <u>the probability is a conserved quantity</u>. $\rho = N\bar{f}$

  <u>the number of stars is a conserved quantity</u>.

  in the phase space

Continuity equation (similar than for hydrodynamics)

V  the time variation of the mass in V : $\frac{dM}{dt} = \sum_{faces} \underbrace{\rho \vec{v}}_{\text{mass flux}} \cdot d\vec{S}$

Gauss

Mass conservation

$$\boxed{\frac{\partial \rho}{\partial t} + \underbrace{\vec{\nabla}_x \cdot (\rho \vec{v})}_{\text{mass flux through the surface of the volume}} = 0}$$

Probability conservation

$$\boxed{\frac{\partial f}{\partial t} + \underbrace{\vec{\nabla}_w \cdot (f \dot{\vec{w}})}_{\text{probability flux through the surface of the volume}} = 0}$$

The Collisionless Boltzmann equation in various coordinates

Generalized coordinates

$$\vec{p} = \frac{\partial L(\vec{q},\vec{p})}{\partial \dot{\vec{q}}}$$

$$\boxed{\frac{\partial f}{\partial t} + \dot{\vec{q}}\cdot\frac{\partial f}{\partial \vec{q}} + \dot{\vec{p}}\cdot\frac{\partial f}{\partial \vec{p}} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{q}}\cdot\frac{\partial H}{\partial \vec{p}} - \frac{\partial f}{\partial \vec{p}}\cdot\frac{\partial H}{\partial \vec{q}} = 0}$$

Cartesian coordinates

$$\begin{cases} p_x = \dot{x} = v_x \\ p_y = \dot{y} = v_y \\ p_z = \dot{z} = v_z \end{cases} \qquad H = \frac{1}{2}\left(v_x^2 + v_y^2 + v_z^2\right) + \Phi(x,y,z)$$

$$\boxed{\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{x}} - \frac{\partial \Phi}{\partial \vec{x}}\cdot\frac{\partial f}{\partial \vec{v}} = 0}$$

Spherical coordinates

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2\dot{\theta} = r v_\theta \\ p_\phi = r^2\sin^2(\theta\dot{\phi}) = r\sin(\theta)\,v_\phi \end{cases}$$

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2(\theta)}\right) + \Phi(R,\theta,\phi)$$

$$\boxed{\frac{\partial f}{\partial t} + p_r\frac{\partial f}{\partial r} + \frac{p_\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{p_\phi}{r^2\sin^2(\theta)}\frac{\partial f}{\partial \phi} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3\sin^2(\theta)}\right)\frac{\partial f}{\partial p_r} - \left(\frac{\partial \Phi}{\partial \theta} - \frac{p_\phi^2\cos(\theta)}{r^2\sin^3(\theta)}\right)\frac{\partial f}{\partial p_\theta} - \frac{\partial \Phi}{\partial \phi}\frac{\partial f}{\partial p_\phi} = 0}$$

Cylindrical coordinates

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2\dot{\phi} = R \\ p_z = \dot{z} = v_z \end{cases}$$

$$H = \frac{1}{2}\left(p_R^2 + \frac{p_\phi^2}{R^2} + p_z^2\right) + \Phi(R,\phi,z)$$

$$\boxed{\frac{\partial f}{\partial t} + p_R\frac{\partial f}{\partial R} + \frac{p_\phi}{R^2}\frac{\partial f}{\partial \phi} + p_z\frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3}\right)\frac{\partial f}{\partial p_R} - \frac{\partial \Phi}{\partial \phi}\frac{\partial f}{\partial p_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial p_z} = 0}$$

Jeans theorems

I. Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion in the given potential.

Demonstration:

If a function is a solution of the steady-state collisionless Boltzmann equation, then, it is an integral of motion, thus the function depends on the phase-space coordinates only through integrals of motion (itself !).

II. Any function of integrals of motion yields a steady-state solution of the collisonless Boltzmann equation.

Extremely useful to generate DFs

Demonstration:

Assume $f(\vec{x},\vec{v}) = f(I_1(\vec{x},\vec{v}), I_2(\vec{x},\vec{v}), I_3(\vec{x},\vec{v}), \ldots)$ and derivate...

Equilibria of collisionless systems

Symmetries and DFs

Choices of DFs and relations with the velocity moments

1. DFs that depend only on H

(no particular symmetry) except time!

$$\phi = \phi(\vec{x}, \cancel{t})$$

Ergodic distribution functions

Example

$$\begin{cases} H(\vec{x}, \vec{v}) = \frac{1}{2}\vec{v}^2 + \phi(\vec{x}) \\ f = f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right) \end{cases}$$

Note: the velocity dependency is only through v^2 (isothropic)

Mean velocity

$$\bar{\vec{v}}(\vec{x}) = \frac{1}{\nu(\vec{x})}\int \vec{v}\, f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right) d^3\vec{v} = 0$$

indeed

$$\bar{v}_x(\vec{x}) = \frac{1}{\nu(\vec{x})} \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_x \underbrace{v_x}_{\text{odd}} \underbrace{f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right)}_{\text{even}} = 0$$

$$\boxed{\text{1. DFs that depend only on } \mathcal{H}}$$

Velocity dispersions

$$\sigma_{ij}^2 = \frac{1}{\nu(\bar{x})} \int \underbrace{(v_i - \cancel{\bar{v}_i})(v_j - \cancel{\bar{v}_j})}_{\text{odd, except if } i=j} \, f\left(\tfrac{1}{2}\bar{v}^2 + \phi(\bar{x})\right) d^3\bar{v}$$

$\bar{v}_i = 0 \qquad \bar{v}_j = 0$

$$= \delta_{ij}\,\sigma^2 \qquad (\sigma_{xx} = \sigma_{yy} = \sigma_{zz})$$

$$\sigma^2 = \frac{1}{\nu(\bar{x})} \int_{-\infty}^{\infty} v_z^2 \, dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \, f\left(\tfrac{1}{2}\bar{v}^2 + \phi(\bar{x})\right)$$

using spherical coord in velocity space : $\left\{ \begin{aligned} dv_x\, dv_y\, dv_z &= v^2 \sin\theta \, dv\, d\varphi\, d\theta \\ v_z^2 &= v^2 \cos^2\theta \\ v^2 &= v_x^2 + v_y^2 + v_z^2 \end{aligned} \right.$

$$\boxed{\sigma^2 = \frac{4}{3}\pi \frac{1}{\nu(\bar{x})} \int_0^{\infty} v^4 \, f\left(\tfrac{1}{2}v^2 + \phi(\bar{x})\right) dv}$$

$$\sigma_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

> isothropic system :
>
> the velocity ellipsoid is a sphere

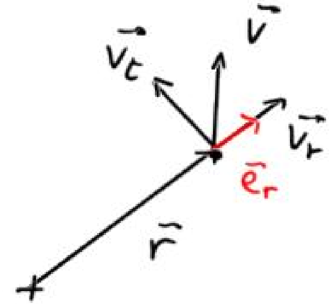
2. DFs that depend on H and $\vec{L}$

(spherical symmetry)

$$\phi = \phi(r)$$

We restrict our study to symmetric DFs : indep. of any direction

$$f(\vec{x}, \vec{v}) = f(H, L) \qquad \vec{L} \to |\vec{L}| = L$$



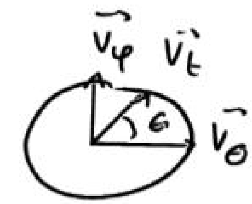
<u>we consider</u>

$$v_r = \vec{v} \cdot \vec{e}_r$$

radial velocity : $\vec{v}_r = v_r \vec{e}_r$

tangential velocity : $\vec{v}_t = \vec{v} - v_r \vec{e}_r$

$$\vec{v}_t^{\,2} = \vec{v}_\theta^{\,2} + \vec{v}_\varphi^{\,2}$$



$$v_\theta = v_t \cos\theta \qquad v_\varphi = v_t \sin\theta$$

$$\left\{ \begin{aligned} L &= r^2 \dot{\theta} = r v_t = r\sqrt{v_\theta^2 + v_\varphi^2} \\ H &= \frac{1}{2} \underbrace{(v_r^2 + v_t^2)}_{v_r^2 + v_\theta^2 + v_\varphi^2} + \phi(r) \end{aligned} \right.$$

2. DFs that depend on H and $\vec{L}$

Mean velocity

$$\bar{v}_r = \frac{1}{\nu(x)} \int v_r \, f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t\right) d^3v$$

$$= \frac{1}{\nu(x)} \int_{-\infty}^{\infty} \underbrace{v_r}_{\text{odd in } v_r} dv_r \int \underbrace{d^2\vec{v}_t \, f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t\right)}_{\text{even in } v_r} = 0$$

$$\bar{\vec{v}}_t = \frac{1}{\nu(x)} \int \underbrace{\vec{v}_t}_{\text{odd in } v_t} d^2\vec{v}_t \int_{-\infty}^{\infty} \underbrace{dv_r \, f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t\right)}_{\text{even in } v_t} = 0$$

Note $v_t \in [0, \infty]$

2. DFs that depend on H and $\vec{L}$

Velocity dispersions

veloc. in cyl. coord

$dv_\theta \, dv_\varphi \rightarrow v_t \, dv_t$

$$\sigma_r^2 = \frac{1}{\nu(x)} \int_{-\infty}^{\infty} v_r^2 \, dv_r \int_{-\infty}^{\infty} dv_\theta \int_{-\infty}^{\infty} dv_\varphi \, f\left(\tfrac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2) + \phi(r), r v_t\right)$$

$$= \frac{2\pi}{\nu(x)} \int_{-\infty}^{\infty} v_r^2 \, dv_r \int_{0}^{\infty} dv_t \, v_t \, f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t\right) \neq 0$$

$$\sigma_\theta^2 = \frac{1}{\nu(x)} \int_{-\infty}^{\infty} v_\theta^2 \, dv_\theta \int_{-\infty}^{\infty} dv_\varphi \int_{-\infty}^{\infty} dv_r \, f\left(\tfrac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2) + \phi(r), r v_t\right)$$

$dv_\theta \, dv_\varphi \rightarrow v_t \, dv_t$

$$= \frac{1}{\nu(x)} \int_{-\infty}^{\infty}\int_{0}^{\infty} v_\theta^2 \, v_t \, dv_t \, dv_r \; f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t\right)$$

$v_\theta^2 \, v_t \, dv_t = v_t^2 \cos^2\theta \, v_t \, dv_t \rightarrow \pi \, v_t^3 \, dv_t$

$$= \frac{\pi}{\nu(x)} \int_{0}^{\infty} dv_t \, v_t^3 \int_{-\infty}^{\infty} dv_r \, f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t\right)$$

2. DFs that depend on H and $\vec{L}$

Velocity dispersions

$$\sigma_\varphi^2 = \frac{1}{\nu(x)} \int_{-\infty}^{\infty} v_\varphi^2 \, dv_\varphi \int_{-\infty}^{\infty} dv_\theta \int_{-\infty}^{\infty} dv_r \, f\left(\tfrac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2) + \phi(r), r v_t \right)$$

$dv_\theta \, dv_\varphi \rightarrow v_t \, dv_t$

$$= \frac{1}{\nu(x)} \int_{-\infty}^{\infty}\int_{0}^{\infty} v_\varphi^2 \; v_t \; dv_t \, dv_r \; f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t \right)$$

$v_\varphi^2 = v_t^2 \sin^2\theta \rightarrow \pi v_t^3 \, d v_t$

$$= \frac{\pi}{\nu(x)} \int_{0}^{\infty} dv_t \, v_t^3 \int_{-\infty}^{\infty} dv_r \, f\left(\tfrac{1}{2}(v_r^2 + v_t^2) + \phi(r), r v_t \right)$$

$$\boxed{\sigma_\varphi^2 = \sigma_\theta^2}$$

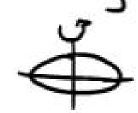
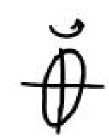
ok, spherical symmetry

$\sigma_{ij} = 0$ if $i \neq j$

Anisothropic system

$$\boxed{\sigma_r^2 \neq \sigma_\theta^2 = \sigma_\varphi^2}$$

The velocity ellipsoid is

oblate  or prolate 

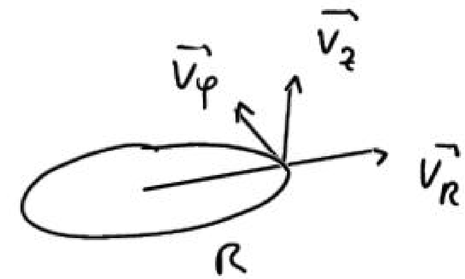
3. DFs that depend on $\mathcal{H}$ and L_z

(cylindrical symmetry)

$$\phi = \phi(R, |z|)$$

$$f(\vec{x}, \vec{v}) = f(\mathcal{H}, L_z)$$

$$\begin{cases} \mathcal{H} = \frac{1}{2}\left(v_r^2 + v_\varphi^2 + v_z^2\right) + \phi(R,z) \\ L_z = R^2\dot{\varphi} = R v_\varphi \qquad (v_\varphi = R\dot{\varphi}) \end{cases}$$



Mean velocity

$$\bar{v}_R = \int dv_R \; v_R \int dv_z \, dv_\varphi \; f\left(\frac{1}{2}\left(v_r^2 + v_\varphi^2 + v_z^2\right) + \phi(R,z), R v_\varphi\right) = 0$$

odd in v_R

$$\bar{v}_z = \int dv_z \; v_z \int dv_R \, dv_\varphi \; f\left(\frac{1}{2}\left(v_r^2 + v_\varphi^2 + v_z^2\right) + \phi(R,z), R v_\varphi\right) = 0$$

odd in v_z

$$\bar{v}_\varphi = \int dv_\varphi \; v_\varphi \int dv_R \, dv_z \; f\left(\frac{1}{2}\left(v_r^2 + v_\varphi^2 + v_z^2\right) + \phi(R,z), R v_\varphi\right)$$

$\neq 0$ in general (net rotation)

$= 0$ only if f is an even function of $L_z = R v_\varphi$

3. DFs that depend on H and L_z

Velocity dispersions

$$\sigma_R^{\,2} = \frac{1}{\nu(x)} \int dv_R\, v_R^2 \int dv_z \int dv_\varphi\, f\left(\frac{1}{2}\left(v_r^{\,2} + v_\varphi^2 + v_z^{\,2}\right) + \Phi(R,z),\, R\, v_\varphi\right)$$

$$\sigma_z^{\,2} = \sigma_R^{\,2}$$

(both variables v_R and v_z can be exchanged)

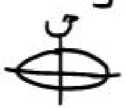
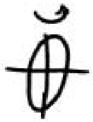
$$\sigma_\varphi^{\,2} = \frac{1}{\nu(x)} \int dv_\varphi \left(v_\varphi - \bar{v}_\varphi\right)^2 \int dv_z\, dv_R\, f\left(\frac{1}{2}\left(v_r^{\,2} + v_\varphi^2 + v_z^{\,2}\right) + \Phi(R,z),\, R\, v_\varphi\right)$$

σ is isothropic in the meridional plane

Anisothropic system

$$\boxed{\sigma_\varphi^{\,2} \neq \sigma_R^{\,2} = \sigma_z^{\,2}}$$

The velocity ellipsoid is

oblate  or prolate 

Interpretation : relation between the DF and the orbits

Example 1 — 1-D potential

$$\begin{cases} E = \frac{1}{2} v^2 + \phi(r) \\ v = \pm \sqrt{2(E - \phi(r))} \end{cases}$$

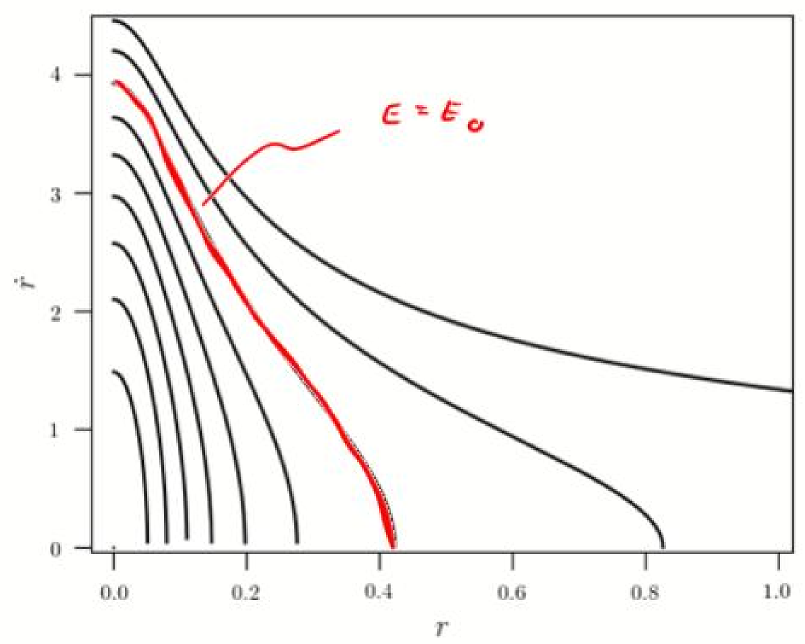
a) $f(x, v) = f(E) = \delta(E - E_0)$

$$\begin{cases} \infty & v = \pm \sqrt{2(E_0 - \phi(r))} \\ 0 & \text{instead} \end{cases}$$

b) $f(x, v) = f(E)$

$\Downarrow$

give a weight to orbits depending on their energy



Example 2 3D - spherical potential

- orbits described in planes, characterized by (E, L)

a) Ergodic DF : $f(\vec{x}, \vec{v}) = f(E)$

$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

- model built-out of all orbits of all planes with a weight that depends on the energy

(radial and circular orbits are weighted the same way) invariant under rotation (isotropic)

b) non Ergodic DF : $f(\vec{x}, \vec{v}) = f(E, L)$

- model built-out of all orbits of all planes with a weight that depends on E and L

(radial and circular orbits are weighted differently)

$$\boxed{\sigma_r^2 \neq \sigma_\theta^2 = \sigma_\varphi^2}$$

c) non Ergodic DF : $f(\vec{x}, \vec{v}) = f(E, \vec{L}) = f_E(E)\, f_L(\vec{L})$

with $f_L(\vec{L}) = 0$ if $\begin{cases} L_x \neq 0 \\ L_y \neq 0 \end{cases}$ $L_z = 0$

- model built-out of orbits lying in the $z = 0$ plane with a weight that depends on E and L_z

$$\boxed{\sigma_\varphi^2 \neq \sigma_R^2 = \sigma_z^2}$$

Questions

Why an ergodic DF <u>where there is a priori no constraints on the symmetry of the potential</u> leads to an <u>isotropic</u> velocity dispersion tensor ?

$$\Phi(x, y, z) \quad f(H) \qquad \Longrightarrow \qquad \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

Connections between barotropic fluids and ergodic stellar systems

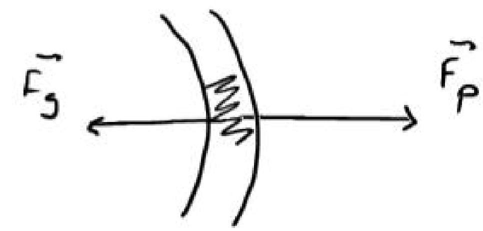
Connections between fluids and stellar systems

In fluid dynamics, the properties of a fluid at rest in a potential is obtained through the Euler equation

$$\boxed{\frac{d\vec{v}}{dt} = \underbrace{-\frac{\vec{\nabla} P}{\rho}}_{\text{pressure force}} \underbrace{- \vec{\nabla}\phi}_{\text{gravity}}}$$

<u>At rest</u> $\quad \frac{d\vec{v}}{dt} = 0$

$$\boxed{\frac{\vec{\nabla} P}{\rho} = -\vec{\nabla}\phi}$$



<u>In 1-D</u> (isothropic case)

$$\boxed{\frac{1}{\rho}\frac{dP}{d\rho} = -\frac{d\phi}{dr}}$$

Eqration of state (EoS) $\quad P = P(\rho, T)$

$P = P(\rho)$: barothropic (depends only on the density)

$P = K\rho^{\gamma}$: polythropic

$P = \frac{k_B T}{m}\rho$: isotherm (T = cte)

Together with the hydrostatic eqration,

$$\boxed{\frac{1}{\rho}\frac{dP}{d\rho} = -\frac{d\phi}{dr}}$$

this relates $\underline{\rho(r) \text{ with } \phi(r)}$.

Self - gravity

The Poisson equation

$$\boxed{\vec{\nabla}^2 \phi = 4\pi G \rho}$$

This constraints the potential $\phi(r)$

or equivalently the density $\rho(r)$

Parallel between gaseous systems and ergodic stellar systems

Note An ergodic DF is such that the velocity dispertion is isothropic

$(\sigma_{\sigma\sigma})$ ≡ similar to a gaseous system

Idea : define a function $P(\rho)$ (an equivalent of the pressure) which is such that :

$$\boxed{\frac{\vec{\nabla} P}{\rho} = - \vec{\nabla} \phi}$$

$$\frac{1}{\rho} \frac{dP}{d\rho} = - \frac{d\phi}{dr}$$

if spherical

If we find $P(\rho)$ for our stellar system, its density will be the same than the one of a gaseous system as the "pressure" will be equivalent.

Ergodic DF $\qquad f(\vec{x}, \vec{v}) = f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right)$

Density $\qquad \rho(\vec{x}) = \int d^3v \; f(\vec{x}, \vec{v})$

$$= \int d^3v \; f\left(\frac{1}{2}\vec{v}^2 + \phi(\vec{x})\right)$$

as ρ depends on $\vec{x}$ only through ϕ, we can write:

$\rho = \rho(\phi)$ and assuming it to be bijective

$$\boxed{\phi = \phi(\rho)}$$

we can then compute $\dfrac{\partial \phi}{\partial \rho}$

Lets define the function $p(\rho)$

$$\boxed{p(\rho) = -\int_0^{\rho} d\rho' \, \rho' \, \frac{\partial \phi}{\partial \rho}(\rho')}$$

Differentiating gives

$$\frac{\partial p}{\partial \rho}(\rho) = -\rho \frac{\partial \phi}{\partial \rho}(\rho)$$

with $\rho = \rho(\vec{x})$ $\quad \frac{\partial p}{\partial \rho} = \vec{\nabla} p \cdot \frac{\partial \vec{x}}{\partial \rho}$, $\quad \frac{\partial \phi}{\partial \rho} = \vec{\nabla} \phi \cdot \frac{\partial \vec{x}}{\partial \rho}$

it becomes:

$$\boxed{\frac{\vec{\nabla} p}{\rho} = - \vec{\nabla} \phi}$$

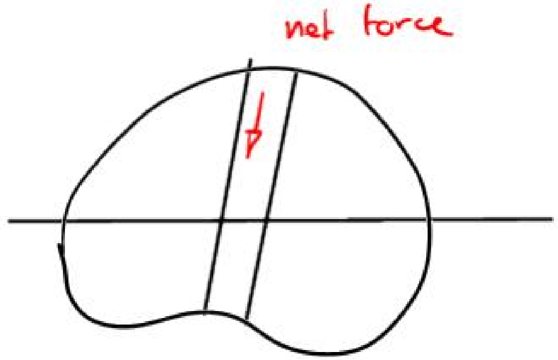
Which is the equation of equilibrium for a barotropic fluid.

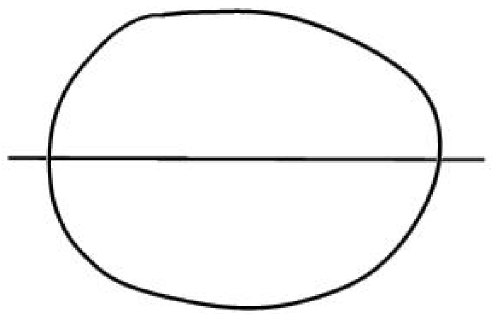
Conclusions

1. To demonstrate the analogy between an ergodic stellar system and a gaseous system, it is sufficient to show that the DF leads to the same pressure form $P(\rho)$

2. An ergodic isolated stellar system is spherical

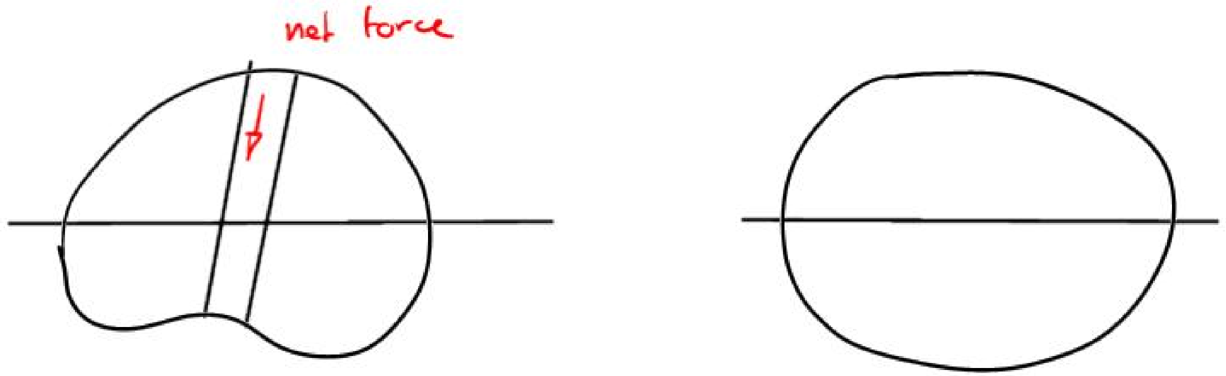
As an isolated finite, static, self-gravitating barotropic fluid must be spherical. (Lichtenstein's theorem)





As a stellar system with an ergodic DF satisfies the same equations, it must be spherical

As an isolated finite, static, self-gravitating barotropic fluid must be spherical. (Lichtenstein's theorem)



Theorem

Any isolated, finite, stellar system with an ergodic distribution function must be spherical.

Self-consistent spherical models with ergodic DFs

Distribution function for spherical systems (Ergodic DFs)

isothropic velocity field

Goal provide a <u>self-consistent</u> model for a spherical stellar system

ex: - elliptical galaxy
- galaxy cluster
- globular cluster

<u>self-consistent</u> = the density obtained from the DF is the one that generates the potential

i.e. is a solution of the Poisson equation

$$\boxed{\rho(\vec{x}) = N m \underbrace{\int d^3v \, f(\vec{x},\vec{v})}_{\nu(\vec{x})} \equiv \frac{1}{4\pi G} \vec{\nabla}^2 \Phi}$$

<u>assumptions</u> : only one type of stars (one stellar population)

i.e. all stars are modeled through the same DF.

Distribution function for spherical systems

- **Method ①**
  - from $\rho(r)$ $\phi(r)$ $\rightarrow$ get $f(\varepsilon) \equiv f\left(\frac{1}{2}v^2 + \phi(r)\right)$
- **Method ②**
  - assume $f(\varepsilon)$ $\rightarrow$ get $\rho(r)$

Spherical systems definded by DFs

Equilibria of collisionless systems

DFs from mass distribution

Determination of the DF from the mass distribution

- We assume that $\rho(r)$ and $\phi(r)$ are known funtions related together by the Poisson equation : $\nabla^2 \phi = 4\pi G \rho$

- The density is related to the DF : $\nu(r) = \dfrac{\rho(r)}{N m} = \dfrac{\rho(r)}{M}$

$$\rho(r) = M \, \nu(r) = \int f(E) \, d^3\vec{v}$$

$$= \int_0^\infty dv \, 4\pi v^2 f\left(\tfrac{1}{2}v^2 + \phi(r)\right)$$

$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}\dot{z}^2 + \phi(r)$

$= \frac{1}{2} v^2 + \phi(r)$

(isothropic in the velocity space)

We are thus looking for DFs f that satisfy :

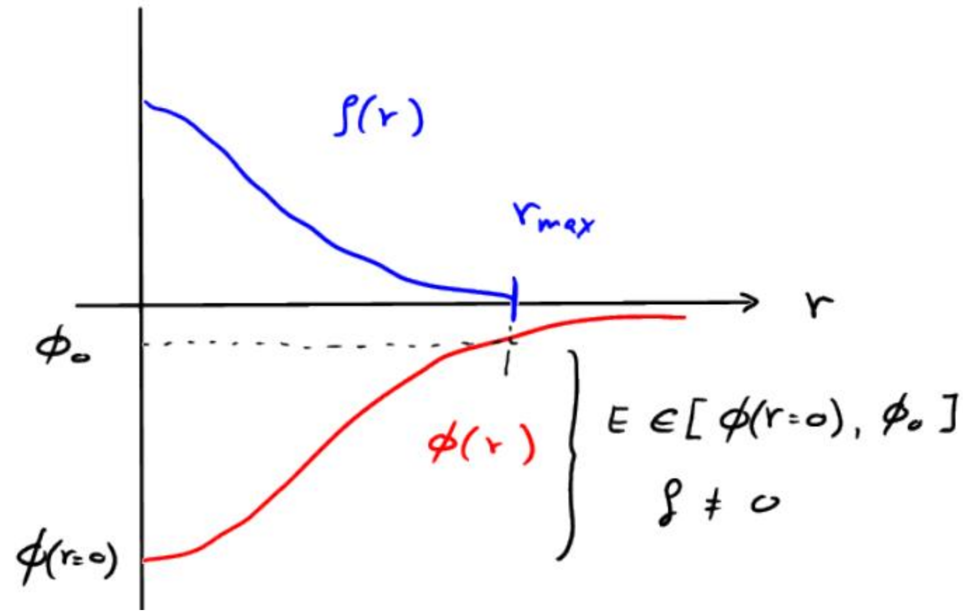
$$\boxed{\nu(r) = 4\pi \int_0^\infty v^2 f\left(\tfrac{1}{2}v^2 + \phi(r)\right) dv}$$

Density and potential

- $\rho(r)$ $\qquad$ $\rho(r > r_{max}) = 0$
- $\phi(r)$ $\qquad$ no limit

Goal: find $\rho = \rho(E)$ with

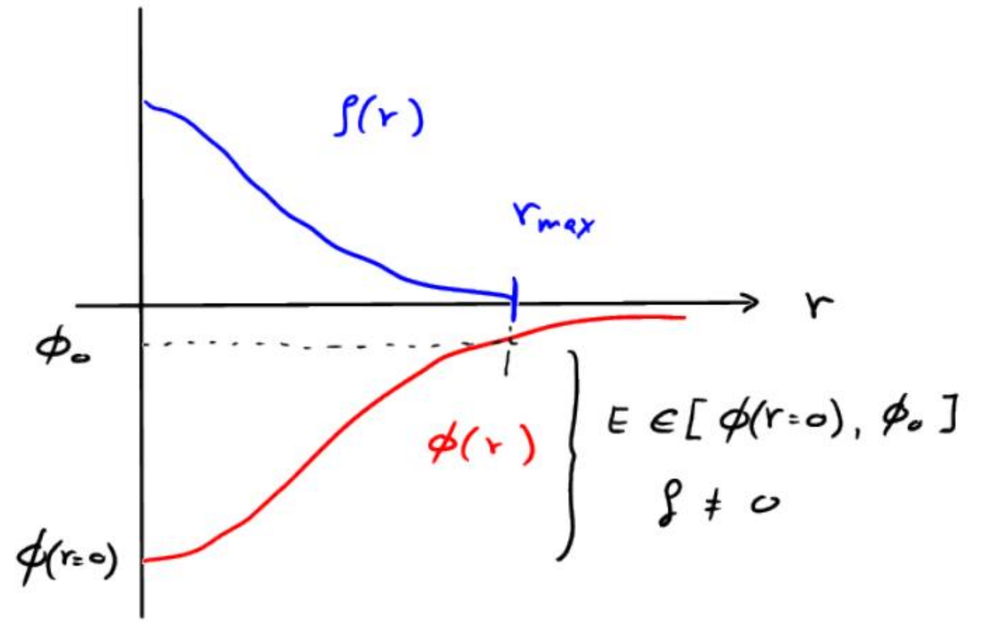
$\rho = 0$ if $r > r_{max}$



Density and potential

- $\rho(r)$ $\qquad \rho(r > r_{max}) = 0$
- $\phi(r)$ $\qquad$ no limit

$$\boxed{\begin{array}{l}\text{Goal: find } \rho = \rho(E) \text{ with} \\ \qquad \rho = 0 \quad \text{if} \quad r > r_{max}\end{array}}$$



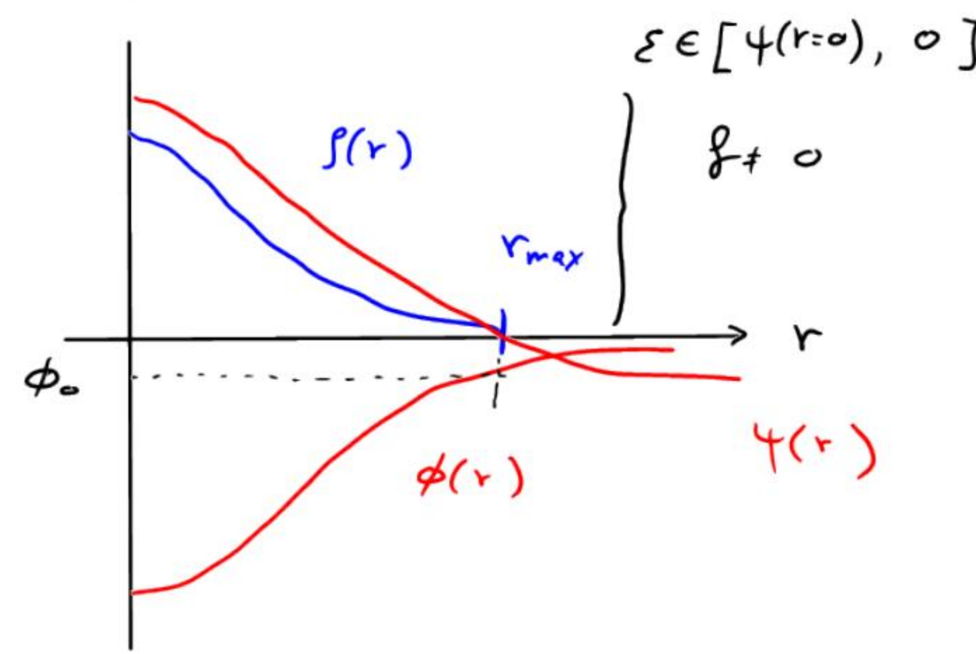
<u>Idea</u> new variables

relative potential

$$\begin{cases} \Psi = -(\phi - \phi_0) = -\phi + \phi_0 \\ \mathcal{E} = -(H - \phi_0) = -H + \phi_0 \end{cases}$$

relative energy $\qquad = \Psi - \frac{1}{2} v^2$

$$\boxed{\rho \rightarrow \rho(\mathcal{E})} \qquad \begin{cases} \mathcal{E} > 0 & \rho > 0 \\ \mathcal{E} \leq 0 & \rho = 0 \end{cases}$$



Idea : Use $\mathcal{E}$ as the main variable and integrate over it.

$\nu(r)$ becomes

$$\nu(r) = 4\pi \int_0^\infty v^2 f(H)\, dv \overset{\mathcal{E} = -H + \phi_0}{=} 4\pi \int_0^\infty v^2 f(\phi_0 - \mathcal{E}(r,v))$$

With

$$\mathcal{E} = -H + \phi_0 = \Psi - \tfrac{1}{2} v^2$$

$$v = \sqrt{2(\Psi - \mathcal{E})} \qquad dv = \frac{-1}{\sqrt{2(\Psi - \mathcal{E})}}\, d\mathcal{E}$$

The integral becomes

$$\nu(r) = 4\pi \int_{\Psi \;\; v=0 \,\to\, \mathcal{E}=\Psi}^{-\infty \;\; v\to\infty \;\; \mathcal{E}\to-\infty} 2(\Psi - \mathcal{E})\, \underbrace{f(\phi_0 - \mathcal{E})}_{= f(\mathcal{E}) \text{ we write } f \text{ as a fct of } \mathcal{E}} \frac{-1}{\sqrt{2(\Psi - \mathcal{E})}}\, d\mathcal{E}$$

$$= 4\pi \int_{-\infty}^{\Psi} \sqrt{2(\Psi - \mathcal{E})}\, f(\mathcal{E})\, d\mathcal{E}$$

$$\nu(r) = 4\pi \int_{-\infty}^{\psi} \sqrt{2(\psi-\mathcal{E})}\, f(\mathcal{E})\, d\mathcal{E}$$

$$= \underbrace{4\pi \int_{-\infty}^{0} \sqrt{2(\psi-\mathcal{E})}\, f(\mathcal{E})\, d\mathcal{E}}_{\substack{=0 \text{ as} \\ f(\mathcal{E}) = 0 \text{ for } \mathcal{E} < 0}} + 4\pi \int_{0}^{\psi} \sqrt{2(\psi-\mathcal{E})}\, f(\mathcal{E})\, d\mathcal{E}$$

$$\boxed{\nu(r) = 4\pi \int_{0}^{\psi} \sqrt{2(\psi-\mathcal{E})}\, f(\mathcal{E})\, d\mathcal{E}}$$

- if ψ is a monotonic function of r (typical potential)

$\psi(r) \rightarrow r(\psi) \Rightarrow \nu(r) = \nu(r(\psi)) = \nu(\psi)$

$$\boxed{\frac{1}{\sqrt{8}\,\pi}\, \nu(\psi) = \int_{0}^{\psi} \sqrt{\psi-\mathcal{E}}\, f(\mathcal{E})\, d\mathcal{E}}$$

Derivating with respect to ψ

$$\frac{1}{\sqrt{8}\,\pi}\,\nu(\psi) = 2\int_0^{\psi}\sqrt{\psi-\varepsilon}\;f(\varepsilon)\,d\varepsilon$$

(!) not obvious

$$\frac{1}{\sqrt{8}\,\pi}\,\frac{d\nu}{d\psi} = \int_0^{\psi} d\varepsilon\,\frac{f(\varepsilon)}{\sqrt{\psi-\varepsilon}}$$

(Abel integral)

Solution : Eddington's formula

$$\boxed{f(\varepsilon) = \frac{1}{\sqrt{8}\,\pi^2}\,\frac{d}{d\varepsilon}\left[\int_0^{\varepsilon}\frac{d\psi}{\sqrt{\varepsilon-\psi}}\,\frac{d\nu}{d\psi}\right]}$$

or

$$\boxed{f(\varepsilon) = \frac{1}{\sqrt{8}\,\pi^2}\left[\int_0^{\varepsilon}\frac{d\psi}{\sqrt{\varepsilon-\psi}}\,\frac{d^2\nu}{d\psi^2} + \frac{1}{\sqrt{\varepsilon}}\left(\frac{d\nu}{d\psi}\right)_{\psi=0}\right]}$$

<u>Note</u> : $f(\varepsilon) > 0$ only if $\int_0^{\varepsilon}\frac{d\psi}{\sqrt{\varepsilon-\psi}}\,\frac{d\nu}{d\psi}$ is an increasing function of ε

How using this formula ?

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\,\pi^2} \frac{d}{d\mathcal{E}} \left[\int_0^{\mathcal{E}} \frac{d\psi}{\sqrt{\mathcal{E}-\psi}} \frac{d\nu}{d\psi} \right]$$

- We start from a given $\rho(r)$, $\phi(r)$

① get r_{max} and compute $\phi_0 = \phi(r_{max})$

② a) get $\nu(r) = \rho(r)/M$

$\psi(r) = -\phi(r) + \phi_0$

b) and $\nu = \nu(\psi)$ if $\psi(r)$ may be inverted

③ if $\frac{\partial \nu}{\partial \psi}$ is analytical, compute $f(\mathcal{E})$ (Eddington's formula)

④ $f(x, v) = f(\mathcal{E}) = f(\phi_0 - \mathcal{E}) = f(\frac{1}{2} v^2 + \phi)$

Note ②a and ③ may be performed numerically

Example: Hernquist model

$$\rho(r) = \frac{\rho_0}{(r/a)(1+r/a)^3}$$

$$M(r) = 2\pi\rho_0 a^3 \frac{(r/a)^2}{(1+r/a)^2}$$

$$M = 2\pi\rho_0 a^3 \quad (r=\infty)$$

$$\phi(r) = -2\pi G\rho_0 \frac{a^2}{(1+r/a)}$$

The density is non zero at $r=\infty$

$\Rightarrow \phi_0 = 0$

$$\psi(r) = -\phi(r)$$

$$\Rightarrow \quad \frac{r}{a} = \frac{2\pi G\rho_0 a^2}{\psi(r)} - 1$$

using $M = 2\pi\rho_0 a^3$

$$= \frac{GM}{\psi(r)\,a} - 1$$

$$\frac{r}{a} = \frac{1}{\tilde{\psi}(r)} - 1 \qquad \text{where } \tilde{\psi}(r) = \frac{\psi(r)}{GM}\,a$$

we can now express ν as $\nu(\Psi)$ eliminating r/a

$$\nu(\Psi) = \frac{\rho}{M} = \frac{1}{2\pi a^3} \frac{\tilde{\Psi}^4}{1-\tilde{\Psi}}$$

then

$$\frac{\partial \nu}{\partial \Psi} = \frac{1}{2\pi a^2 GM} \frac{\tilde{\Psi}^3(4-3\tilde{\Psi})}{(1-\tilde{\Psi})^2}$$

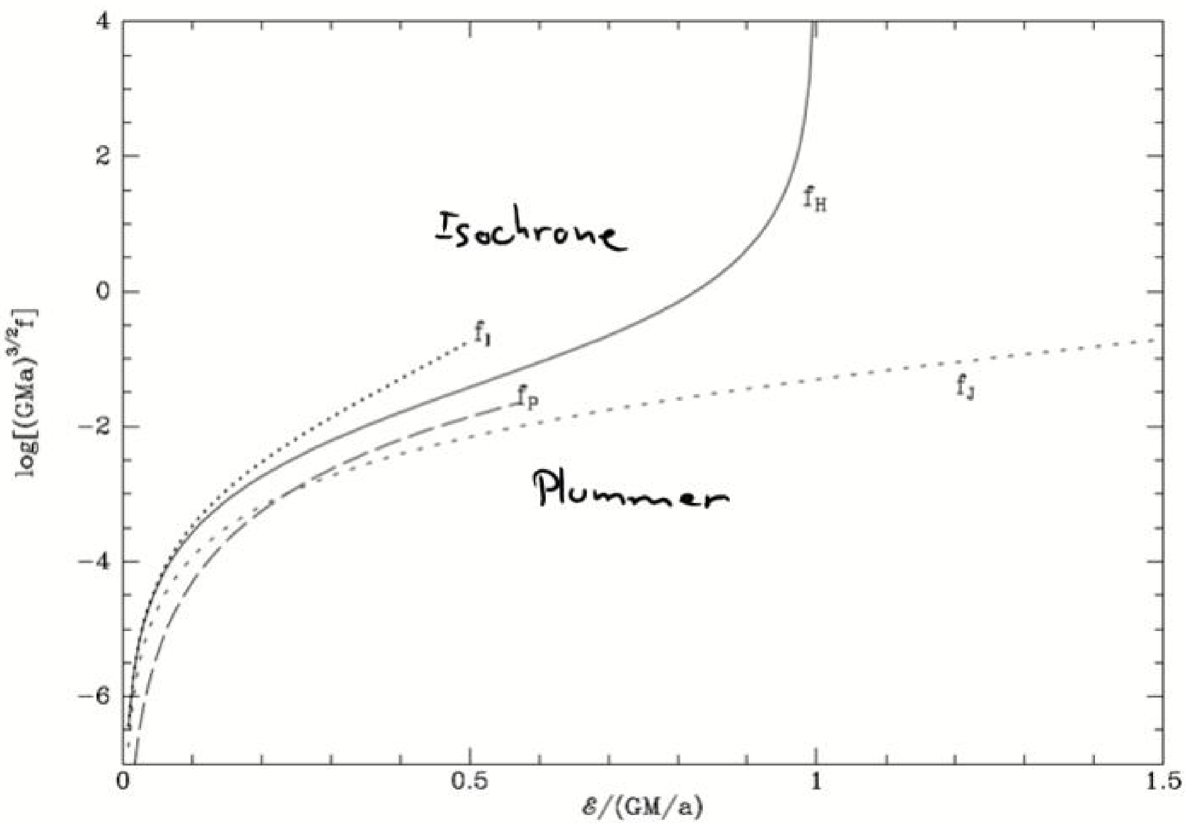
And the DF becomes using $\tilde{\mathcal{E}} = -\frac{\mathcal{E} a}{GM}$

$$f(\mathcal{E}) = \frac{\sqrt{2}}{(2\pi)^3 (GM)^2 a} \int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \frac{2\tilde{\Psi}^2(6-8\tilde{\Psi}+3\tilde{\Psi}^2)}{(1-\tilde{\Psi})^3}$$

$$= \frac{1}{\sqrt{2}(2\pi)^3 (GMa)^{3/2}} \frac{\sqrt{\tilde{\mathcal{E}}}}{(1-\tilde{\mathcal{E}})^2} \left[(1-2\tilde{\mathcal{E}})(8\tilde{\mathcal{E}}^2-8\tilde{\mathcal{E}}-3) + \frac{3\arcsin(\sqrt{\tilde{\mathcal{E}}})}{\sqrt{\tilde{\mathcal{E}}(1-\tilde{\mathcal{E}})}} \right]$$

Note: It is possible to do the same for the Plummer, Isochrone an Jaffe models

Hernquist • $\rho(0) \to \infty$



• no minimal energy

Jaffe $\mathcal{E} \to \infty$

• $\rho(0) \to \infty$

Plummer model

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + b^2}}$$

$$\rho(r) = \frac{3M}{4\pi b^3} \left(1 + \frac{r^2}{b^2}\right)^{-5/2}$$

Isochrone model

$$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}$$

$$\rho(r) = M \frac{3(b + \sqrt{b^2 + r^2})(b^2 + r^2) - r^2(b + 3\sqrt{b^2 + r^2})}{4\pi(b + \sqrt{b^2 + r^2})^3 (b^2 + r^2)^{3/2}}$$

Jaffe model

$$\Phi(r) = -4\pi G\rho_0 a^2 \ \ln(1 + a/r)$$

$$\rho(r) = \frac{\rho_0}{(r/a)^2(1 + r/a)^2}$$

Hernquist model

$$\Phi(r) = -4\pi G\rho_0 a^2 \ \frac{1}{2(1 + r/a)}$$

$$\rho(r) = \frac{\rho_0}{(r/a)(1 + r/a)^3}$$

Equilibria of collisionless systems

Models defined from DFs

Distribution function for spherical systems

- Method ①

  - from $\rho(r)$ $\phi(r)$ $\rightarrow$ get $f(\varepsilon) = f\left(\frac{1}{2}v^2 + \phi(r)\right)$

- Method ②

  - assume $f(\varepsilon)$ $\rightarrow$ get $\rho(r)$

Spherical systems defined by DFs

Density

should be $N \cdot m \cdot f$

spherical integration in velocity space

$$\rho(r) = \int d^3\vec{v}\; f\left(\psi(r) - \tfrac{1}{2}v^2\right) = 4\pi \int_0^\infty dv\; v^2 f\left(\psi(r) - \tfrac{1}{2}v^2\right)$$

Conditions for $f > 0$: $\mathcal{E} > 0$ as $\mathcal{E} = \psi - \frac{1}{2}v^2 > 0$

$$\boxed{\sqrt{2\psi} > v}$$

$$\rho(r) = 4\pi \int_0^\infty dv\; v^2 f\left(\psi(r) - \tfrac{1}{2}v^2\right)$$

note we don't integrate over the energy

$$= 4\pi \int_0^{\sqrt{2\psi}} dv\; v^2 f\left(\psi(r) - \tfrac{1}{2}v^2\right) + \underbrace{4\pi \int_{\sqrt{2\psi}}^\infty dv\; v^2 f\left(\psi(r) - \tfrac{1}{2}v^2\right)}_{}$$

$= 0$ as $v > \sqrt{2\psi}$

$\Rightarrow f = 0$

$$\boxed{\rho(r) = 4\pi \int_0^{\sqrt{2\psi}} dv\; v^2 f\left(\psi(r) - \tfrac{1}{2}v^2\right)}$$

Models defined from DFs: Polytropes

Polythropes and Plummer models

$$
\boxed{f(\mathcal{E}) = \begin{cases} F\,\mathcal{E}^{\,n-3/2} & (\mathcal{E} > 0) \\ 0 & (\mathcal{E} \leq 0) \end{cases}}
$$

F, a constant

$f = 0$ if $\mathcal{E} > 0$

$\rho = 0$

Corresponding density

$$
\rho(r) = 4\pi F \int_0^{\sqrt{2\psi}} dv\; v^2 \left(\psi(r) - \tfrac{1}{2} v^2\right)^{n-3/2}
$$

smart substitution : introduce the variable $\theta(v)$ such that

$$
v^2 = 2\psi \cos^2\theta \quad , \quad \theta = \arccos\left(\frac{v}{\sqrt{2\psi}}\right)
$$

$$
2v\,dv = -4\psi \cos\theta \sin\theta\, d\theta
$$

$$
\Rightarrow \quad dv = -\frac{2\psi \cos\theta \sin\theta\, d\theta}{\sqrt{2\psi}\,\cos\theta} = -\sqrt{2\psi}\,\sin\theta\, d\theta
$$

$$\begin{cases} v = 0 \to \theta = \frac{\pi}{2} \\ v = \sqrt{2\psi} \to \theta = 0 \end{cases} \quad \to \qquad \psi - \frac{1}{2}v^2 = \psi - \psi\cos^2\theta = \psi\sin^2\theta$$

$$\rho(r) = 4\pi F \int_0^{\pi/2} \left(\sqrt{2\psi}\,\sin\theta\, d\theta\right)\cdot\left(2\psi\cos^2\theta\right)\cdot\left(\psi\sin^2\theta\right)^{n-\frac{3}{2}}$$

$$= 4\pi F \int_0^{\pi/2} 2\cdot 2^{\frac{1}{2}}\,\psi^{\frac{1}{2}}\,\psi\,\psi^{n-\frac{3}{2}} \cdot \cos^2\theta\,\sin\theta^{2n-2}\,d\theta$$

$$= 8\pi F\sqrt{2}\,\psi^n \int_0^{\frac{\pi}{2}} \underbrace{\cos^2\theta}_{1-\sin^2} \sin\theta^{2n-2}\,d\theta$$

So, we get

$$\boxed{\rho(r) = C_n\,\psi(r)^n} \qquad (\text{for } \psi > 0)$$

relation between ρ and ϕ

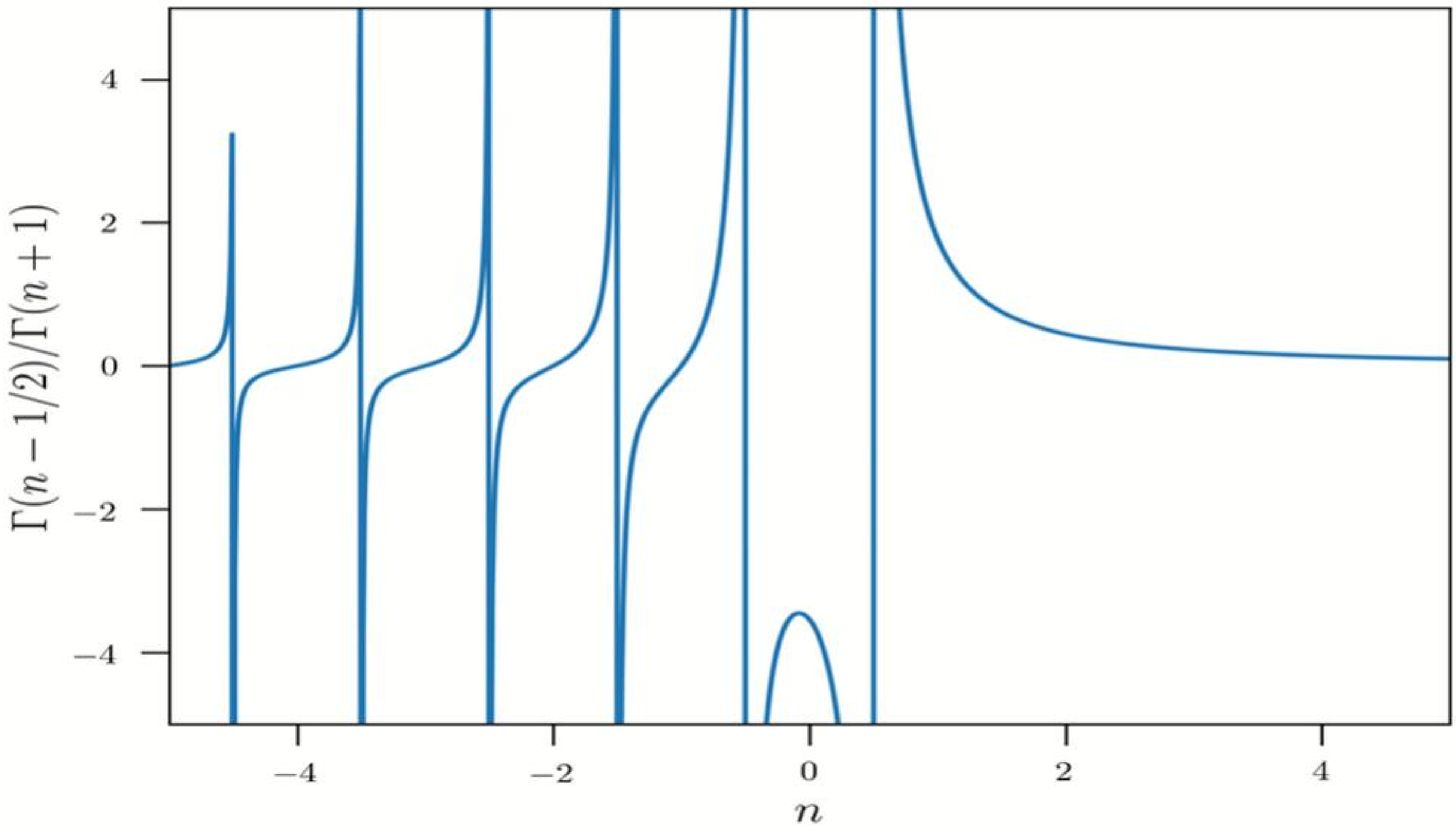
$$C_n = \frac{(2\pi)^{3/2}\left(n-\frac{3}{2}\right)!\,F}{n!} = \frac{(2\pi)^{3/2}\,\Gamma\left(n-\frac{1}{2}\right)F}{\Gamma(n+1)}$$

$$n! = \Gamma(n+1) = \int_0^\infty dt\, t^n e^{-t}$$

$$c_n \sim \frac{(n-\frac{3}{2})!}{n!} = \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n+1)}$$

$n = \frac{1}{2}$

$n > \frac{1}{2}$, $c_n > 0$, $f > 0$



"Pressure"

$$P(\rho) = - \int_0^\rho d\rho' \, \rho' \, \frac{\partial \phi}{\partial \rho}(\rho')$$

$$\rho = C_n \psi^n$$

$$\psi = \frac{1}{C_n^{1/n}} \rho^{\frac{1}{n}} \qquad \frac{\partial \psi}{\partial \rho} = \frac{1}{C_n^{1/n}} \frac{1}{n} \rho^{\frac{1}{n}-1}$$

$$\frac{\partial \phi}{\partial \rho} = - \frac{1}{C_n^{1/n}} \frac{1}{n} \rho^{\frac{1}{n}-1}$$

$$P(\rho) = \frac{1}{C_n^{1/n}} \frac{1}{n} \int_0^\rho d\rho' \, \rho^{\frac{1}{n}} = \frac{1}{C_n^{1/n}} \frac{1}{n+1} \rho^{\frac{1}{n}+1}$$

$$\boxed{P(\rho) = K \rho^\gamma}$$

$$\begin{cases} \gamma = \frac{1}{n} + 1 \\ K = \frac{1}{C_n^{1/n}} \frac{1}{n+1} \end{cases} \qquad n = \frac{1}{\gamma - 1} \qquad C_n = \left(\frac{\gamma - 1}{K \gamma} \right)^{\frac{1}{\gamma - 1}}$$

= Polytropic EoS

Conclusion

The density of a stellar system described by and ergodic DF

$$f(\epsilon) \sim \epsilon^{n-3/2}$$

Is the same as a polytropic gas sphere in hydrostatic equilibrium, with:

$$P(\rho) \sim \rho^{\gamma}$$

This is why these DFs are called <u>polytropes</u>.

Note: from $\rho(r) = C_n \, \psi(r)^n$

if $\rho = \text{cte} \implies n = 0$

But from $C_n = \dfrac{(2\pi)^{3/2}\, \Gamma(n - \frac{1}{2})\, F}{\Gamma(n+1)} \implies C_n < 0 \quad \rho < 0$

① No finite ergodic stellar system is homogeneous.

② No self-gravitating homogeneous system equivalent to a self-gravitating sphere of incompressible fluid exists.

Indeed: the hydrostatic solution of an incompressible fluid of constant density requires

$$\frac{dP}{dr} = -\rho_0 \frac{d\phi}{dr} = -\frac{4}{3}\pi G \rho_0^2 r$$

not a polytropic EOS $\longleftarrow$ $\quad P = P_0 - \frac{2}{3}\pi G \rho_0^2 r^2$

Self-gravity !

$$\vec{\nabla}^2(\Phi) = 4\pi G\rho$$

The Poisson equation for spherical systems (with Ψ)

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = -4\pi G\rho(r)}$$

$$\rho = C_n \Psi^n$$

$$\rho^{\frac{n-1}{n}} = C_n^{\frac{n-1}{n}} \Psi^{n-1}$$

With $\rho = C_n \Psi^n$

$$\frac{d\rho}{dr} = C_n\, n\, \Psi^{n-1}\frac{d\Psi}{dr} = C_n\, n\left(\frac{1}{C_n}\rho\right)^{\frac{n-1}{n}}\frac{d\Psi}{dr}$$

thus

$$\frac{d\Psi}{dr} = \frac{1}{C_n^{1/n}\, n}\,\rho^{\frac{1-n}{n}}\frac{d\rho}{dr}$$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{n\, C_n^{1/n}}\rho^{\frac{1-n}{n}}\right) + 4\pi G\rho = 0}$$

or eliminating ρ, using $\rho(r) = C_n \Psi(r)^n$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + 4\pi G C_n \Psi^n = 0}$$

Solutions

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + 4\pi G c_n \Psi^n = 0}$$

A. Power laws

$$\left\{\begin{array}{l} \rho(r) \sim r^{-\alpha} \\ \Psi(r) \sim r^{-\alpha/n} \end{array}\right. \qquad \rho \sim \Psi^n$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) \sim r^{-\frac{\alpha}{n}-2}$$

Poisson

$$\underbrace{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right)}_{r^{-\frac{\alpha}{n}-2}} + \underbrace{4\pi G \rho(r)}_{r^{-\alpha}} = 0$$

$$-\frac{\alpha}{n}-2 \sim -\alpha$$

$$\Longrightarrow \boxed{\alpha = \frac{2n}{n-1}}$$

As the potential may not decrease faster than the Kepler potential $\frac{1}{r}$

$$\left(\Psi \sim r^{-\frac{\alpha}{n}}\right) \qquad \underline{\frac{\alpha}{n} \leq 1} \implies \boxed{n \geq 3}$$

B Models with finite potential and density

Define new variables $s = \frac{r}{b}$ $\qquad \psi' = \frac{\psi}{\psi_0}$

where $\begin{cases} b = \left(\frac{4}{3}\pi G \psi_0^{n-1} c_n\right)^{1/2} \\ \psi_0 = \psi(0) \end{cases}$

The Poisson equation becomes $\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) + 4\pi G c_n \psi^n = 0}$

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -3\psi'^n}$$

+ boundary conditions

$\begin{cases} \cdot\ \psi'(0) = 1 & \text{normalisation} \\ \cdot\ \left.\frac{d\psi'}{dr}\right)_0 = 0 & \text{no force at the center (smooth)} \end{cases}$

Lane - Emden Equation

(In general, non trivial solutions)

Two analytical solutions $\quad n = 1, \quad n = 5$

$$\boxed{n = 1} \qquad \boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -3\psi'}$$

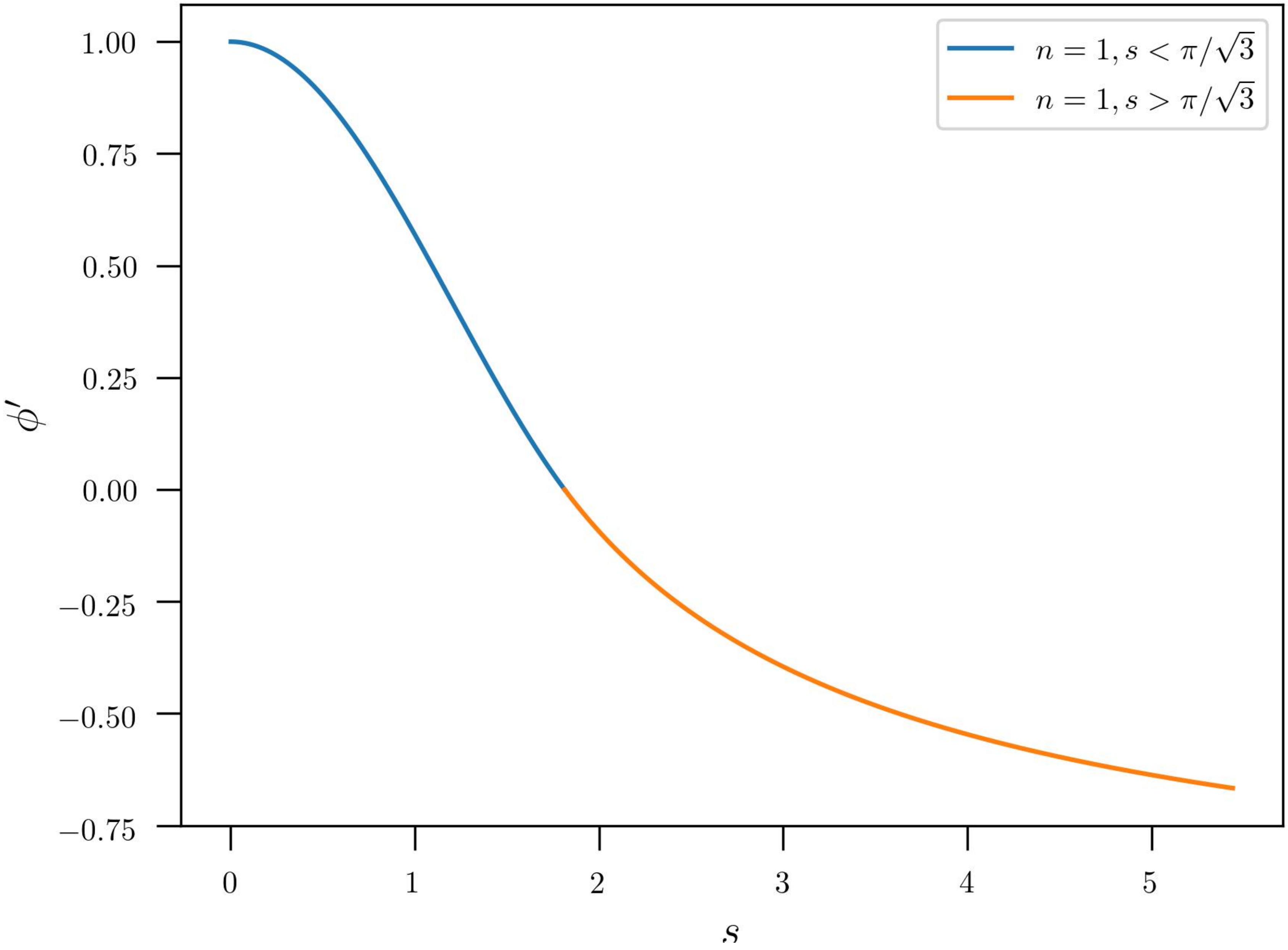
linear Helmholtz Equation

$$\psi'(s) = \begin{cases} \dfrac{\sin(\sqrt{3}\, s)}{\sqrt{3}\, s} & s < \dfrac{\pi}{\sqrt{3}} \\[2ex] \dfrac{\pi}{\sqrt{3}\, s} - 1 & s \geq \dfrac{\pi}{\sqrt{3}} \end{cases}$$

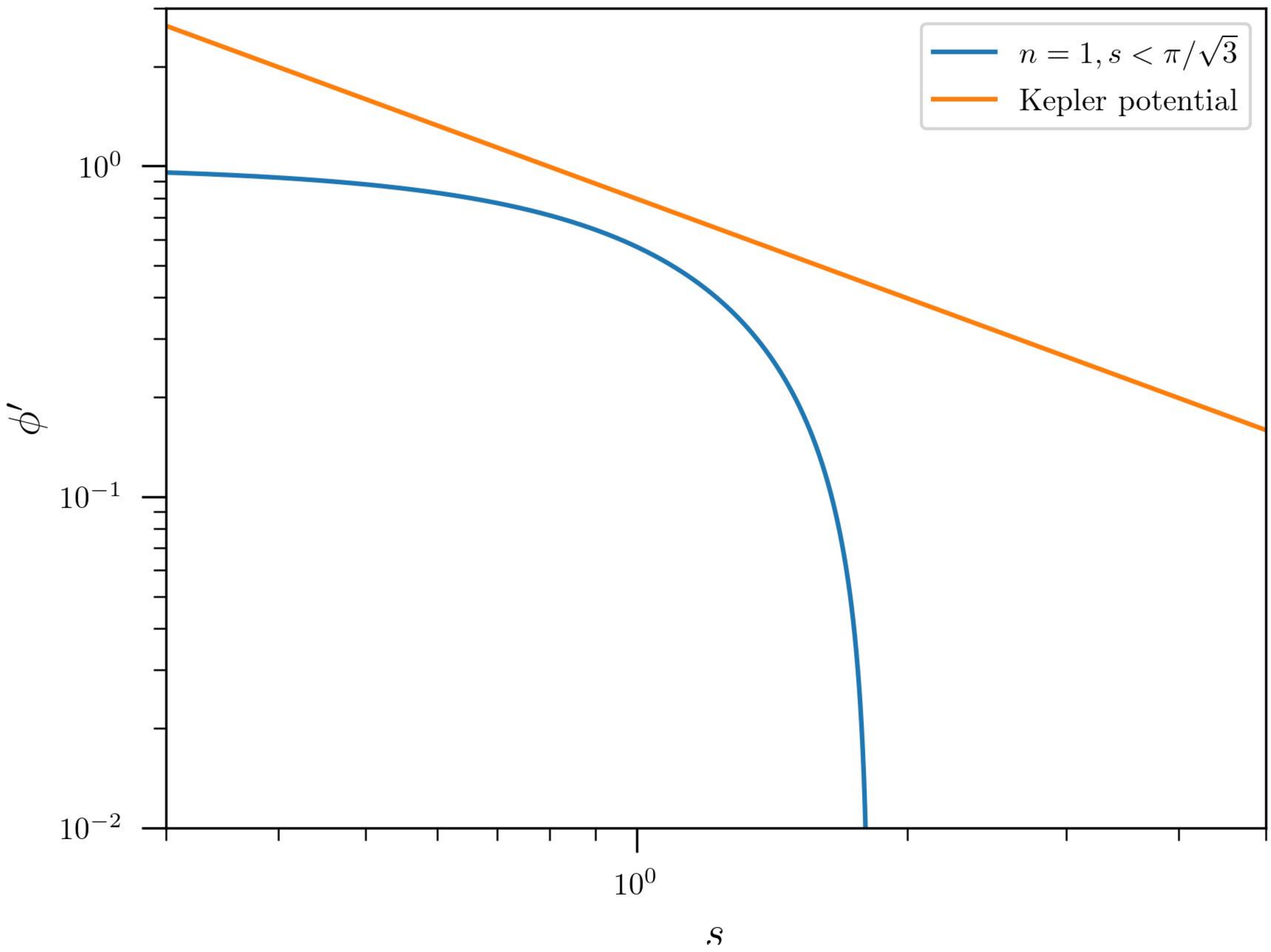
⚠ $n = 1 < 3$

non physical solution

Solution of the Lane-Emden Equation for n=1



Solution of the Lane-Emden Equation for n=1



$$\boxed{n = 5}$$

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -3\psi'^5}$$

consider $\psi'(s) = \frac{1}{\sqrt{1+s^2}}$

The Poisson Equation becomes

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi'}{ds}\right) = -\frac{1}{s^2}\frac{d}{ds}\left(\frac{s^3}{(1+s^2)^{3/2}}\right) = -\frac{3}{(1+s^2)^{5/2}} = -3\psi'^5$$

$\rightarrow$ $\psi'(s)$ is a solution !

$\boxed{n = 5}$ $\boxed{\frac{1}{s^2} \frac{d}{ds}\left(s^2 \frac{d\psi'}{ds} \right) = -3\psi'^5}$

consider $\psi'(s) = \frac{1}{\sqrt{1+s^2}}$

The Poisson Equation becomes

$$\frac{1}{s^2} \frac{d}{ds}\left(s^2 \frac{d\psi'}{ds} \right) = -\frac{1}{s^2} \frac{d}{ds}\left(\frac{s^3}{(1+s^2)^{3/2}} \right) = -\frac{3}{(1+s^2)^{5/2}} = -3\psi'^5$$

$\rightarrow$ $\psi'(s)$ is a solution !

and corresponds to the <u>Plummer model</u>

$\boxed{\phi(r) = -\frac{GM}{\sqrt{r^2+a^2}}}$ $\boxed{\rho(r) = \frac{3M}{4\pi a^3}\left(1 + \frac{r^2}{a^2} \right)^{-5/2}}$

Then : what do we learn concerning the Plummer model?

We have access to its DF:

$$f(\mathcal{E}) \sim \mathcal{E}^{n-3/2} \sim \left(\frac{GM}{\sqrt{r^2+a^2}} + \frac{1}{2} v^2 \right)^{7/2}$$

We have access to the kinematics structure :

① Velocity distribution function

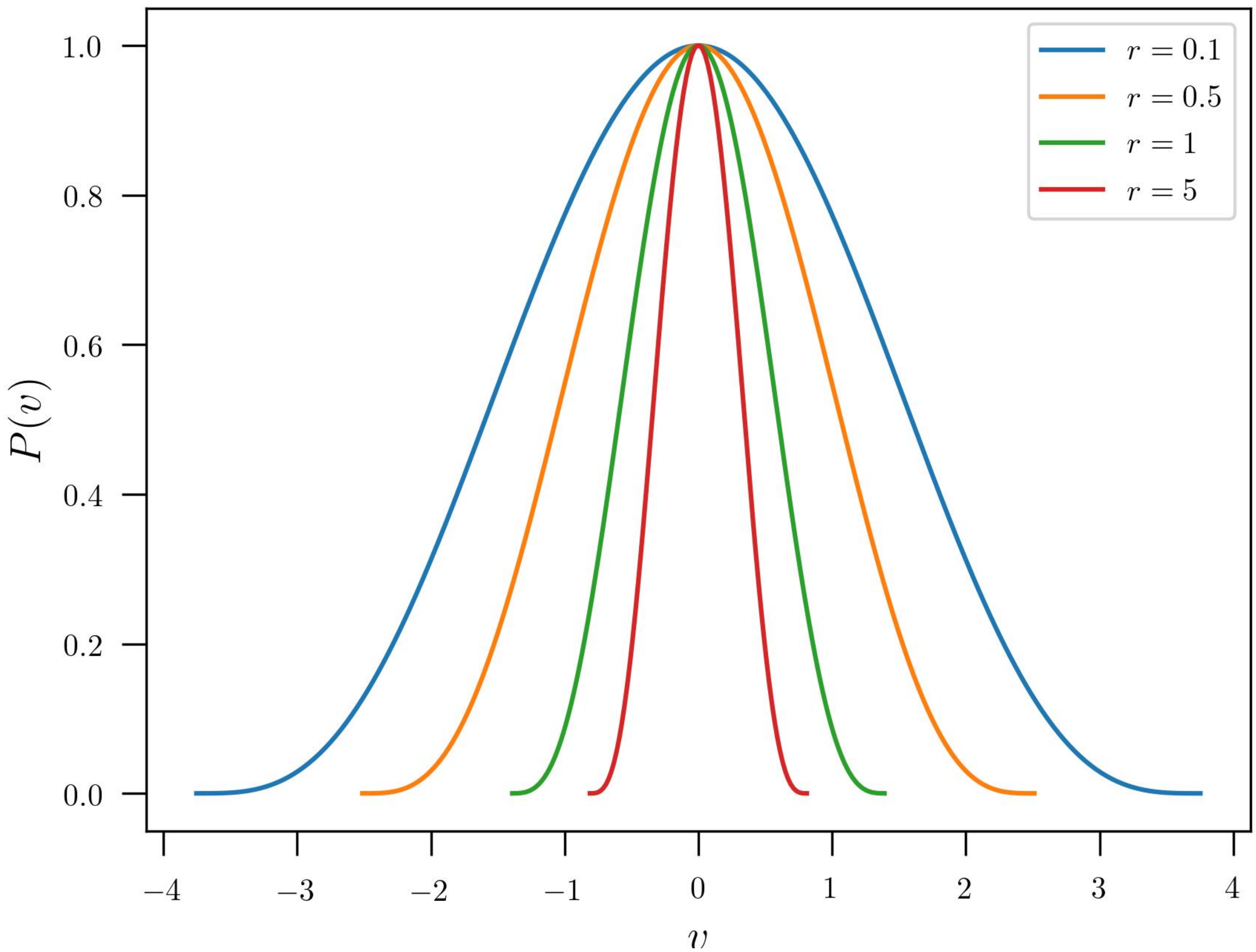
$$P_r(v) = \frac{f(\frac{1}{2}v^2 + \phi(r))}{\nu(r)} \sim \underbrace{\left(1+\frac{r^2}{a^2}\right)^{5/2}}_{\frac{1}{\rho}} \underbrace{\left(\frac{GM}{\sqrt{r^2+a^2}} + \frac{1}{2} v^2 \right)^{7/2}}_{\mathcal{E}^{7/2}}$$

② Velocity dispersion

$$\sigma^2 = \frac{4}{3}\pi \frac{1}{\nu(x)} \int_0^{v_{max} = \sqrt{2\psi}} v^4 f\left(\frac{1}{2}v^2 + \phi(\vec{x})\right) dv$$

$$= \frac{4}{3}\pi \frac{1}{\nu(r)} \int_0^{v_{max}} v^4 \left(\frac{1}{2}v^2 - \frac{GM}{\sqrt{r^2+a^2}} \right)^{7/2} dv$$

The Plummer velocity distribution function



The End