Derivation of Euler Equations

1) Kinetic Theory

1.1) Distribution Functions

A distribution function $f(\underline{x}, \underline{u}, t)$ is used for the description at a microscopic level. It is a phase space distribution function: $(\underline{x}, \underline{u}) \in \mathbb{R}^6$, $\underline{x}$ and $\underline{u}$ are independent variables.

Examples:

Let $f \equiv$ mass density $\Rightarrow \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(\underline{x}, \underline{u}, t)\, d^3x\, d^3u = M$

Let $f \equiv$ probability density $\Rightarrow \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(\underline{x}, \underline{u}, t)\, d^3x\, d^3u = 1$

A particle-by-particle description of the system is not possible. Instead, we use a statistical description with distributions.

Let us define $f(\underline{x}, \underline{u}, t)$ as the average number of particles contained at time t in a volume element d^3x about $\underline{x}$ and a velocity-space element d^3u about $\underline{u}$.

Furthermore, we demand:

- $f \geq 0$ everywhere
- $u_i \to \infty$, $f \to 0$ sufficiently rapidly such that a finite amount of particles has a finite energy

Let us define the moment of a distribution.
Generally, a moment $Q(\underline{x}, t)$ is defined as:

$$Q(\underline{x}, t) = \int_{\mathbb{R}^3} f(\underline{x}, \underline{u}, t)\, q(\underline{u})\, d^3u$$

p-th moment: $Q_p(\underline{x}, t) = \int_{\mathbb{R}^3} f(\underline{x}, \underline{u}, t)\, u^p\, d^3u$

0-th moment: $Q_0(\underline{x}, t) = \int_{\mathbb{R}^3} f(\underline{x}, \underline{u}, t)\, d^3u = n(\underline{x}, t)$ number density

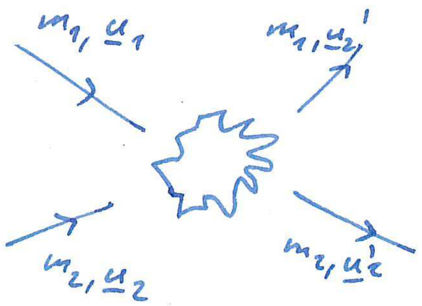
1-st moment: $Q_1(\underline{x}, t) = \int_{\mathbb{R}^3} f(\underline{x}, \underline{u}, t)\, u\, d^3u = n(\underline{x}, t) \cdot v(\underline{x}, t)$ average velocity

2-nd moment: $Q_2 = \frac{1}{m} E(\underline{x}, t) = n(\underline{x}, t) \cdot \underline{v}^2(\underline{x}, t)$

1.2) Binary Collisions

The wave packets of particles are highly localized. To a very high degree of approximation we can consider the gas to be a collection of classical point particles. We can describe the motion of an (electrically not charged) particle as a sequence of straight lines, each interrupted by a brief collision with another particle. Because the probability of collision is small, we neglect the possibility of a collision between three or more particles and consider only binary collisions.

Binary collisions are collisions between two particles. They conserve energy and are characterized by a collision cross section.



Conserved quantities:

Mass: $M = m_1 + m_2 = m_1' + m_2'$

Momentum: $m_1 \underline{u}_1 + m_2 \underline{u}_2 = m_1 u_1' + m_2 u_2'$

Energy: $\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2$

Furthermore, the relative velocity of the two particles only changes the direction after a collision:

Let $\underline{V} = \frac{1}{M}(m_1 \underline{u}_1 + m_2 \underline{u}_2)$ CoM - velocity

$\underline{v} = \underline{u}_1 - \underline{u}_2$, $\underline{v}' = \underline{u}_1' - \underline{u}_2'$ relative velocity

$\mu = \frac{m_1 m_2}{m_1 + m_2}$ reduced mass

Then:

$$E = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

Using that

$$V^2 = \frac{1}{M^2}(m_1^2 u_1^2 + m_2^2 u_2^2 + 2 m_1 u_1 m_2 u_2)$$

and

$$v^2 = u_1^2 + u_2^2 - 2 u_1 u_2$$

we can write

$$E = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2M}(M m_1 u_1^2 + M m_2 u_2^2) =$$

$$= \frac{1}{2M}((m_1 + m_2) m_1 u_1^2 + (m_1 + m_2) m_2 u_2^2)$$

$$= \frac{1}{2M}(m_1^2 u_1^2 + m_1 m_2 u_1^2 + m_1 m_2 u_2^2 + m_2^2 u_2^2)$$

$$= \frac{1}{2M}(m_1^2 u_1^2 + m_1 m_2 u_1^2 + m_1 m_2 u_2^2 + m_2^2 u_2^2 + 2 m_1 m_2 u_1 u_2 - 2 m_1 m_2 u_1 u_2)$$

$$= \frac{1}{2M}(m_1^2 u_1^2 + 2 m_1 m_2 u_1 u_2 + m_2^2 u_2^2) + \frac{1}{2M}(m_1 m_2 u_1^2 + m_1 m_2 u_2^2 - 2 m_1 m_2 u_1 u_2$$

$$= \frac{1}{2} M V^2 + \frac{1}{2} \mu v^2$$

Since we conserve energy and momentum we know:

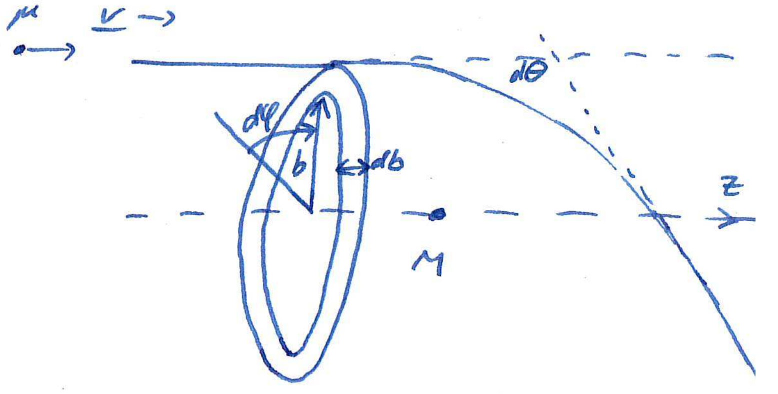
$$E = E' \, , \quad \|V\| = \|V'\|$$

$$\Rightarrow \|v\| = \|v'\|$$

$\Rightarrow$ only the direction of the relative velocity of the particles changes after the collision.

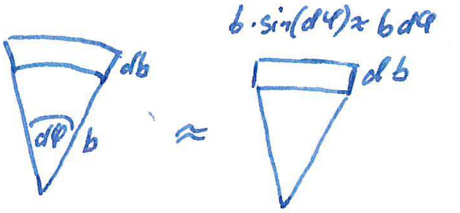
1.3) Differential Cross Sections

Choose a particle to act as a collision center and bombard it with a flux of particles.



The rate of collisions R_1 that have the impact parameter between $(b, b+db)$ within an increment of $d\varphi$ is

$$R_1 = j \cdot b \cdot db \, d\varphi$$



where j is the incident flux.

We can also assign the process a differential cross section σ defined as the rate at which particles are scattered out of the incident beam into an increment of solid angle $d\Omega$ around some direction $\vec{n}$, specified by the angles (θ, φ). The rate will be

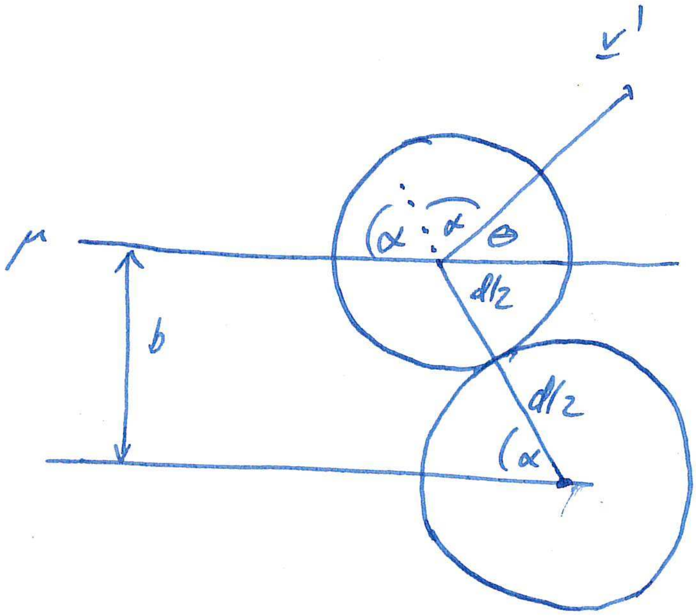
$$R_2 = j \sigma \, d\Omega$$

Because such a collision must have an unique solution, we can relate those two rates:

$$R_1 = R_2 \iff j b \, d\varphi \, db = j \sigma \, d\Omega$$

With $d\Omega = \sin\theta \, d\varphi \, d\theta$:

$$\Rightarrow \sigma = \frac{b}{\sin\theta} \frac{db}{d\theta}$$



For two rigid spheres with diameter d:

$$b = 2 \cdot \frac{d}{2} \cdot \sin\alpha = d \sin\alpha$$

$$\theta = \pi - 2\alpha$$

$$\Rightarrow \sigma = \frac{b}{\sin\theta} \frac{db}{d\theta} = \frac{d \sin\alpha}{\sin(\pi - 2\alpha)} \frac{db}{d\alpha} \frac{d\alpha}{d\theta}$$

$$= \frac{d \sin\alpha}{2 \sin\alpha \cos\alpha} \, d \cos\alpha \cdot \frac{1}{2} = \underline{\underline{\frac{1}{4} d^2}}$$

Then the total cross section is

$$\sigma_{tot} = \int \sigma \, d\Omega = 4\pi\sigma = \underline{\pi d^2}$$

Differential cross sections are:

- time reversal invariant:

$$\sigma(u_1, u_2; u_1', u_2') = \sigma(-u_1', -u_2'; -u_1, -u_2)$$

each particle must retrace its original trajectory

- rotation/reflection invariant:

The collision only depends on the magnitude and relative velocities

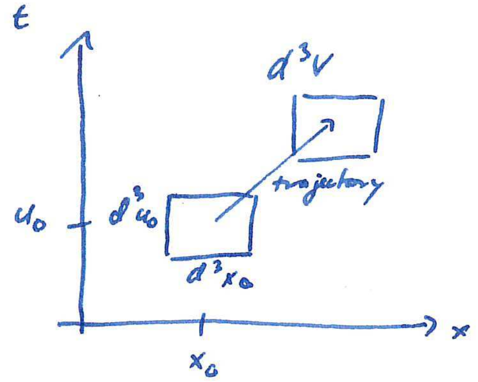
- reverse collision invariant

exchanging $u_1, u_2 \leftrightarrow u_1', u_2'$

(essentially time reversal + 180° rotation)

2) Boltzmann Equation

2.1) Vlassov Equation



Interpret the timely evolution of a phase-space element as a coordinate transformation:

Neglecting second order terms, we have:

$$\begin{cases} \underline{x} \simeq \underline{x}_0 + \underline{u}_0 \, dt \\ \underline{u} \simeq \underline{u}_0 + \underline{a} \, dt \end{cases}$$

assuming $\frac{\partial a}{\partial u} = 0$. The phase space element $d^3x_0 \, d^3u_0$ "evolves" to $d^3x \, d^3u$.

The Jacobian of the transformation gives:

$$J = \begin{vmatrix} \partial x / \partial x_0 & \partial x / \partial u_0 \\ \partial u / \partial x_0 & \partial u / \partial u_0 \end{vmatrix} = \begin{vmatrix} 1 & dt \\ \frac{\partial a}{\partial x} dt & 1 \end{vmatrix} \simeq 1 - \frac{\partial a}{\partial x} dt^2 \simeq 1$$

$\Rightarrow$ The volume element is conserved (to first order)

Let δN_0 be the number of particles in dV_0.
Assuming we have no collisions that might remove or add particles, then

$$\delta N_0 \stackrel{!}{=} \delta N$$

$$\delta N_0 = f(\underline{x}_0, \underline{u}_0, t)\, d^3x_0\, d^3u_0 \stackrel{!}{=} f(\underline{x}_0 + \underline{u}_0 dt, \underline{u}_0 + \underline{a}\, dt, t_0 + dt)\, d^3x\, d^3u$$

As shown, $d^3x_0\, d^3u_0 = d^3x\, d^3u$ since we're interpreting it as a coordinate transformation and the Jacobian $J = 1$

$$\Rightarrow f(\underline{x}_0, \underline{u}_0, t_0) = f(\underline{x}_0 + \underline{u}_0 dt, \underline{u}_0 + \underline{a}\, dt, t_0 + dt)$$

$\Rightarrow$ The distribution function is the same for all particles everywhere (provided we have no collisions)

By expanding the l.h.s to first order in the manner

$$f(x + dx) = f(x) + \frac{\partial f}{\partial x} dx + \mathcal{O}(dx^2) \approx f(x) + \frac{\partial f}{\partial x} dx$$

we get:

$$f(\underline{x}_0 + \underline{u}_0 dt, \underline{u}_0 + \underline{a}\, dt, t_0 + dt) - f(\underline{x}_0, \underline{u}_0, t_0) = 0$$

$$\boxed{\frac{\partial f}{\partial t} + \underline{u}_0 \cdot \frac{\partial f}{\partial \underline{x}} + \underline{a} \cdot \frac{\partial f}{\partial \underline{u}} = 0}$$

where $\frac{\partial f}{\partial \underline{x}} = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)^T$

This equation is known as the collisionless Boltzmann equation, or as the Vlasov equation

2.2) The Collision Integral

Now let's consider the case where binary collisions change the number of particles of a phase-space element.

Notation: Index 1: Collider particles
Index 2: Target particles
Prime : State after collision

The change in particle numbers in the volume element dV_2 can be described as follows:

$$\delta N_2 = \#\text{ particles inside} \cdot [\#\text{ particles incoming} \cdot \text{probability to collide}]$$

$$= f_2 \, d^3v \cdot [f_1 \cdot \underbrace{(\sigma d\Omega) v \cdot dt}]$$

Volume element containing the probability of collisions through cross sections

$$= f_1 f_2 \, \sigma v \, d^3v \, d\Omega$$

where v is the relative velocity.

$$\Rightarrow \boxed{\frac{\delta N_2}{dt} = f_1 f_2 \, \sigma v \, d^3v \, d\Omega} \quad \text{for outgoing collisions}$$

By demanding the process to be reversible, and using the same arguments we can write for incoming collisions:

$$\boxed{\frac{\delta N_1}{dt} = f_1' f_2' \sigma v \, d\Omega \, d^3v}$$ (also using $v' = v$)

Using these two expressions, we find the collision integral:

$$d\left(\frac{Df}{Dt}\right)_{coll} = \text{Sources} - \text{sinks} = \frac{\delta N_2}{dt} - \frac{\delta N_1}{dt} =$$

$$= [f_1' f_2' - f_1 f_2] \sigma v \, d\Omega \, d^3v$$

$$\Rightarrow \boxed{\left(\frac{Df}{Dt}\right)_{coll} = \iint [f_1' f_2' - f_1 f_2] \sigma v \, d\Omega \, d^3v}$$

2.3) Collision Invariants

Invariants of the collision integral are also invariants of the Boltzmann equation; Finding them gives us conservation laws.

A moment $Q(\underline{u}_1)$ is an invariant if:

$$I(\underline{x},t) \equiv \iiint Q(\underline{u}_1)\left[(f_1'f_2' - f_1f_2)\sigma v\, d\Omega\, d^3v_2\right] d^3v_1 = 0 \Leftrightarrow Q \text{ invariant}$$

The particles must be interchangeable:

$$\Rightarrow I(\underline{x},t) = \iiint Q(\underline{u}_2)\left[(f_1'f_2' - f_1f_2)\sigma v\, d\Omega\, d^3v_1\right] d^3v_2$$

$$\Rightarrow I = \frac{1}{2}(I+I) = \frac{1}{2}\iiint \left(Q(\underline{u}_1) + Q(\underline{u}_2)\right)\left[(f_1'f_2' - f_1f_2)\sigma v\, d\Omega\right] d^3v_1\, d^3v_2$$

Reverse collisions also have to hold:

$$\Rightarrow I = \frac{1}{2}\iiint \left(Q(\underline{u}_1') + Q(\underline{u}_2')\right)\left[(f_1f_2 - f_1'f_2')\sigma v\, d\Omega\right] d^3v_1\, d^3v_2$$

$$= \frac{1}{2}(I+I)$$

$$= \frac{1}{4}\iiint \left[Q(\underline{u}_1) + Q(\underline{u}_2) - Q(\underline{u}_1') - Q(\underline{u}_2')\right]\left[(f_1'f_2' - f_1f_2)\sigma v\, d\Omega\right] d^3v_1\, d^3v_2$$

By setting $I(\underline{x},t) = 0$, we see that a moment Q is invariant if

$$Q(\underline{u}_1) + Q(\underline{u}_2) = Q(\underline{u}_1') + Q(\underline{u}_2')$$

2.4) Equilibria

LTE: Local Thermodynamic Equilibrium. The internal state of a system in which no macroscopic flows of matter or energy are present over a timescale of interest. $\left(\frac{Df}{Dt}\right)_{coll} = 0$.

Detailed Balance: At equilibrium, each elementary process (collision) is balanced by its reverse process. $f_1' f_2' = f_1 f_2$

Global Thermodynamic Equilibrium:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} = 0$$

For LTE, $\left(\frac{Df}{Dt}\right)_{coll} = 0$ suffices.

If a system is in LTE, it follows that it is in detailed balance, but not the way around!

2.5 Maxwell-Boltzmann Distribution

The Maxwell-Boltzmann distribution is a distribution for a system in LTE. We will obtain the Euler equations from the moments of the Boltzmann equation when the distribution is a Maxwell-Boltzmann distribution.

Since we have LTE, we demand detailled balance:

$$f_1' f_2' = f_1 f_2$$

$$\Rightarrow \ln f_1' + \ln f_2' = \ln f_1 + \ln f_2$$

If now $\ln f_i$ is a collision invariant moment, then the condition for detailled balance is satisfied and we obtain conservation laws.

$\Rightarrow$ Ansatz: We have three collision invariants (m, $m\underline{v}$, $\frac{1}{2}m\underline{v}^2$) and three equations; So f must be a linear combination of those moments so the equations won't be overdetermined:

$$\Rightarrow \ln f_0 = \alpha_1 + \underline{\alpha}_2 \cdot \underline{u} + \alpha_3 \tfrac{1}{2} \underline{u}^2$$

$$= -\tfrac{1}{2} \beta m (\underline{u} - \underline{v})^2 + \gamma$$

β must be > 0 to ensure that $f \to 0$ for $\underline{u} \to \infty$; the factor $\frac{1}{2} m$ was added to simplify upcoming results. $\underline{v}$ is the mean velocity because the distribution function must be isotropic in the frame in which the material is at rest.

We can decompose the particle velocity:

$$\underline{u} \equiv \underline{v} + \underline{w}$$

where $\underline{w}$ is called the random velocity.

In terms of random velocities, we then have

$$\ln f_0 = -\frac{1}{2}\beta m \underline{w}^2 + \gamma$$

$$\Rightarrow f_0(\underline{w}) = A \exp\left(-\frac{1}{2}\beta m \underline{w}^2\right)$$

We can determine the normalisation A by using

$$n = \int_{\mathbb{R}^3} f_0(\underline{w})\, d^3 w$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} \exp\left(-\frac{1}{2}\beta m w^2\right) w^2\, dw$$

$$= 4\pi \int_0^{\infty} \exp\left(-\frac{1}{2}\beta m w^2\right) w^2\, dw$$

To evaluate this integral, first consider

$$I_k = \int_0^{\infty} x^k e^{-x^2}\, dx$$

and integrate by parts: $\int u' v\, dx = uv - \int u v'\, dx$

let $u' = x^k \quad \Rightarrow \quad u = \frac{1}{k+1} x^{k+1}$

$v = e^{-x^2} \quad \Rightarrow \quad v' = -2x e^{-x^2}$

Then: $I_k = \int_0^\infty x^k e^{-x^2} dx = \left[\frac{x^{k+1}}{k+1} e^{-x^2} \right]_0^\infty - \frac{1}{k+1} \int_0^\infty x^{k+1} (-2x e^{-x^2}) d$

$$= \frac{1}{k+1} [0 - 0] + \frac{2}{k+1} \int x^{k+2} e^{-x^2} dx$$

$$\Rightarrow I_k = \frac{2}{k+1} I_{k+2}$$

Using $n = k+2$:

$$I_{n-2} = \frac{2}{n-1} I_n \quad \Rightarrow I_n = \frac{n-1}{2} I_{n-2}$$

For our case here, we have $n = 2$, so only one recursion suffices. We still need to compute I_0:

$$I_0 = \int_0^\infty e^{-x^2} dx \quad \Rightarrow I_0^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$$

$$\Rightarrow I_0^2 = \iint e^{-(x^2+y^2)} dx\, dy$$

switching to polar coordinates:
$x = r \sin \theta$
$y = r \cos \theta$
$dx\, dy = r\, dr\, d\theta$

$$= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\, dr\, d\theta$$

$$= \frac{\pi}{2} \int e^{-r^2} r\, dr$$

Now let $s \equiv -r^2 \Rightarrow ds = -2r\, dr \Rightarrow r\, dr = -\frac{1}{2} ds$

$$\hookrightarrow I_0^2 = \frac{\pi}{2} \cdot \left(-\frac{1}{2}\right) \int e^{+s} ds = \frac{-\pi}{4} e^s = -\frac{\pi}{4} e^{-r^2} \Big|_{r=0}^{r=\infty}$$

$$= -\frac{\pi}{4} [0 - 1] = \frac{\pi}{4}$$

$$\Rightarrow I_0 = \frac{\sqrt{\pi}}{2}$$

Then we get for I_2:

$$I_2 = \frac{1}{2} I_0 = \frac{\sqrt{\pi}}{4}$$

Back to the Maxwell-Boltzmann amplitude:

$$n = 4\pi \int_0^\infty \exp(-\tfrac{1}{2}\beta m w^2) w^2 dw \qquad \text{let } s = \sqrt{\tfrac{1}{2}\beta m}\, w$$

$$\Rightarrow w = \frac{s}{\sqrt{1/2\,\beta m}}$$

$$= 4\pi \int_0^\infty \exp(-s^2) \frac{s^2}{1/2\,\beta m} \frac{ds}{\sqrt{1/2\,\beta m}}$$

$$= \frac{4\pi}{(1/2\,\beta m)^{3/2}} \int_0^\infty \exp(-s^2) s^2 ds$$

$$= \frac{4\pi}{(1/2\,\beta m)^{3/2}} I_2 = \frac{4\pi}{(1/2\,\beta m)^{3/2}} \frac{\sqrt{\pi}}{4} = \frac{1}{(\beta m/2\pi)^{3/2}}$$

$$\Rightarrow \boxed{A = \frac{n}{(\beta m/2\pi)^{3/2}}}$$

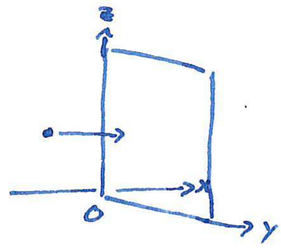
What remains is to find an expression for β in order to fully determine the Maxwell-Boltzmann distribution f_0.

To evaluate β, we use $f_0(\underline{w})$ to calculate a directly measurable quantity: the pressure.

By definition:

$$\text{pressure } p = \frac{\text{momentum transfer from particle to wall}}{\text{unit area} \cdot \text{unit time}}$$

Suppose you have a perfectly reflecting wall in the (y,z) plane and confine the gas to the region $x \leq 0$ so that particles hit the wall only if $w_x > 0$.



If an incoming particle has velocity (w_x, w_y, w_z) after hitting the wall its velocity is $(-w_x, w_y, w_z)$ and the momentum transferred to the wall is $\Delta p = 2 m w_x$.

Since f_0 gives the average number of particles in the phase space volume around $(\underline{x}, t)$, the flux of particles hitting the wall is given by $w_x f_0$.

The pressure is then given by

only $w_x > 0$ considered

$$p = \int_{\mathbb{R}} dw_z \int_{\mathbb{R}} dw_y \int_0^\infty \Delta p \, w_x f_0 \, dw_x$$

$$= \int dw_z \int dw_y \int_0^\infty (2 m w_x) \, w_x \left(A \exp(-\tfrac{1}{2} \beta m w^2) \right) dw_x$$

$$= Am \int dw_z \int dw_y \int_0^\infty 2 \, w_x^2 \exp(-\tfrac{1}{2} \beta m w^2) \, dw_x$$

$$= Am \int_{\mathbb{R}} dw_z \int_{\mathbb{R}} dw_y \int_{-\infty}^{\infty} w_x^2 \exp(-\tfrac{1}{2} \beta m w^2) \, dw_x$$

Since integrating over all $\underline{w}$ gives us the average value of a variable, we see that

$$\langle w_x^2 \rangle = \iiint_{\mathbb{R}^3} w_x^2 f_0 \, d^3 w$$

By symmetry, we must have

$$\langle w_x^2 \rangle = \langle w_y^2 \rangle = \langle w_z^2 \rangle$$

Furthermore, because

$$\langle w^2 \rangle = \langle w_x^2 \rangle + \langle w_y^2 \rangle + \langle w_z^2 \rangle$$
$$= 3 \langle w_x^2 \rangle$$

$$\Rightarrow \langle w_x^2 \rangle = \frac{1}{3} \langle w^2 \rangle$$

Inserting that into the pressure integral gives:

$$p = \frac{1}{3} m A \iiint_{\mathbb{R}^3} w^2 e^{-\frac{1}{2}\beta m w^2} d^3 w$$

$$= \frac{1}{3} m A \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} w^4 e^{-1/2 \beta m w^2} dw$$

$$= \frac{4\pi}{3} m A \int_0^{\infty} w^4 e^{-1/2 \beta m w^2} dw$$

Let $s^2 = \frac{1}{2} \beta m w^2$

$\rightarrow ds = \sqrt{\frac{1}{2} \beta m} \, dw$

$$= \frac{4\pi}{3} m A \int_0^{\infty} \frac{s^4}{(\frac{1}{2}\beta m)^2} e^{-s^2} \frac{1}{\sqrt{1/2 \beta m}} ds$$

$$= \frac{4\pi}{3} m A \left(\frac{1}{2}\beta m\right)^{-5/2} \int_0^{\infty} s^4 e^{-s^2} ds$$

We recognize the remaining integral as

$$I_4 = \int_0^\infty s^4 e^{-s^2} ds = \frac{4-1}{2} I_2 = \frac{3}{2}\left(\frac{2-1}{2} I_0\right) = \frac{3}{4}\frac{\sqrt{\pi}}{2}$$

$$= \frac{3\sqrt{\pi}}{8}$$

$$\Rightarrow p = \frac{4}{3}\pi m A \frac{1}{(1/2\beta m)^{5/2}} \cdot \frac{3\sqrt{\pi}}{8}$$

$$= \frac{1}{2}\pi^{3/2} m A \frac{1}{(1/2\beta m)^{5/2}} \qquad \Big| A = \left(\frac{\beta m}{2\pi}\right)^{3/2} n$$

$$= \frac{1}{2}\pi^{3/2} m \left(\frac{\beta m}{2\pi}\right)^{3/2} \frac{1}{(1/2\beta m)^{5/2}} n$$

$$= \frac{\frac{1}{2}\left(\frac{1}{2}\right)^{3/2} \beta^{3/2} m^{5/2}}{(1/2)^{5/2} \beta^{5/2} m^{5/2}} n = \frac{1}{\beta} n$$

Using the ideal gas law:

$$pV = NkT \Rightarrow p = \frac{N}{V} kT = nkT = \frac{n}{\beta}$$

$$\Rightarrow \boxed{\beta = \frac{1}{kT}}$$

And we obtain the Maxwell-Boltzmann distribution

$$\boxed{f_0(\underline{w}) = \frac{n}{(kT/2\pi m)^{3/2}} \exp\left(-\frac{1}{2}\frac{m}{kT} \underline{w}^2\right)}$$

3) Moments of the Boltzmann Equation

Consider the Boltzmann equation

$$\frac{\partial f}{\partial t} + \underline{u}\frac{\partial f}{\partial \underline{x}} + \underline{a}\frac{\partial f}{\partial \underline{u}} = \left(\frac{Df}{Dt}\right)_{coll}$$

If we multiply both sides by a moment (m, $m\underline{u}$, $\frac{1}{2}m\underline{u}^2$) and integrate over all velocity space, we obtain conservation laws, since the moments are collision invariants:

$$\int_{\mathbb{R}^3} \left(\frac{Df}{Dt}\right)_{coll} \cdot Q(\underline{u})\, d^3\underline{u} = 0$$

To compute the moments of the Boltzmann equation, we will make use of the following:

- $f(\underline{x},\underline{u},t)$ is a distribution function defined in phase space; $\underline{x}$, $\underline{u}$, t are independent variables. Interchange the integration order as you please.

- f is a distribution; It sinks to zero more rapidly than any power law:
  $$\lim_{\alpha\to\infty} \alpha^n f(\alpha) = 0 \quad \forall n$$

- $\underline{a} = \underline{a}(x)$, not $\underline{a}(\underline{x},\underline{u})$
  $$\Rightarrow \int \underline{a}\, g(\underline{u})\, d^3u = \underline{a}\int g(u)\, d^3u$$

- Known integrals:

$$n = \int_{\mathbb{R}^3} f d^3u$$

$$n\underline{v} = \int f \underline{u}\, d^3u$$

$$n\underline{v}^2 = \int f \underline{u}^2 d^3u$$

- We can separate the velocity:

$$\underline{u} = \underline{v} + \underline{w}$$

$\underline{v}$ is the average bulk velocity coming from $n\underline{v} = \int f\underline{u}\, d^3u$

$\underline{v} = \underline{v}(x)$; $\frac{\partial v_i}{\partial u_j} = 0 \quad \forall i,j$; $\int v_i\, g(u) d^3u = v_i \int g(u) d^3u$

$\underline{w}$ is the random thermal velocity. We have

$$\langle \underline{w} \rangle = \int w_i f(u) d^3u = \int (u_i - v_i) f d^3u = \int u_i f d^3u - v_i \int f d^3u$$

$$= n v_i - v_i \cdot n = 0$$

3.1. Mass Conservation

We obtain the mass conservation law by using the first moment: $Q(\underline{u}) = m$

$$\underbrace{\int_{\mathbb{R}^3} m \frac{\partial f}{\partial t} d^3u}_{1} + \underbrace{\int_{\mathbb{R}^3} m \frac{\partial f}{\partial x_i} u_i d^3u}_{2} + \underbrace{\int m \frac{\partial f}{\partial u_i} a_i d^3u}_{3} = 0$$

1 $\displaystyle \int m \frac{\partial f}{\partial t} d^3u = m \frac{\partial}{\partial t} \int f d^3u = m \frac{\partial}{\partial t} n = \frac{\partial}{\partial t}(mn) = \frac{\partial \rho}{\partial t}$

2 $\displaystyle \int m \frac{\partial f}{\partial x_i} \underline{u}_i d^3u = m \frac{\partial}{\partial x_i} \int f u_i d^3u = m \frac{\partial}{\partial x_i}(n v_i) = \frac{\partial}{\partial \underline{x}}(\rho \underline{v})$

3 $\displaystyle \int m \frac{\partial f}{\partial \underline{u}_i} a_i d^3u = m \int \frac{\partial}{\partial u_i}(f a_i) d^3u = m f a_i \Big|_{-\infty}^{\infty} = 0$

Where we used the fact that

$$\frac{\partial}{\partial \underline{u}}(f \underline{a}) = \frac{\partial f}{\partial \underline{u}} \underline{a} + f \underbrace{\frac{\partial \underline{a}}{\partial \underline{u}}}_{=0} = \frac{\partial f}{\partial \underline{u}} \underline{a}$$

This gives us:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \underline{x}}(\rho \underline{v}) = 0}$$

3.2) Momentum Conservation

We obtain the momentum conservation equation law by using the second moment: $Q(\underline{u}) = m\underline{u}$

For any component i:

$$\underbrace{\int_{\mathbb{R}} m\, u_i \frac{\partial f}{\partial t} d^3u}_{①} + \underbrace{\int_{\mathbb{R}^3} m\, u_i u_j \frac{\partial f}{\partial x_j} d^3u}_{②} + \underbrace{\int_{\mathbb{R}^3} m\, u_i \frac{\partial f}{\partial u_j} a_j d^3u}_{③} = 0$$

① $$\int m\, u_i \frac{\partial f}{\partial t} d^3u = \frac{\partial}{\partial t}\left(m \int f u_i d^3u\right) = \frac{\partial}{\partial t}(m n v_i) = \underline{\frac{\partial}{\partial t} \varrho v_i}$$

② $$\int_{\mathbb{R}^3} m\, u_i u_j \frac{\partial f}{\partial x_j} d^3u = \quad \text{Substitute } u_i = v_i + w_i$$

$$= m \frac{\partial}{\partial x_j} \int (v_i + w_i)(v_j + w_j) f d^3u =$$

$$= m \frac{\partial}{\partial x_j} \left[\int v_i v_j f d^3u + \int w_i w_j f d^3u + \int v_i w_j f d^3u + \int v_j w_i f d^3u\right]$$

$$= m \frac{\partial}{\partial x_j} \left[v_i v_j \int f d^3u + \int w_i w_j f d^3u + v_i \underbrace{\int w_j f d^3u}_{=0} + v_j \underbrace{\int w_i f d^3u}_{=0}\right]$$

$$= m \frac{\partial}{\partial x_j} \left[n v_i v_j + \int w_i w_j f d^3u\right]$$

$$= \frac{\partial}{\partial x_j}\left(\varrho v_i \cdot v_j + \underline{\underline{P}}_{ij}\right) \qquad \text{with } \underline{\underline{P}}_{ij} = m \int w_i w_j f d^3u$$

③ $$\int_{\mathbb{R}^3} m\, u_i \frac{\partial f}{\partial u_j} a_j \, d^3u = m\, a_j \int u_i \frac{\partial f}{\partial u_j} d^3u =$$

integrate by parts
$$= m\, a_j \left[u_i f \Big|_{\mathbb{R}^3} - \int \frac{\partial u_i}{\partial u_j} f \, d^3u \right]$$

$$= -m\, a_j \int \delta^i_j f(u)\, d^3u = -m_i n a_j = \underline{\underline{-\rho a_i}}$$

This gives us:

$$\boxed{\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + \underline{\underline{P}}_{ij}) = \rho a_i}$$

with $\underline{\underline{P}}_{ij} = \int_{\mathbb{R}^3} w_i w_j f(u)\, d^3u$

3.3) Energy Conservation

We obtain the energy conservation law by using the third momentum: $Q(\underline{u}) = \frac{1}{2} m \underline{u}^2$

$$\int_{\mathbb{R}^3} \frac{1}{2} m u^2 \frac{\partial f}{\partial t} d^3u + \int \frac{1}{2} m u^2 u_i \frac{\partial f}{\partial x_i} d^3u + \int a_i \frac{\partial f}{\partial u_i} \cdot \frac{1}{2} m u^2 d^3u = 0$$

(3)

(1) (2)

(1) $$\int \frac{1}{2} m u^2 \frac{\partial f}{\partial t} d^3u = \frac{1}{2} m \int (\underline{v} + \underline{w})^2 \frac{\partial f}{\partial t} d^3u =$$

$$= \frac{1}{2} m \int (v^2 + w^2 + 2 v_i w_i) \frac{\partial f}{\partial t} d^3u$$

$$= \frac{1}{2} m \frac{\partial}{\partial t} \left[\int v^2 f d^3u + \int w^2 f d^3u + 2 \int v_i w_i f d^3u \right]$$

$$= \frac{1}{2} m \frac{\partial}{\partial t} \left[v^2 \int f d^3u + \int w^2 f d^3u + 2 v_i \underbrace{\int w_i f d^3u}_{=0} \right]$$

$$= \frac{1}{2} m \frac{\partial}{\partial t} \left[n v^2 + \int w^2 f d^3u \right]$$

$$= \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + \rho \varepsilon \right) \equiv \underline{\underline{\frac{\partial E}{\partial t}}}$$

with $E \equiv \frac{1}{2} \rho v^2 + \rho \varepsilon$

$$\rho \varepsilon \equiv \int_{\mathbb{R}^3} w^2 f d^3u$$

② $\int \frac{1}{2} m u^2 u_i \frac{\partial f}{\partial x_i} d^3u =$

$$= \frac{1}{2} m \frac{\partial}{\partial x_i} \int (\underline{v} + \underline{w})^2 (v_i + w_i) f d^3u$$

$$= \frac{1}{2} m \frac{\partial}{\partial x_i} \int (v^2 + w^2 + 2 v_j w_j)(v_i + w_i) f d^3u$$

$$= \frac{1}{2} m \frac{\partial}{\partial x_i} \int (v^2 v_i + v^2 w_i + w^2 v_i + w^2 w_i + 2 v_i v_j w_j + 2 v_j w_i w_j) f d^3u$$

$$= \frac{1}{2} m \frac{\partial}{\partial x_i} \Big[v^2 v_i \int f d^3u + v^2 \underbrace{\int w_i f d^3u}_{=0} + v_i \int w^2 f d^3u + \int w^2 w_i f d^3u + $$
$$+ 2 v_i v_j \underbrace{\int w_j f d^3u}_{=0} + 2 v_j \int w_i w_j f d^3u \Big]$$

$$= \frac{\partial}{\partial x_i} \Big[\frac{1}{2} m v^2 v_i n + \frac{1}{2} m v_i \int w^2 f d^3u + \frac{1}{2} m \int w^2 w_i f d^3u + m v_j \int w_i w_j f d^3u \Big]$$

$$\equiv \frac{\partial}{\partial x_i} \Big[\frac{1}{2} \rho v^2 v_i + v_i \underbrace{\int \frac{1}{2} m w^2 f d^3u}_{= \rho \varepsilon} + \underbrace{\int \frac{1}{2} m w_i w^2 f d^3u}_{\equiv Q_i} + v_j \underbrace{\int w_i w_j f d^3u}_{\underline{\underline{P}}_{ij}} \Big]$$

$$= \frac{\partial}{\partial x_i} \Big[v_i \underbrace{(\tfrac{1}{2} \rho v^2 + \rho \varepsilon)}_{= E} + Q_i + \underline{\underline{P}}_{ij} v_j \Big]$$

$$= \frac{\partial}{\partial x_i} \Big[E v_i + Q_i + \underline{\underline{P}}_{ij} v_j \Big]$$

with $E = \frac{1}{2} \rho v^2 + \rho \varepsilon$

$\rho \varepsilon = \frac{1}{2} m \int w^2 f d^3u$

$P_{ij} = m \int w_i w_j f d^3u$

$Q_i = \frac{1}{2} m \int w_i w^2 f d^3u$

③ $\int a_i \frac{\partial f}{\partial u_i} \cdot \frac{1}{2} m u^2 d^3u = \frac{1}{2} m a_i \int \frac{\partial f}{\partial u_i} u^2 d^3u =$

$= \frac{1}{2} m a_i \left[u^2 f \Big|_{\mathbb{R}^3} - 2 \int u_i f d^3u \right] =$

$= - m a_i n v_i = \underline{\underline{- \rho \underline{a} \cdot \underline{v}}}$

Put together, we get:

$$\boxed{\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} \left(E v_j + Q_j + \underline{\underline{P}}_{ij} \cdot v_i \right) = \rho a_i v_i}$$

with

$E = \frac{1}{2} \rho v^2 + \rho \varepsilon$

$\varepsilon \equiv \frac{1}{2} m \int w^2 f d^3u$ internal thermal energy

$\underline{\underline{P}}_{ij} = m \int w_i w_j f d^3u$ pressure tensor

$Q_i = \frac{1}{2} m \int w_i w^2 f d^3u$ heat flux

3.4) The Euler Equations

The Euler equations are the moments of the Boltzmann equation where the distribution function is the Maxwell-Boltzmann distribution, i.e. when detailed balance holds:

$$f = f_0 = f_0(\underline{w}) = \frac{n}{\sqrt{kT/2\pi m}} \exp\left(-\frac{1}{2}\frac{m}{kT}\underline{w}^2\right)$$

In this case, the pressure tensor $\underline{\underline{P}}_{ij}$ and the heat flux $\underline{Q}$ simplify:

$$\underline{\underline{P}}_{ij} \equiv m \int_{\mathbb{R}^3} w_i w_j f \, d^3u$$

$$Q_i = \frac{1}{2} m \int_{\mathbb{R}^3} w_i w^2 f \, d^3u$$

Note that f_0 is an even function in w_i:

$$f_0(\underline{w}_i) = f_0(-\underline{w}_i)$$

and we only integrate over integer powers of w_i from $-\infty$ to ∞

$\Rightarrow$ if there is an odd power of w_i in the integral, the integral is zero.

Then:

$$\underline{\underline{P}}_{ij} = m\int w_i w_j f_0 \, d^3u = m\int w^2 \delta_{ij} f_0 \, d^3u$$

To simplify further comutations, let us define

$$\sigma = \sqrt{\frac{kT}{m}} \quad \text{and} \quad g_i(\underline{w}) = \frac{1}{\sqrt{kT/2\pi m}} \int_{-\infty}^{\infty} \exp\left(-\frac{m}{kT}\frac{1}{2} w_i^2\right) du_i$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{w_i^2}{2\sigma^2}\right) du_i$$

Then $f_0(\underline{w}) = n \cdot g_1(\underline{w})\, g_2(\underline{w})\, g_3(\underline{w})$

We have seen that $\underline{\underline{P}}_{ij}$ is diagonal. Let us now compute $\underline{\underline{P}}_{ii}$:

$$\underline{\underline{P}}_{ii} = m\int_{\mathbb{R}^3} w_i^2 f_0 \, d^3u = m\int_{\mathbb{R}^3} w_i^2 n\, g_i(\underline{w})\, g_j(\underline{w})\, g_k(\underline{w})\, du_i\, du_j\, du_k$$

$$= mn \underbrace{\int_{-\infty}^{\infty} g_j(\underline{w})\, du_j}_{1} \underbrace{\int_{-\infty}^{\infty} g_k(\underline{w})\, du_k}_{1} \underbrace{\int_{-\infty}^{\infty} w_i^2 g_i(w)\, du_i}_{2}$$

① $$\int_{-\infty}^{\infty} g_j(\underline{u})\, du_j = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{w_j^2}{2\sigma^2}\right) du_j$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp\left(-\frac{(u_j - v_j)^2}{2\sigma^2}\right) du_j$$

Let $s^2 \equiv \frac{(u_j - v_j)^2}{2\sigma^2} \Rightarrow \sqrt{2\sigma^2}\, ds = du_j$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp(-s^2) \sqrt{2\sigma^2}\, ds =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-s^2)\, ds$$

$$= \frac{1}{\sqrt{\pi}} \sqrt{\int_{-\infty}^{\infty} \exp(-s^2)\, ds \int_{-\infty}^{\infty} \exp(-t^2)\, dt}$$

$$= \frac{1}{\sqrt{\pi}} \sqrt{\iint_{-\infty}^{\infty} \exp(-(s^2+t^2))\, ds\, dt}$$

Let $s = r\cos\varphi$
$t = r\sin\varphi$
$ds\,dt = r\,d\varphi\, dr$

$$= \frac{1}{\sqrt{\pi}} \sqrt{\int_0^{2\pi}\int_0^{\infty} \exp(-r^2)\, r\,dr\, d\varphi}$$

Let $\ell \equiv -r^2$
$d\ell = -2r\,dr$
$r\,dr = -\frac{1}{2} d\ell$

$$= \frac{1}{\sqrt{\pi}} \sqrt{-2\pi \int \exp(+\ell)\, \frac{1}{2}\, d\ell}$$

$$= \frac{1}{\sqrt{\pi}} \sqrt{-\pi\left[\exp(\ell)\right]} = \frac{1}{\sqrt{\pi}} \sqrt{-\pi\left[\exp(-r^2)\Big|_{r=\infty} - \exp(-r^2)\Big|_{r=0}\right]}$$

$$= \frac{1}{\sqrt{\pi}} \sqrt{\pi} = \underline{\underline{1}}$$

② $\int w_i^2 g_i(\underline{w})\, du_i = \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp\left(-\frac{(u_i - v_i)^2}{2\sigma^2}\right)(u_i - v_i)^2 du_i$

$\left[\text{Let } s^2 = \frac{(u_i - v_i)^2}{2\sigma^2} \Rightarrow du_i = \sqrt{2\sigma^2}\, ds\right]$

$= \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp(-s^2)\, 2\sigma^2 s^2 \sqrt{2\sigma^2}\, ds =$

$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} s^2 \exp(-s^2) ds$

Using again that $I_n = \int_0^\infty x^n \exp(-x^2) dx =$

$= \frac{1}{2} \int_{-\infty}^{\infty} x^n \exp(-x^2) dx$

for even n

$= \frac{n-1}{2} I_{n-2}$

and $I_0 = \frac{\sqrt{\pi}}{2} \Rightarrow \int_{-\infty}^{\infty} s^2 \exp(-s^2) ds = 2 \cdot I_2 = 2 \frac{2-1}{2} I_0$

$= I_0 = \frac{\sqrt{\pi}}{2}$

$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \sigma^2$

Combined, this gives us

$\underline{P}_{ii} = mn \cdot 1 \cdot 1 \cdot \sigma^2 = \underline{\underline{\rho\sigma^2}}$.

This is valid for all i:

$\Rightarrow \boxed{\underline{P}_{ij} = \underline{\mathbb{1}}\, p = \underline{\mathbb{1}}\, \rho\sigma^2}$ with $\sigma^2 = \frac{kT}{m}$

This essentially gives us the ideal gas law: $p = \frac{N}{V} kT = \frac{\rho}{m} kT$

We can also simplify the expression for the specific internal energy:

$$\varrho \varepsilon = \int_{\mathbb{R}^3} \frac{1}{2} m w^2 f \, d^3 w = \frac{1}{2} \mathrm{Tr}\,(\underline{\underline{P_{ij}}}) =$$

$$= \frac{3}{2} p$$

This also gives us that we're working with monoatomic gas:

$$\varepsilon \overset{\text{perfect gas}}{=} \frac{p}{(\gamma-1)\varrho} \overset{!}{=} \frac{3}{2} \frac{p}{\varrho}$$

$$\Rightarrow (\gamma - 1) = \frac{2}{3} \qquad \Rightarrow \gamma = \frac{5}{3} = \frac{f+2}{f} \quad \Rightarrow f = 3$$

Lastly, the heat flux Q_i also simplifies:

$$Q_i = \frac{1}{2} m \int_{\mathbb{R}^3} w_i w^2 f \, d^3 u = 0$$

again because we integrate over an odd power of $\underline{w_i}$.

The Euler equations are then given by:

$$\boxed{\begin{aligned}
&\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x_j}(\varrho v_j) = 0 \\
&\frac{\partial}{\partial t}(\varrho v_i) + \frac{\partial}{\partial x_j}(\varrho v_i v_j + p_i) = \varrho a_i \\
&\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}(E v_j + p v_j) = \varrho a_j v_j
\end{aligned}}$$

with $E = \frac{1}{2}\varrho v^2 + \varrho \varepsilon$

$$\varepsilon = \frac{3}{2} p$$

$$p = \varrho \sigma^2 = \frac{\varrho k T}{m} = n k T$$

and where we used that

$$\underline{\underline{P}}_{ij} v_j = \delta_{ij} p v_j \quad \text{such that}$$

$$\frac{\partial}{\partial x_i}(\underline{\underline{P}}_{ij} v_j) = \frac{\partial}{\partial x_i}(\delta_{ij} p v_j) = \frac{\partial}{\partial x_j}(p \cdot v_j)$$