Equilibria of collisionless systems

3rd part

Outlines

Models defined from Dfs

- The isothermal sphere

Anisotropic distribution function in spherical systems

- Motivation
- General concepts
- Example of an anisotropic DF
- Application to the Hernquist model

The Jeans Equations

- Motivations
- The Jeans Equations and conservation laws
- The Jeans Equations in Spherical coordinates
- The Jeans Equations in Cylindrical coordinates

Equilibria of collisionless systems

Models defined from DFs: Polytropes

Then : what do we learn concerning the Plummer model ?

We have access to its DF :

$$f(\mathcal{E}) \begin{cases} \sim \mathcal{E}^{n-3/2} \sim \left(\dfrac{GM}{\sqrt{r^2+a^2}} - \dfrac{1}{2} v^2 \right)^{7/2} \\ = 0 \quad \text{if} \quad \dfrac{GM}{\sqrt{r^2+a^2}} - \dfrac{1}{2} v^2 < 0 \end{cases}$$

We have access to the kinematics structure :

① Velocity distribution function

$$P_r(v) = \frac{f\left(\frac{1}{2} v^2 + \phi(r)\right)}{\nu(r)} \sim \underbrace{\left(1 + \frac{r^2}{a^2}\right)^{5/2}}_{\frac{1}{\rho}} \underbrace{\left(\frac{GM}{\sqrt{r^2+a^2}} - \frac{1}{2} v^2 \right)^{7/2}}_{\mathcal{E}^{7/2}}$$

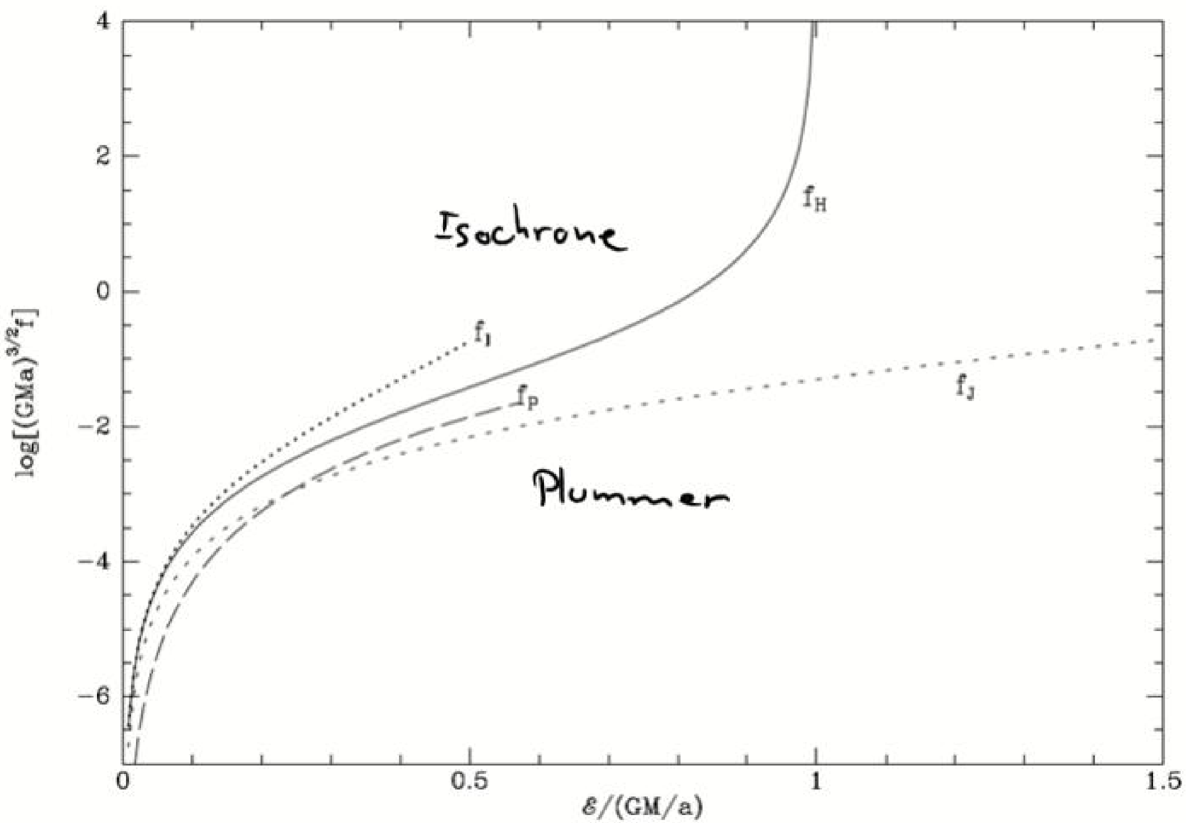
② Velocity dispersion

$$\sigma^2 = 4\pi \, \frac{1}{\nu(x)} \int_0^{v_{max} = \sqrt{2\psi}} v^4 \, f\left(\frac{1}{2} v^2 + \phi(x)\right) dv$$

$$= 4\pi \, \frac{1}{\nu(r)} \int_0^{v_{max}} v^4 \left(\frac{1}{2} v^2 - \frac{GM}{\sqrt{r^2+a^2}} \right)^{7/2} dv$$

Note: It is possible to do the same for the Plummer, Isochrone an Jaffe models

Hernquist • $\rho(0) \to \infty$

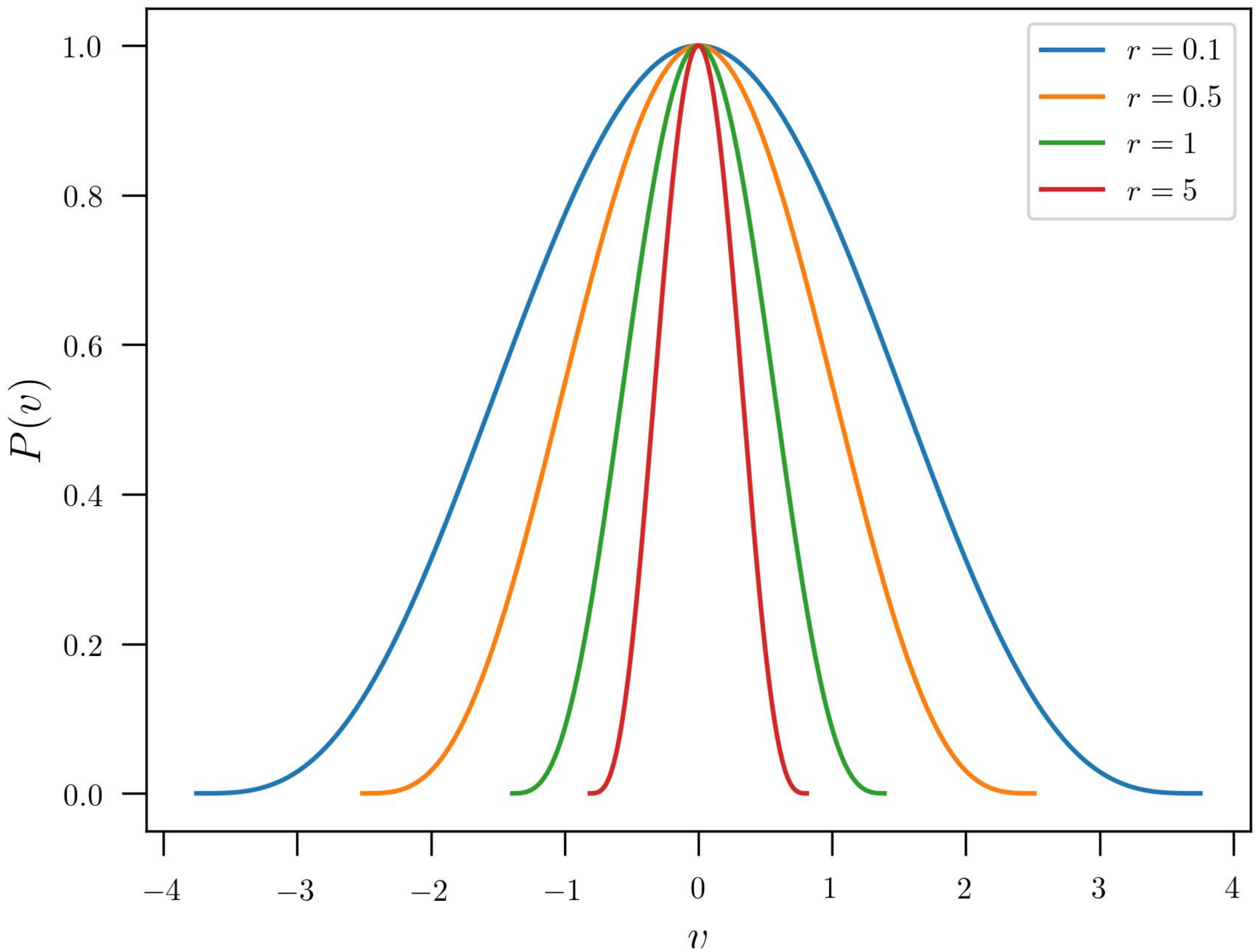


• no minimal energy

Jaffe $\quad \mathcal{E} \to \infty$

• $\rho(0) \to \infty$

Plummer velocity distribution function



Equilibria of collisionless systems

Models defined from DFs: Isothermal spheres

Stellar system with the DF (Isothermal)

$$\boxed{f(\varepsilon) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{\frac{\varepsilon}{\sigma^2}}}$$

with $\varepsilon = \psi - \frac{1}{2}v^2$

$$\rho(r) = 4\pi \int_0^\infty v^2 \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{\frac{\psi - \frac{1}{2}v^2}{\sigma^2}} = \rho_1 e^{\frac{\psi}{\sigma^2}}$$

$$\left(\int_0^\infty \frac{v^2 \, e^{\frac{\psi - \frac{1}{2}v^2}{\sigma^2}}}{(2\pi\sigma^2)^{3/2}} dv = \frac{e^{\frac{\psi}{\sigma^2}}}{4\pi} \right)$$

$$\boxed{\rho(r) = \rho_1 e^{\frac{\psi}{\sigma^2}}}$$

$$\boxed{\rho(\psi) = \rho_1 e^{\frac{\psi}{\sigma^2}}}$$

"Pressure" $\quad P(\rho) = \int_0^{\rho} d\rho' \rho' \frac{\partial \phi}{\partial \rho'} \equiv -\int_0^{\rho} d\rho' \rho' \frac{\partial \psi}{\partial \rho'}$

Derivating $\quad \rho(\psi) = \rho_1 \, e^{\frac{\psi}{\sigma^2}}$ with respect to ρ

$$\frac{\partial \rho}{\partial \rho} = 1 = \rho_1 \, e^{\frac{\psi}{\sigma^2}} \frac{1}{\sigma^2} \frac{\partial \psi}{\partial \rho} = \frac{1}{\sigma^2} \rho \frac{\partial \psi}{\partial \rho}$$

$\Rightarrow \quad \rho \frac{\partial \psi}{\partial \rho} = \sigma^2$ and $\boxed{P(\rho) = \sigma^2 \rho}$

Isothermal EoS

$$\sigma^2 \equiv \frac{k_B T}{m}$$

The structure of an isothermal self-gravitating sphere of gas with an EoS

$$\boxed{P(\rho) = \frac{k_B T}{m} \rho}$$

is identical to the one of a collisionless self-gravitating system with a DF

$$\boxed{f(\varepsilon) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{\frac{\varepsilon}{\sigma^2}}} \qquad \text{if} \quad \sigma^2 \equiv \frac{k_B T}{m}$$

wich leads to $\quad P(\rho) = \sigma^2 \rho$

Velocity distribution function

- collisionless isothermal sphere

$$P_r(v) = \frac{f(\varepsilon)}{\nu(\varepsilon)} \sim \frac{e^{\frac{1}{\sigma^2}\left(-\frac{1}{2}v^2+\psi(r)\right)}}{e^{\frac{1}{\sigma^2}\psi}} \sim e^{-\frac{v^2}{2\sigma^2}}$$

similar

- Gas sphere : (elastic collisions between particles)

$\Rightarrow$ Maxwell - Boltzman distribution $\quad P_r(v) \sim e^{-\frac{mv^2}{2k_BT}} \equiv e^{-\frac{v^2}{2\sigma^2}}$

Note The correspondance between gaseous polythrope and stellar collisionless systems is not always as close a for the isothermal sphere

- gaseous polytrope : σ is allways Maxwellian and isothrope
- stellar system : σ given by f is no necessarily Maxwellian and may be anisothrope (if not ergodic)

Velocity dispersion

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \frac{1}{\nu} \int d^3v \; v^2 \, \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \, e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}}$$

spherical coord in vel. space

$$= \frac{\frac{4}{3}\pi \int_0^\infty v^4 \, e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}} dv}{4\pi \int_0^\infty v^2 \, e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}} dv} = \frac{2\sigma^2}{3} \frac{\int_0^\infty dx \, x^4 e^{-x^2}}{\int_0^\infty dx \, x^2 e^{-x^2}} = \underline{\underline{\sigma^2}}$$

$$-x^2 = \frac{\Psi - \frac{1}{2}v^2}{\sigma^2}$$

σ^2 is indep. of r

What is the corresponding density / potential $\rho(r)$, $\phi(r)$ of the system?

Self-gravity !

$$\vec{\nabla}^2(\Phi) = 4\pi G\rho$$

The Poisson Equation

$$\frac{1}{r}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) = -4\pi G \rho(r)$$

$$\ln\rho = \ln\rho_1 + \frac{\psi}{\sigma^2}$$

$$\frac{d\ln\rho}{dr} = \frac{1}{\sigma^2}\frac{d\psi}{dr}$$

yields

$$\boxed{\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = -\frac{4\pi G}{\sigma^2} r^2 \rho(r)}$$

Solutions of the Poisson equation

$$\boxed{\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = -\frac{4\pi G}{\sigma^2} r^2 \rho(r)}$$

A. Power law

$$\rho \sim r^{-b}$$

Poisson $\Rightarrow$ $-b = -\dfrac{4\pi G}{\sigma^2} r^{2-b}$

$$b = 2$$

$$\boxed{\rho(r) = \frac{\sigma^2}{2\pi G r^2}}$$ Singular isothermal sphere

Notes

① The specific energy (σ^2) is constant every where

② The velocity dispersion is isotropic

Maximal equilibrium ?

But ρ and ϕ diverges at $r=0$!
$M(r)$ diverges at $r=\infty$

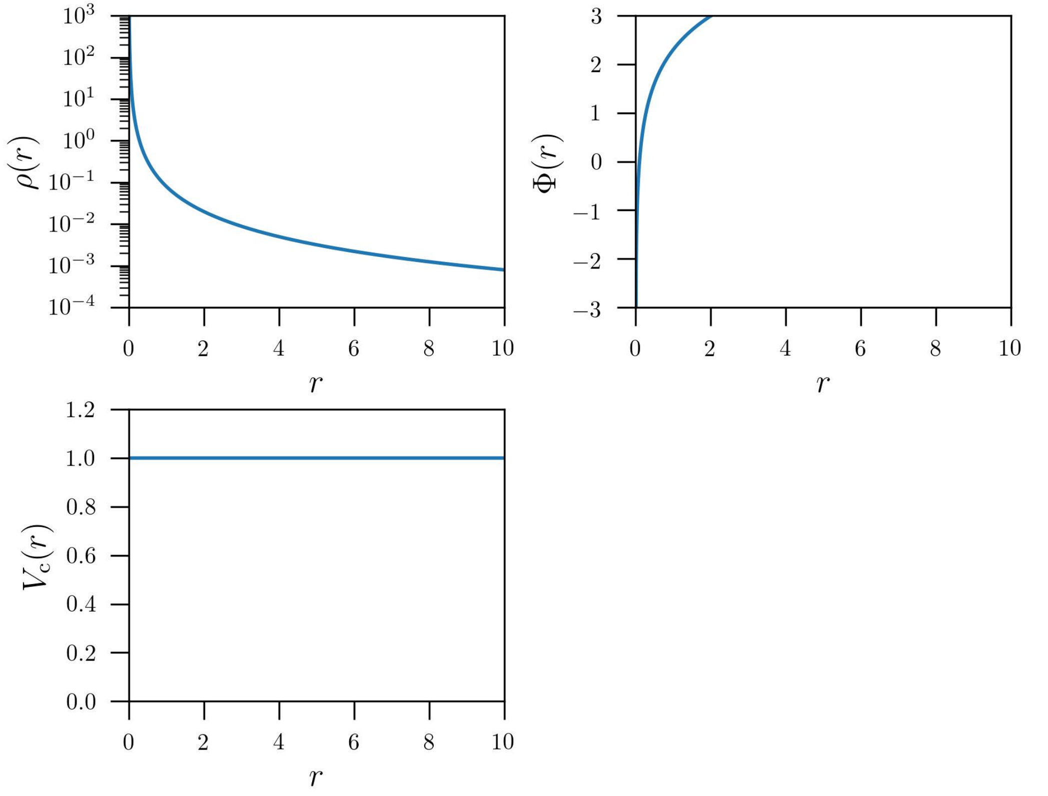
Isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \ln\left(\frac{r}{a}\right)$$

$$M(r) = 4\pi \rho_0 a^2 \, r$$

$$V_c^2(r) = 4\pi G \rho_0 a^2$$



- often used for gravitational lens models
- But !
  - diverge towards the centre !
  - Infinite mass !

B Models with finite potential and density

$$\tilde{\rho} = \frac{\rho}{\rho_0} \qquad \tilde{r} = \frac{r}{r_0} \qquad r_0 = \sqrt{\frac{9\sigma^2}{4\pi G \rho_0}}$$ (King radius)

The Poisson equation becomes

$$\boxed{\frac{d}{d\tilde{r}}\left(\tilde{r}^2 \frac{d\ln\tilde{\rho}}{d\tilde{r}}\right) = -9\hat{r}\hat{\rho}}$$

Requires numerical integration

+ boundary conditions

$$\begin{cases} \cdot\ \tilde{\rho}(0) = 1 & \text{normalisation} \\ \cdot\ \left.\frac{d\hat{\rho}}{d\tilde{r}}\right)_0 = 0 & \text{smooth} \end{cases}$$

Numerical solution of the non-singular isothermal sphere

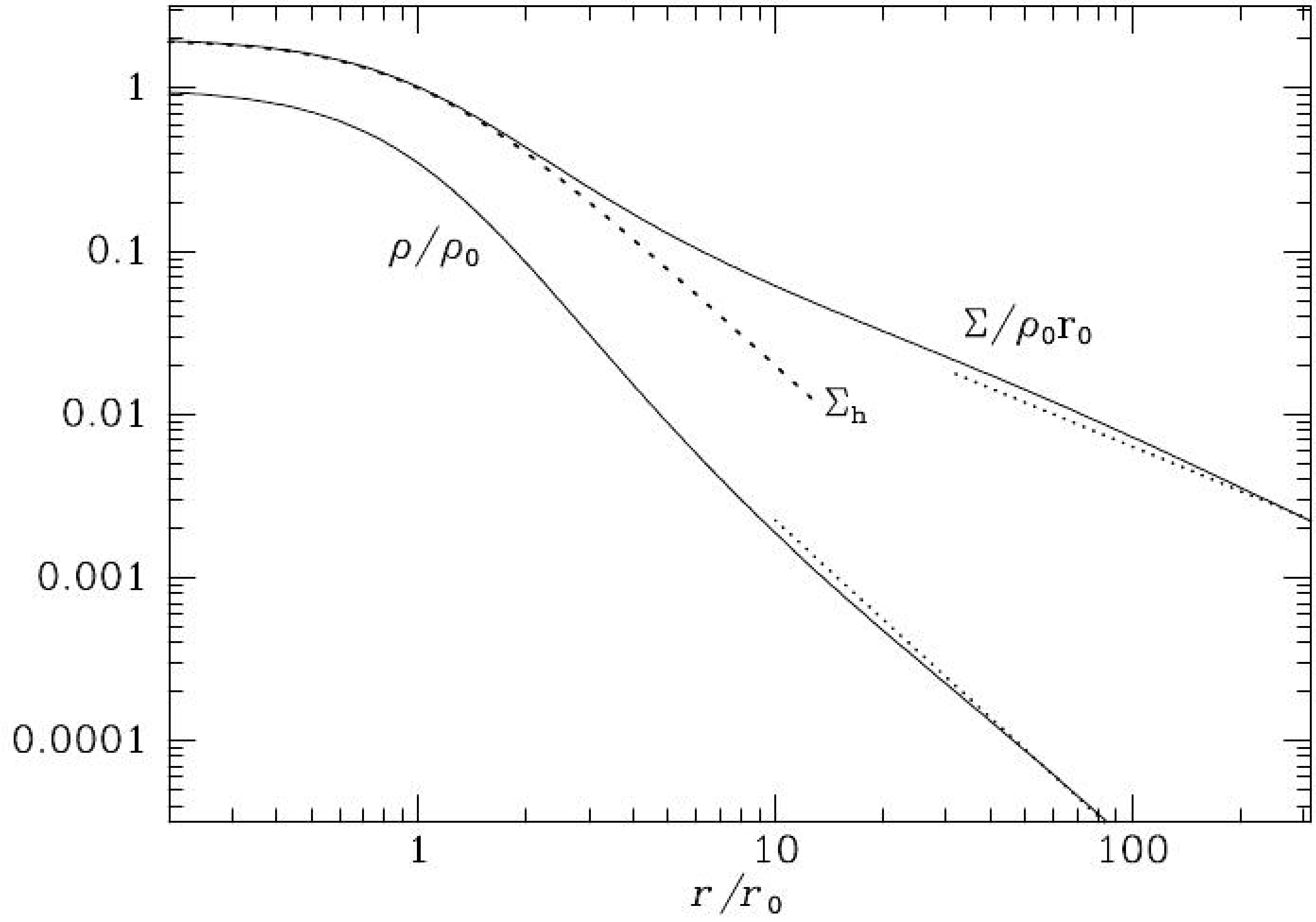
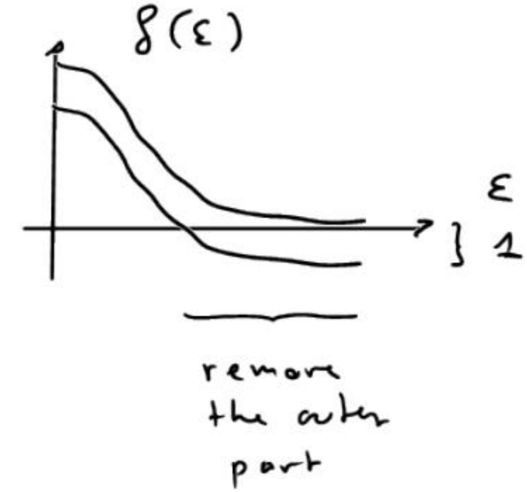


Figure 4.6 Volume (ρ/ρ_0) and projected ($\Sigma/\rho_0 r_0$) mass densities of the isothermal sphere. The dotted lines show the volume- and surface-density profiles of the singular isothermal sphere. The dashed curve shows the surface density of the modified Hubble model (4.109a).

King models

Similer to the isothermal sphere, but avoid the mass divergeance

$$
f_K(\mathcal{E}) = \begin{cases} \dfrac{\rho_1}{(2\pi\sigma^2)^{3/2}}\left(e^{\frac{\mathcal{E}}{\sigma^2}} - 1\right) & \mathcal{E} > 0 \\ 0 & \mathcal{E} \leq 0 \end{cases}
$$



Goal: decrease f for low $\mathcal{E}$, i.e in the outer parts.

→ Possible to solve the Poisson equation and obtain self-consistent models.

$$
\begin{aligned}
\rho_{\mathrm{K}}(\Psi) &= \frac{4\pi\rho_1}{(2\pi\sigma^2)^{3/2}} \int_0^{\sqrt{2\Psi}} \mathrm{d}v\, v^2 \left[\exp\left(\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}\right) - 1\right] \\
&= \rho_1 \left[\mathrm{e}^{\Psi/\sigma^2} \operatorname{erf}\left(\frac{\sqrt{\Psi}}{\sigma}\right) - \sqrt{\frac{4\Psi}{\pi\sigma^2}}\left(1 + \frac{2\Psi}{3\sigma^2}\right)\right],
\end{aligned}
$$

Density profiles for the King model

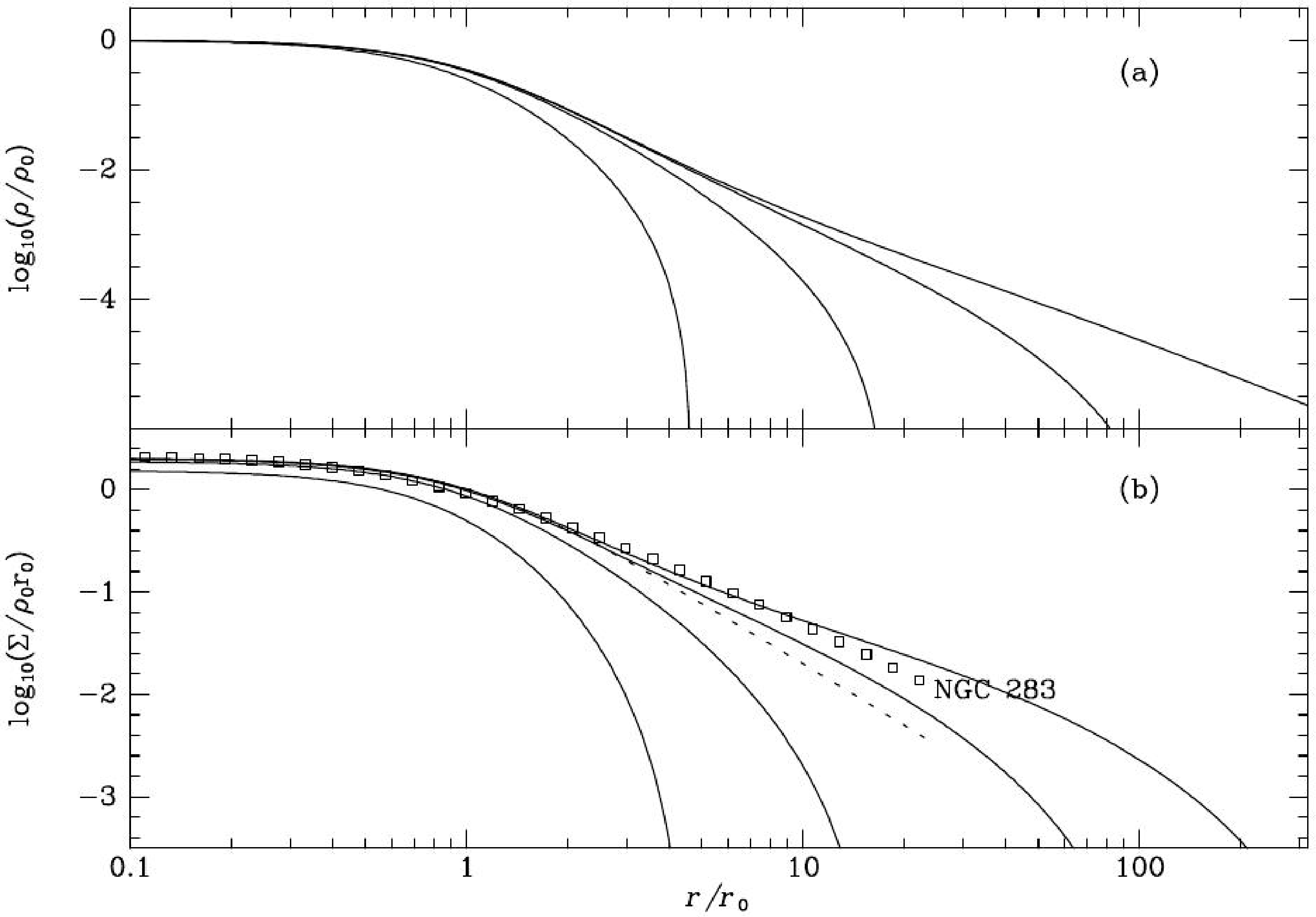


Figure 4.8 (a) Density profiles of four King models: from top to bottom the central potentials of these models satisfy $\Psi(0)/\sigma^2 = 12, 9, 6, 3$. (b) The projected mass densities of these models (full curves), and the projected modified Hubble model of equation (4.109b) (dashed curve). The squares show the surface brightness of the elliptical galaxy NGC 283 (Lauer et al. 1995).

Anisotropic DFs in spherical systems

Spherical systems with anisothropic velocities

Ergodic DF : $f(\mathcal{E}) \Rightarrow \sigma_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$

If we know $\nu(r)$: Eddington's formula

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\,\pi^2} \frac{d}{d\mathcal{E}} \left[\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \frac{d\nu}{d\Psi} \right]$$

or

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\,\pi^2} \left[\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \frac{d^2\nu}{d\Psi^2} + \frac{1}{\sqrt{\mathcal{E}}} \left(\frac{d\nu}{d\Psi} \right)_{\Psi=0} \right]$$

<u>Note</u> : $f(\mathcal{E}) > 0$ only if $\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} \frac{d\nu}{d\Psi}$ is an increasing function of $\mathcal{E}$

⚠ for a given $\nu(r)$: no guarentee that $f(\mathcal{E}) > 0$ ⚠

By relaxing the assumption that $f = f(\mathcal{E})$ (isothropic in v)

Ex: $f = f(\mathcal{E}, L = |\vec{L}|)$, we can ensure $f > 0$

Idea:

① Build a model based on circular orbits only.

By giving the appropriate weight to orbits at every radius, we can obtain a model with the desired $\nu(r)$

② Add it to an ergodic DF that generates $\nu(r)$

We can ensure that the sum of both DFs is positive.

Model based on circular orbits

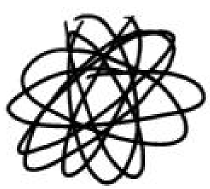
We split the model into a set of shells of radius r

- at each radius, we consider the corresponding circular orbits. For a given density and potential :

$$\begin{cases} - \text{ energy} & \mathcal{E}_{c,r} \\ - \text{ angular momentum} & L_c(\mathcal{E}_{c,r}) \end{cases}$$

- The DF of a spherical shell is thus :

$$\boxed{f_{s,r}(\mathcal{E}, L) = \underbrace{\delta(\mathcal{E} - \mathcal{E}_{c,r})}_{\substack{\text{Select the} \\ \text{right energy}}} \; \underbrace{\delta(L - L_c(\mathcal{E}_{c,r}))}_{\substack{\text{select the} \\ \text{right ang. momentum}}}}$$



Note each shell contains orbit from all inclinaison (no selection on the direction)

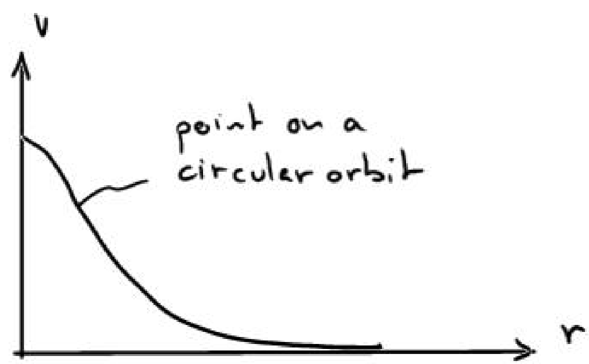
Total DF Sum the contribution of all shells (integration over the radius) but as there is a bijective relation between r and $\varepsilon_{c,r}$ we can integrate over $\varepsilon_{c,r}$:

$$f_c(\varepsilon, L) = \int_0^{\varepsilon_{max}} d\varepsilon_{c,r}\, \delta(\varepsilon - \varepsilon_{c,r})\, \delta(L - L_c(\varepsilon_{c,r}))\, \underbrace{F(\varepsilon_{c,r})}_{\text{weight}}$$

$$\boxed{f_c(\varepsilon, L) = \delta(L - L_c(\varepsilon))\, F(\varepsilon)}$$

($= 0$ except when L corresponds to the angular momentum of the circular orbit of energy ε)

Phase space (1-D as all planes are equivalent



circular orbit

$$V(r) = \sqrt{r \frac{\partial \phi}{\partial r}} = v_t$$

$$L = r v_t$$

$$\boxed{\text{With a suitable weight } F(\mathcal{E}) \quad f_c(\mathcal{E}, L) \text{ generates } \nu(r)}$$

$$\nu(r) = \int d^3v \; F(\varepsilon)\, \delta\big(L - L_c(\varepsilon)\big) = 4\pi \int_0^{\infty} dv\, v^2 F(\varepsilon)\, \delta\big(L - L_c(\varepsilon)\big)$$

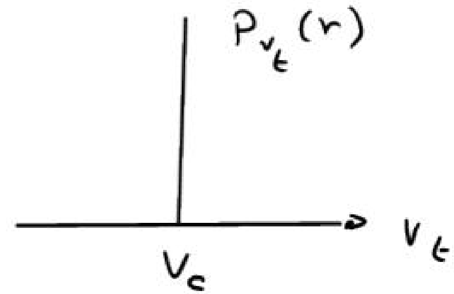
$$= 4\pi \int_{-\infty}^{\psi} \sqrt{2(\psi - \varepsilon)}\; F(\varepsilon)\, \delta\big(L - L_c(\varepsilon)\big)\, d\mathcal{E} = 4\pi \underbrace{\sqrt{2\big(\psi - \mathcal{E}_{c,r}\big)}}_{V_c(r)} \; F(\mathcal{E}_{c,r})$$

$$= 4\pi \sqrt{r \frac{\partial \psi}{\partial r}}\; F\big(\mathcal{E}_{c,r}(r)\big)$$

$$\mathcal{E} = -\tfrac{1}{2} v^2 + \psi$$

Velocity dispersion $\quad P_v(\varepsilon) = \dfrac{1}{4\pi V_c}\, \delta\big(L - L_c(\varepsilon)\big)$

- All orbits are purely tangential (circular)



- $V_r = 0$
- $\sigma_r = 0$

Idea : If $f_i(\varepsilon)$ is an ergodic DF

we can define new DFs : (Note: we ensure $\nu(r) = \int f_\alpha \, d^3v$)

$$\boxed{f_\alpha(\varepsilon, L) = \alpha f_i(\varepsilon) + (1-\alpha) f_c(\varepsilon, L)}$$

$\alpha = 0$: circular orbits $\sigma_\theta = \sigma_\varphi \neq 0$, $\sigma_r = 0$

eccentricity of orbits increases

$\alpha = 1$: ergodic (isothrope) $\sigma_\theta = \sigma_\varphi = \sigma_r$

$\alpha > 1$: more elongated orbits "radiale" $\sigma_\theta = \sigma_\varphi < \sigma_r$

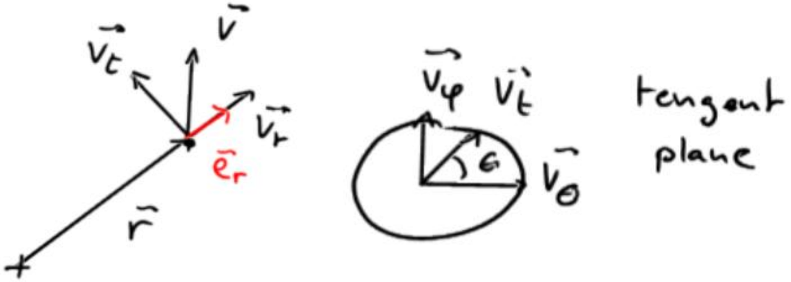
⚠ as long as $f_\alpha > 0$

If $f_i(\varepsilon) < 0$ we can then ensure $f_\alpha(\varepsilon, L) > 0$ as

1) $f_c(\varepsilon, L) > 0$

2) $(1-\alpha) > 0$ $\alpha \in [0, 1]$

i.e giving more weight to circular orbits



<u>Definition</u>: anisotropy parameter

$$\beta := 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\sigma_r^2} = 1 - \frac{\sigma_t^2}{2\sigma_r^2}$$

$\beta = -\infty$

- Circular orbits

  $\sigma_\theta = \sigma_\phi \neq 0, \sigma_r = 0$

$\beta = 0$

- Isotrope ergodic

  $\sigma_\theta = \sigma_\phi = \sigma_r = \frac{1}{\sqrt{2}}\sigma_t$

$\beta = 1$

- Radial orbits

  $\sigma_\theta = \sigma_\phi = 0, \sigma_r \neq 0$

- tangentially biased orbits
  $\sigma_\theta = \sigma_\phi > \sigma_r$

- radially biased orbits
  $\sigma_\theta = \sigma_\phi < \sigma_r$

Models with constant anisotropy

$$\boxed{f(\mathcal{E}, L) = f_1(\mathcal{E})\, L^{\gamma} = f_1(\mathcal{E})\, L^{-2\beta}} \qquad f_1(\mathcal{E}) > 0$$

Can we find an expression for $f_1(\mathcal{E})$, for a given $\phi(r)$ and $\rho(r)$?

<u>Density</u> : $\quad \nu(r) = \int d^3\vec{v}\; f_1(\mathcal{E})\, L^{-2\beta}$

integration using polar coord. in velocity space :

$$\begin{cases} v_r = v \cos\eta \\ v_\theta = v \sin\eta \cos\varphi \\ v_\varphi = v \sin\eta \sin\varphi \end{cases} \qquad \begin{aligned} L &= r\sqrt{v_\theta^2 + v_\varphi^2} = r v \sin\eta \\ d^3\vec{v} &= dv_r\, dv_\theta\, dv_\varphi\, v^2 \sin\eta \end{aligned}$$

$$\nu(r) = \int d^3\vec{v} \; f_1(\varepsilon) \, L^{-2\beta}$$

$$= 2\pi \int_0^{\pi} d\eta \, \sin\eta \int_0^{\infty} dv \, v^2 \; f_1\left(\psi(r) - \tfrac{1}{2}v^2\right) L^{-2\beta}$$

$$= \frac{2\pi}{r^{2\beta}} \underbrace{\int_0^{\pi} d\eta \, \sin^{1-2\beta}\eta}_{\sqrt{\pi}\,\frac{(-\beta)!}{(\frac{1}{2}-\beta)!} \;:=\; I_\beta} \int_0^{\infty} dv \; v^{2-2\beta} f_1\left(\psi(r) - \tfrac{1}{2}v^2\right)$$

$$(\,!\;\beta < 1\,)$$

And integrating through the energy $\varepsilon = \psi - \frac{1}{2}v^2$

$$\begin{cases} v = \sqrt{2(\psi-\varepsilon)} & dv = \dfrac{-1}{\sqrt{2(\psi-\varepsilon)}}\, d\varepsilon \\ \frac{1}{2}v^2 + \phi = \phi_0 - \varepsilon \end{cases}$$

+ $\nu(r)$ is a monotonic function of ψ

$$\boxed{\frac{2^{\beta-\frac{1}{2}}}{4\pi I_\beta} r^{2\beta}\, \nu(\Psi) = \int_0^{\Psi} d\mathcal{E} \, \frac{\rho_1(\mathcal{E})}{(\Psi-\mathcal{E})^{\beta-\frac{1}{2}}}}$$

Note: Differenciating with respect to Ψ, we can obtain an Abel equation and the equivalent of the Eddington formula.

$\boxed{\text{Solution for } \quad \beta = \frac{1}{2}}$ i.e $\sigma_\theta^2 = \sigma_\phi^2 = \frac{1}{2}\sigma_r^2$ (radially biased)

$$\frac{1}{2\pi^2} r \nu(\Psi) = \int_0^\Psi d\mathcal{E} \; f_1(\mathcal{E})$$ and differentiating by Ψ

$$\boxed{f_1(\Psi) = \frac{1}{2\pi^2} \frac{d}{d\Psi}(r\nu)}$$

$\boxed{\text{Solution for } \quad \beta = -\frac{1}{2}}$ i.e $\sigma_\theta^2 = \sigma_\phi^2 = \frac{3}{2}\sigma_r^2$ (tangentially biased)

$$\frac{1}{2\pi^2} \frac{\nu(\Psi)}{r} = \int_0^\Psi d\mathcal{E} \; f_1(\mathcal{E}) \, (\Psi - \mathcal{E})$$ and differentiating twice by Ψ

$$\boxed{f_1(\Psi) = \frac{1}{2\pi^2} \frac{d^2(\nu/r)}{d\Psi^2}}$$

Application to the Hernquist model

$$\frac{r}{a} = \frac{1}{\tilde{\psi}} - 1 \qquad \text{where } \tilde{\psi}(r) = \frac{\psi(r)}{GM}\,a$$

- $\beta = \frac{1}{2}$

$$\boxed{f_1(\mathcal{E}) = \frac{3\,\tilde{\mathcal{E}}^2}{4\pi^3 G M a}} \qquad \text{with } \tilde{\mathcal{E}} = \frac{\mathcal{E} a}{GM}$$

- $\beta = -\frac{1}{2}$

$$\boxed{f_1(\mathcal{E}) = \frac{1}{4\pi^3 (GMa)^2}\,\frac{d^2}{d\hat{\mathcal{E}}^2}\left(\frac{\hat{\mathcal{E}}^5}{(1-\hat{\mathcal{E}})^2}\right)}$$

Line of sight velocity of Hernquist models with three different anisotropies (β)

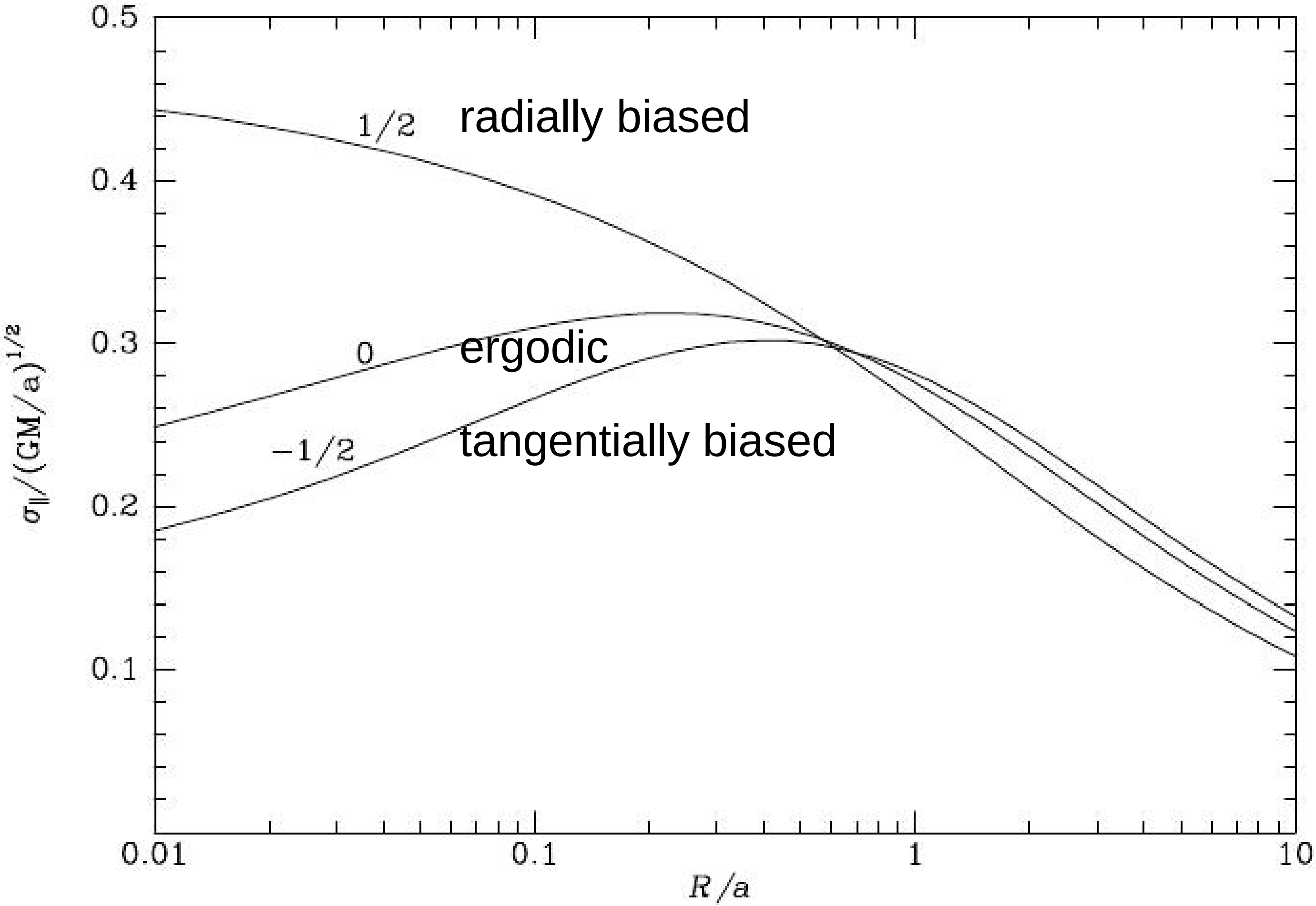
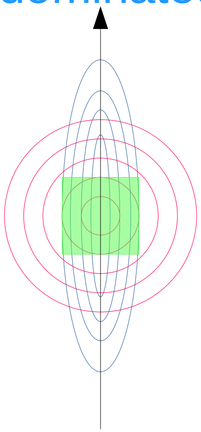
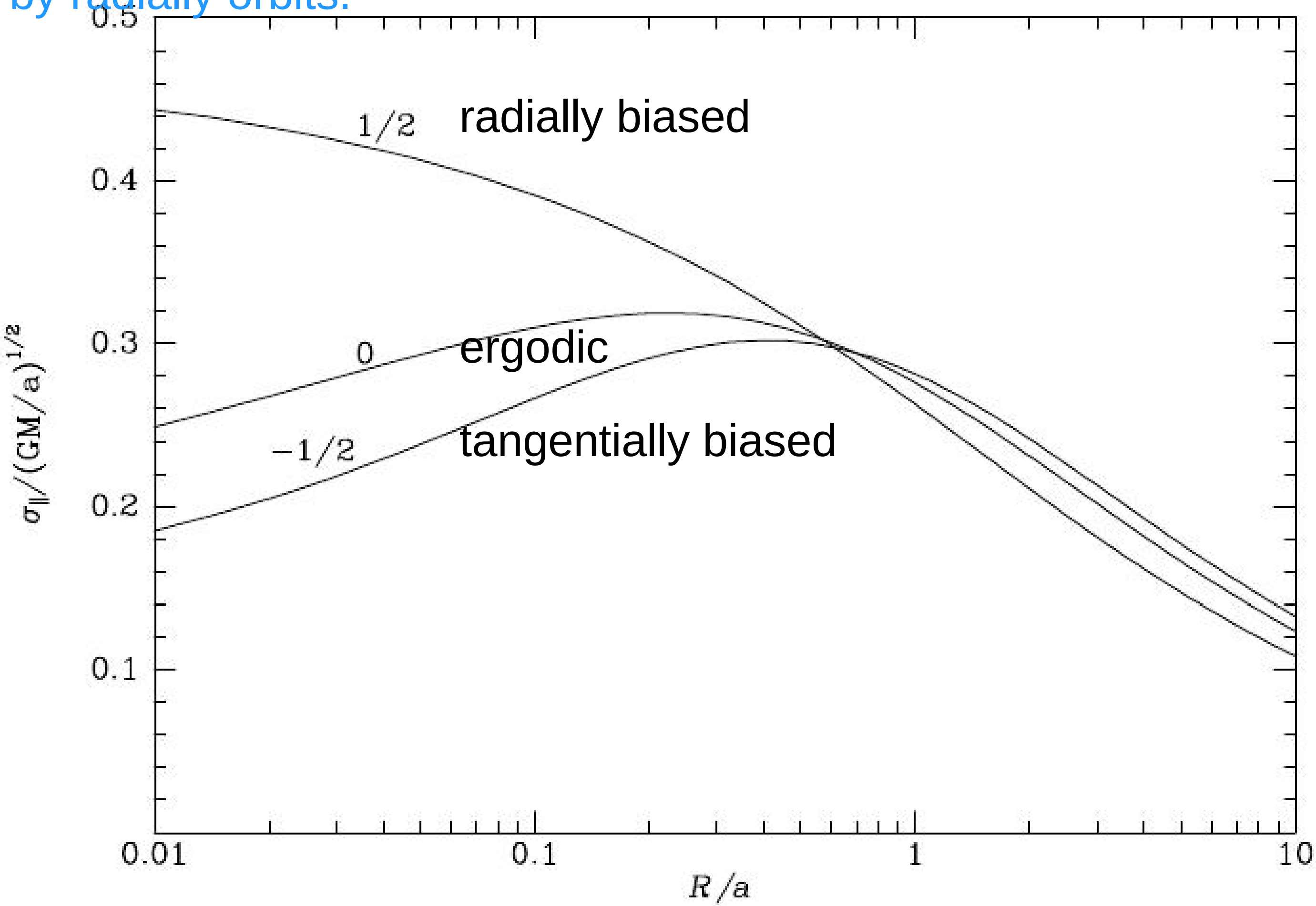


Figure 4.4 Line-of-sight velocity dispersion as a function of projected radius, from spatially identical systems that have different DFs. In each system the density and potential are those of the Hernquist model and the anisotropy parameter β of equation (4.61) is independent of radius. The curves are labeled by the relevant value of β. In the isotropic system, the velocity dispersion falls as one approaches the center (cf. Problem 4.14).

Line of sight velocity of Hernquist models with three different anisotropies (β)

At the centre, the kinematics of the the galaxy is dominated by radially orbits.





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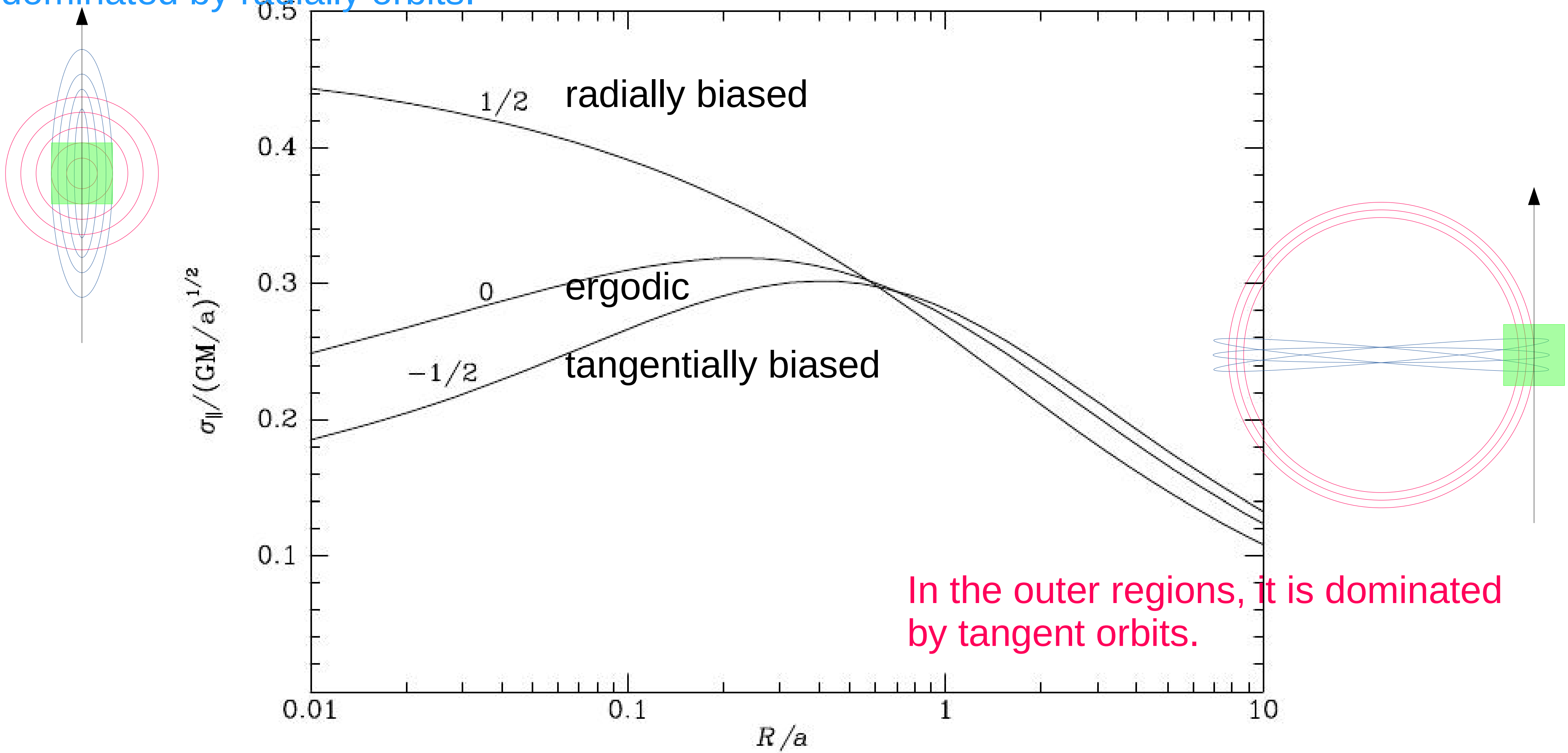

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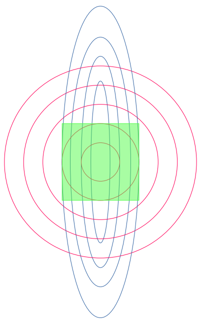
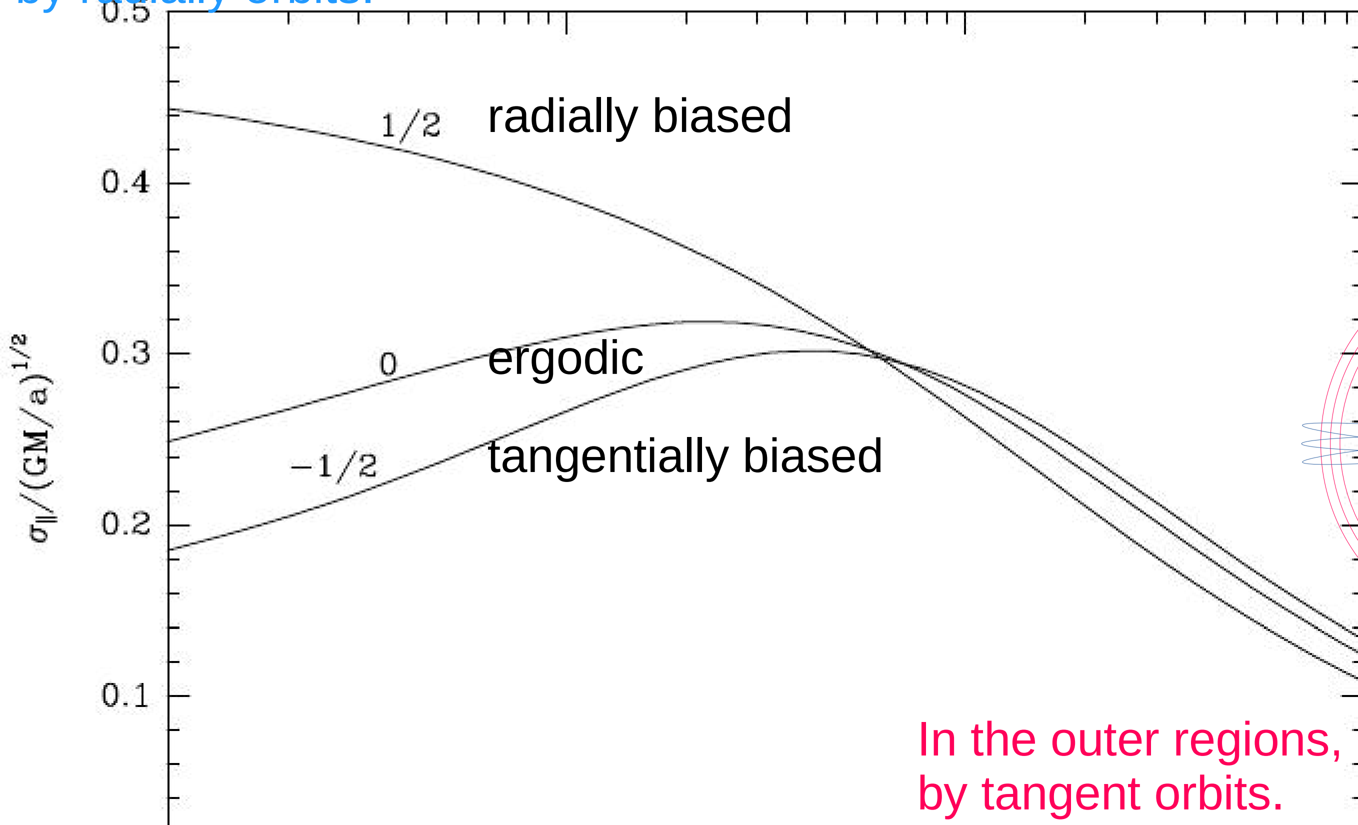
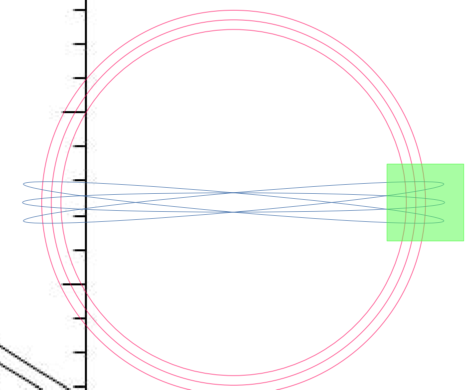






In the outer regions, it is dominated by tangent orbits.

Velocity dispersions trace the build-up of the galaxies

Figure 4.4 Li[...]om spatially identical [...]potential are those of the [...] (4.61) is independent of radius. The curves are labeled by the relevant value of β. In the isotropic system, the velocity dispersion falls as one approaches the center (cf. Problem 4.14).

Equilibria of collisionless systems

Jeans Equations

The Jeans Equations

- From observations, we usually obtain velocity moments:

  Examples: mean velocity $\bar{v}_i$

  velocity dispersions $\overline{v_i v_j} \equiv \sigma_{ij}$

- Computing moments from a DF is "easy":

$$\bar{v}_i = \frac{1}{\nu(\vec{x})} \int v_i \, f(\vec{x}, \vec{v}) \, d^3\vec{v}$$

- Obtaining a DF compatible with an observed $\nu(\vec{x})$ ($\rho(\vec{x})$) is less easy and solutions are often not unique.

Our goal

Find a method that let infer moments from stellar systems, without recovering the DF.

Idea

Compute moments of the collisionless Boltzman equation.

In carthesian coordinates

$$\frac{\partial f}{\partial t} + \vec{v}\,\frac{\partial f}{\partial \vec{x}} - \vec{\nabla}\phi\,\frac{\partial f}{\partial \vec{v}} = 0$$

$$\boxed{\frac{\partial}{\partial t} f + \sum_i v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0}$$

Zeroth moment

$$\boxed{\frac{\partial}{\partial t} f + \sum_i v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0}$$

integrate over velocities

$$\int \frac{\partial}{\partial t} f \, d^3 v + \sum_i \int d^3 v \, v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i} \int d^3 v \frac{\partial f}{\partial v_i} = 0$$

$$\frac{\partial}{\partial t} \underbrace{\int f \, d^3 v}_{\nu(\vec{x})} + \sum_i \frac{\partial}{\partial x_i} \int d^3 v \, v_i f - \sum_i \frac{\partial \phi}{\partial x_i} \oint_S d^2 s_i \underbrace{f}_{=0} = 0$$

v_i does not dep. on x_i (canonical coords)

div. theorem ⊛ + $f(x, v, t) = 0$ for $v \to \infty$ $= 0$

We get

$$\boxed{\frac{\partial}{\partial t} \nu(\vec{x}) + \sum_i \frac{\partial}{\partial x_i} \left(\nu \bar{v}_i \right) = 0}$$

continuity equation for $\nu(x)$

$$\boxed{\frac{\partial}{\partial t} \nu + \vec{\nabla} \cdot (\nu \vec{\bar{v}}) = 0}$$

$$\left(\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot (\rho \vec{v}) \right) \qquad \nu \equiv \rho, \quad \vec{\bar{v}} \equiv \vec{v}$$

⊛ div. theorem $\int d^3 x \, \vec{\nabla} \vec{F} = \int d^2 \vec{S} \cdot \vec{F}$

for $\vec{F} = f \vec{e}_j$ $\quad \int d^3 x \frac{\partial f}{\partial x_j} = \int d^2 S_j \, f$

First moment

$$\frac{\partial}{\partial t} f + \sum_i v_i \frac{\partial f}{\partial x_i} - \sum_i \frac{\partial \phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0$$

multiply by v_j and integrate over velocities

$$\frac{\partial}{\partial t} \underbrace{\int d^3v \; v_j \; f}_{\nu \, \bar{v}_j} + \underbrace{\int d^3v \sum_i v_i v_j \frac{\partial f}{\partial x_i}}_{1} - \sum_i \frac{\partial \phi}{\partial x_i} \underbrace{\int d^3v \; v_j \frac{\partial f}{\partial v_i}}_{2 \; = \; \frac{\partial \phi}{\partial x_j}\nu} = 0$$

1 $\quad \displaystyle\int d^3v \sum_i v_i v_j \frac{\partial f}{\partial x_i} = \sum_i \frac{\partial}{\partial x_i} \int d^3v \; v_i v_j f = \sum_i \frac{\partial}{\partial x_i} \left(\overline{v_i v_j} \; \nu \right)$

2 $\quad \displaystyle\int d^3v \frac{\partial}{\partial v_i}\left(v_j f \right) = \underbrace{\int d^3v \; v_i \frac{\partial f}{\partial v_i}}_{2} + \underbrace{\int d^3v \; f \underbrace{\frac{\partial v_j}{\partial v_i}}_{\delta_{ij}}}_{\delta_{ij}\,\nu}$

$\quad \displaystyle\oint d^2S_i \; v_j f = 0$

$$\boxed{\frac{\partial}{\partial t}\left(\bar{v}_j \nu \right) + \sum_i \frac{\partial}{\partial x_i} \left(\overline{v_i v_j} \; \nu \right) + \nu \frac{\partial \phi}{\partial x_j} = 0}$$

Using the continuity equation multiplied by $\bar{v}_j$

$$\bar{v}_j \left(\frac{\partial}{\partial t} \nu(\vec{x}) + \sum_i \frac{\partial}{\partial x_i} (\nu \bar{v}_i) \right) = 0$$

and substracting it from the previous result

$$\underbrace{\frac{\partial}{\partial t}(\bar{v}_j \nu) - \bar{v}_j \frac{\partial}{\partial t} \nu}_{\nu \frac{\partial}{\partial t}(\bar{v}_j)} + \underbrace{\sum_i \frac{\partial}{\partial x_i} (\overline{v_i v_j}\, \nu) - \bar{v}_j \sum_i \frac{\partial}{\partial x_i} (\nu \bar{v}_i)}_{1} + \nu \frac{\partial \phi}{\partial x_j} = 0$$

with $\sigma_{ij}^2 = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$

$$1 = \sum_i \frac{\partial}{\partial x_i} (\sigma_{ij}^2 \nu) + \underbrace{\sum_i \frac{\partial}{\partial x_i} (\bar{v}_i \bar{v}_j \nu)}_{\nu \sum_i \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j + \sum_i \bar{v}_j \frac{\partial}{\partial x_i}(\nu \bar{v}_i)} - \bar{v}_j \sum_i \frac{\partial}{\partial x_i} (\nu \bar{v}_i)$$

$$\nu \sum_i \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j + \underbrace{\sum_i \bar{v}_j \frac{\partial}{\partial x_i}(\nu \bar{v}_i) - \bar{v}_j \sum_i \frac{\partial}{\partial x_i} (\nu \bar{v}_i)}_{=0}$$

$$\nu \frac{\partial}{\partial t}(\bar{v}_j) + \nu \sum_i \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j = - \sum_i \frac{\partial}{\partial x_i}(\sigma_{ij}^2 \nu) - \nu \frac{\partial \phi}{\partial x_j}$$

Jeans 1919

Interpretation

Euler equation in hydrodynamics

Lagrangian form

$$\frac{d}{dt}\vec{v} = -\frac{\vec{\nabla} P}{\rho} - \vec{\nabla}\phi$$

Eulerian form

$$(*) \quad \frac{\partial}{\partial t}\vec{v} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{\vec{\nabla} P}{\rho} - \vec{\nabla}\phi$$

$$\rho \frac{\partial}{\partial t}\vec{v} + \rho\,\vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{\nabla} P - \rho\,\vec{\nabla}\phi$$

"j" component only

$$\boxed{\rho \frac{\partial}{\partial t} v_j + \rho \sum_i v_i \frac{\partial v_j}{\partial x_i} = -\frac{\partial P}{\partial x_j} - \rho \frac{\partial \phi}{\partial x_j}}$$

$$(*) \quad \frac{dv_i(x,y,z)}{dt} = \frac{\partial v_i}{\partial t} + \sum \frac{\partial v_i}{\partial x}\dot{x}$$

Both equations are similar

if $\boxed{\rho \equiv \nu}$ $\boxed{V_i \equiv \bar{V}_i}$ $\boxed{\frac{\partial p}{\partial x_j} \equiv \sum_j \frac{\partial}{\partial x_i} \left(\sigma_{ij}^2 \nu \right)}$

$$\begin{pmatrix} p & & \\ & p & \\ & & p \end{pmatrix} \equiv \begin{pmatrix} \sigma_{xx}^2 & \sigma_{xy}^2 & \sigma_{xz}^2 \\ \sigma_{yx}^2 & \sigma_{yy}^2 & \sigma_{yz}^2 \\ \sigma_{zx}^2 & \sigma_{zy}^2 & \sigma_{zz}^2 \end{pmatrix} \nu$$

anisotropic stress tensor

(symmetric)

diagonal in an appropriate rest frame $\begin{pmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_3^2 \end{pmatrix} \nu$

Note: it is possible to show that for an ergodic system,

$$P = \int_0^{\rho} d\rho' \, \rho' \, \frac{\partial f}{\partial \rho'}$$

leads to

$$P = \sigma^2 \nu$$

Comments

$f(\bar{x}, \bar{v})$ is unknown

2 known quantities : $\rho(\bar{x})$, $\phi(\bar{x})$

6 unknown quantities : $\bar{v}_x$ $\bar{v}_y$ $\bar{v}_z$, σ_{xx}, σ_{yy}, σ_{zz} (assuming it is diagonal)

4 equations : zeroth moment (1) + first moment (3)

The Jeans equations are not closed!

- if we multiply the CB by $v_i v_j$ → new terms $\overline{v_i v_j v_k}$
  → not a solution
- we need to do some assumptions (closure conditions)

example: σ_{ij} (3) → σ (1) ok if f is ergodic

Equilibria of collisionless systems

Jeans Equations
for spherical systems

The Jeans equations for spherical systems

Canonical momenta

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2\dot{\theta} = r v_\theta \\ p_\phi = r^2 \sin^2(\theta\dot{\phi}) = r\sin(\theta)\, v_\phi \end{cases}$$

The static Collisionless Boltzmann Equation, for spherical systems

$$\cancel{\frac{\partial f}{\partial t}} + p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2}\frac{\partial f}{\partial \theta} + \cancel{\frac{p_\phi}{r^2\sin^2(\theta)}}\frac{\partial f}{\partial \phi} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3\sin^2(\theta)}\right)\frac{\partial f}{\partial p_r} - \left(\cancel{\frac{\partial \Phi}{\partial \theta}} - \frac{p_\phi^2\cos(\theta)}{r^2\sin^3(\theta)}\right)\frac{\partial f}{\partial p_\theta} - \cancel{\frac{\partial \Phi}{\partial \phi}}\frac{\partial f}{\partial p_\phi} = 0$$

f can depend on θ as $p_\phi = r\sin(\theta)\, v_\phi$

Zeroth order moment of the Jeans Equation

$$\frac{\partial}{\partial r}\left(\sin(\theta)\nu\overline{v_r}\right) = \frac{\partial}{\partial \theta}\left(\sin(\theta)\nu\overline{v_\theta}\right)$$

Canonical momenta

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2\dot{\theta} = r v_\theta \\ p_\phi = r^2 \sin^2(\theta\dot{\phi}) = r\sin(\theta)\, v_\phi \end{cases}$$

The static Collisionless Boltzmann Equation, for spherical systems

$$\cancel{\frac{\partial f}{\partial t}} + p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2}\frac{\partial f}{\partial \theta} + \cancel{\frac{p_\phi}{r^2\sin^2(\theta)}}\frac{\partial f}{\partial \phi} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3\sin^2(\theta)}\right)\frac{\partial f}{\partial p_r} - \left(\cancel{\frac{\partial \Phi}{\partial \theta}} - \frac{p_\phi^2\cos(\theta)}{r^2\sin^3(\theta)}\right)\frac{\partial f}{\partial p_\theta} - \cancel{\frac{\partial \Phi}{\partial \phi}}\frac{\partial f}{\partial p_\phi} = 0$$

f can depend on θ as $p_\phi = r\sin(\theta)\, v_\phi$

Zeroth order moment of the Jeans Equation

$$\frac{\partial}{\partial r}\left(\sin(\theta)\nu\overline{v_r}\right) = \frac{\partial}{\partial \theta}\left(\sin(\theta)\nu\overline{v_\theta}\right)$$

if $f = f(H)$ or $f(H, L) \Rightarrow \overline{v_r} = \overline{v_z} = \overline{v_\theta} = 0$

$$\overline{v_r^2} = \sigma_r^2 \;\; \overline{v_\theta^2} = \sigma_\theta^2 \;\; \overline{v_\phi^2} = \sigma_\phi^2$$

The Jeans equations for spherical systems

Canonical momenta

$$\begin{cases} p_r = \dot{r} = v_r \\ p_\theta = r^2\dot{\theta} = r v_\theta \\ p_\phi = r^2 \sin^2(\theta\dot{\phi}) = r\sin(\theta)\, v_\phi \end{cases}$$

The static Collisionless Boltzmann Equation, for spherical systems

$$\cancel{\frac{\partial f}{\partial t}} + p_r \frac{\partial f}{\partial r} + \frac{p_\theta}{r^2}\frac{\partial f}{\partial \theta} + \cancel{\frac{p_\phi}{r^2\sin^2(\theta)}\frac{\partial f}{\partial \phi}} - \left(\frac{\partial \Phi}{\partial r} - \frac{p_\theta^2}{r^3} - \frac{p_\phi^2}{r^3\sin^2(\theta)}\right)\frac{\partial f}{\partial p_r} - \left(\cancel{\frac{\partial \Phi}{\partial \theta}} - \frac{p_\phi^2\cos(\theta)}{r^2\sin^3(\theta)}\right)\frac{\partial f}{\partial p_\theta} - \cancel{\frac{\partial \Phi}{\partial \phi}\frac{\partial f}{\partial p_\phi}} = 0$$

f can depend on θ as $p_\phi = r\sin(\theta)\, v_\phi$

Zeroth order moment of the Jeans Equation

$$0 = 0$$

if $f = f(H)$ or $f(H, L) \Rightarrow \overline{v_r} = \overline{v_z} = \overline{v_\theta} = 0$

$$\overline{v_r^2} = \sigma_r^2 \quad \overline{v_\theta^2} = \sigma_\theta^2 \quad \overline{v_\phi^2} = \sigma_\phi^2$$

First order moment of the Jeans Equation

$$\frac{\partial}{\partial r}\left(\nu\overline{v_r^2}\right) + \nu\left(\frac{\partial \Phi}{\partial r} + \frac{2\overline{v_r^2} - \overline{v_\theta^2} - \overline{v_\phi^2}}{r}\right) = 0$$

or

$$\boxed{\frac{\partial}{\partial r}\left(\nu\overline{v_r^2}\right) + 2\frac{\beta}{r}\nu\overline{v_r^2} = -\nu\frac{\partial \Phi}{\partial r}}$$

where

$$\beta = 1 - \frac{\overline{v_\theta^2} + \overline{v_\phi^2}}{2\overline{v_r^2}} = 1 - \frac{\overline{v_t^2}}{2\overline{v_r^2}}$$

Discussion

$$\boxed{\frac{\partial}{\partial r}\left(\nu\,\sigma_r^2\right) + \nu\left(\frac{\partial\phi}{\partial r} + \frac{2\sigma_r^2 - \sigma_\theta^2 - \sigma_\varphi^2}{r}\right) = 0}$$

Case $\boxed{\sigma_r = \sigma_\varphi = \sigma_\theta}$

Ergodic

$$\Rightarrow \frac{1}{\nu}\frac{\partial}{\partial r}\left(\nu\,\sigma_r^2\right) = -\frac{\partial\phi}{\partial r}$$

$$\equiv \frac{\vec{\nabla} p}{\rho} = \vec{F}_{grav}$$

Note: for $\sigma = cte$, we should recover the isothermal sphere

Discussion

$$\boxed{\frac{\partial}{\partial r}\left(\nu\, \sigma_r^2\right) + \nu\left(\frac{\partial \phi}{\partial r} + \frac{2\sigma_r^2 - \sigma_\theta^2 - \sigma_\varphi^2}{r}\right) = 0}$$

Case $\boxed{\sigma_r = 0}$ $\Rightarrow$ $\underline{\sigma_t^2 = r\,\frac{\partial \phi}{\partial r}}$

interpretation only circular orbits

$$V_t^2 = r\,\frac{\partial \phi}{\partial r}$$

but from all possible planes

Demonstration

associated dispersion : in the tangential plane

$$V_\varphi = V_t \cos\eta \qquad \sigma_\varphi^2 = \frac{1}{2\pi}\int V_t^2 \cos^2\eta \; d\eta = \frac{1}{2} V_t^2$$

$$V_\theta = V_t \sin\eta \qquad \sigma_\theta^2 = \qquad\qquad = \frac{1}{2} V_t^2$$

thus $\sigma_t^2 := \sigma_\varphi^2 + \sigma_\theta^2 = V_t^2$

#

Discussion

$$\boxed{\frac{\partial}{\partial r}\left(\nu\,\sigma_r^2\right) + \nu\left(\frac{\partial\phi}{\partial r} + \frac{2\sigma_r^2 - \sigma_\theta^2 - \sigma_\varphi^2}{r}\right) = 0}$$

Case $\boxed{\sigma_t = 0}$ $\Rightarrow$ $\frac{1}{\nu}\frac{\partial}{\partial r}\left(\nu\,\sigma_r^2\right) + \frac{2\sigma_r^2}{r} = -\frac{\partial\phi}{\partial r}$

purely radial orbits

The Jeans equations for spherical systems

$$\frac{\partial}{\partial r}\left(\nu\sigma_r^2\right) + 2\frac{\beta}{r}\nu\sigma_r^2 = -\nu\frac{\partial\Phi}{\partial r}$$

$$r^{-2\beta}\frac{\partial}{\partial r}\left(\nu\sigma_r^2 r^{2\beta}\right) = -\nu\frac{\partial\Phi}{\partial r}$$

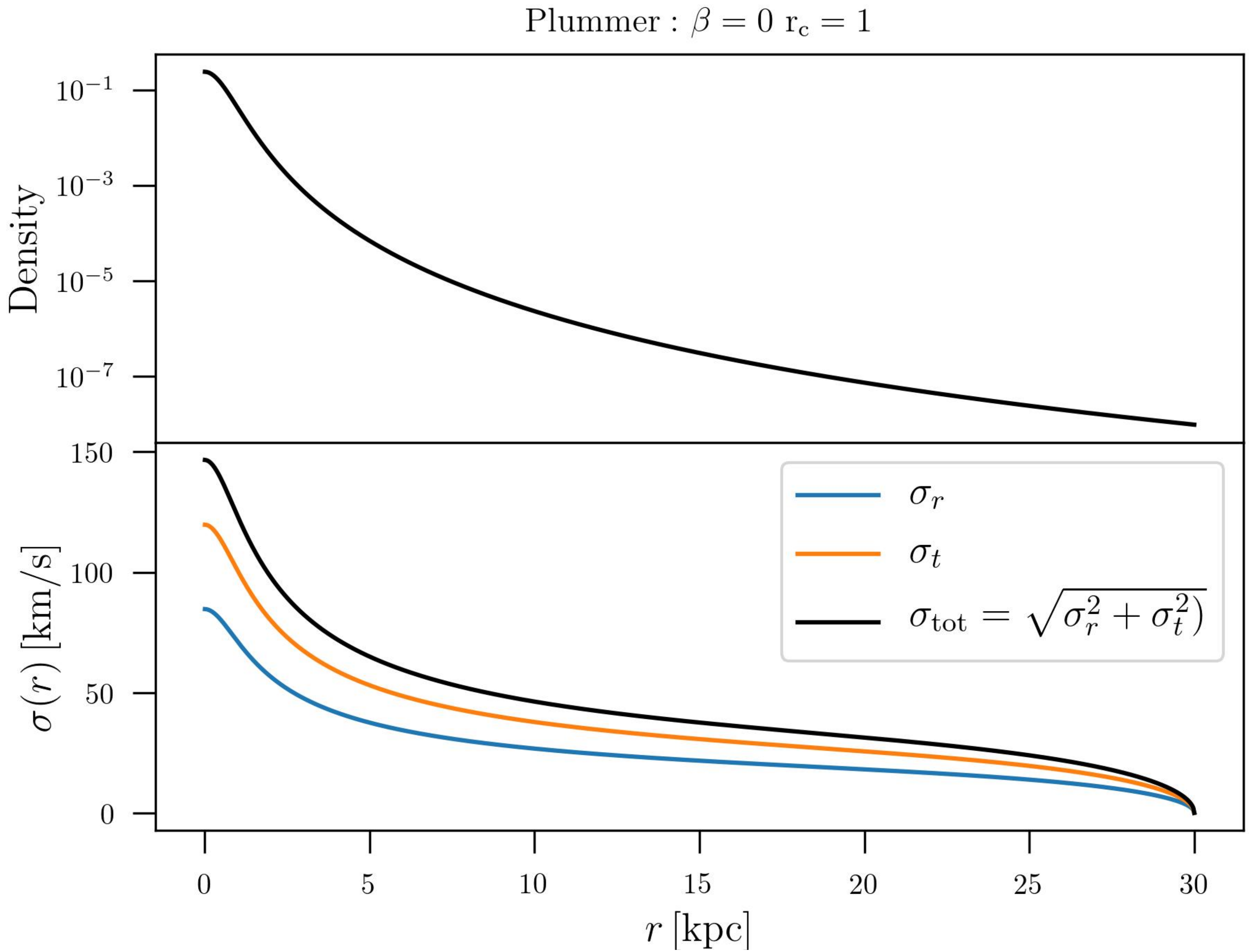
If the system has a constant anisotrpy parameter $\beta = cte$

$$\sigma_r^2(r) = \frac{1}{r^{2\beta}\nu(r)}\int_r^\infty \mathrm{d}r' r'^{2\beta}\nu(r')\frac{\partial\Phi}{\partial r'} = \frac{G}{r^{2\beta}\nu(r)}\int_r^\infty \mathrm{d}r' r'^{2\beta-2}\nu(r')M(r')$$

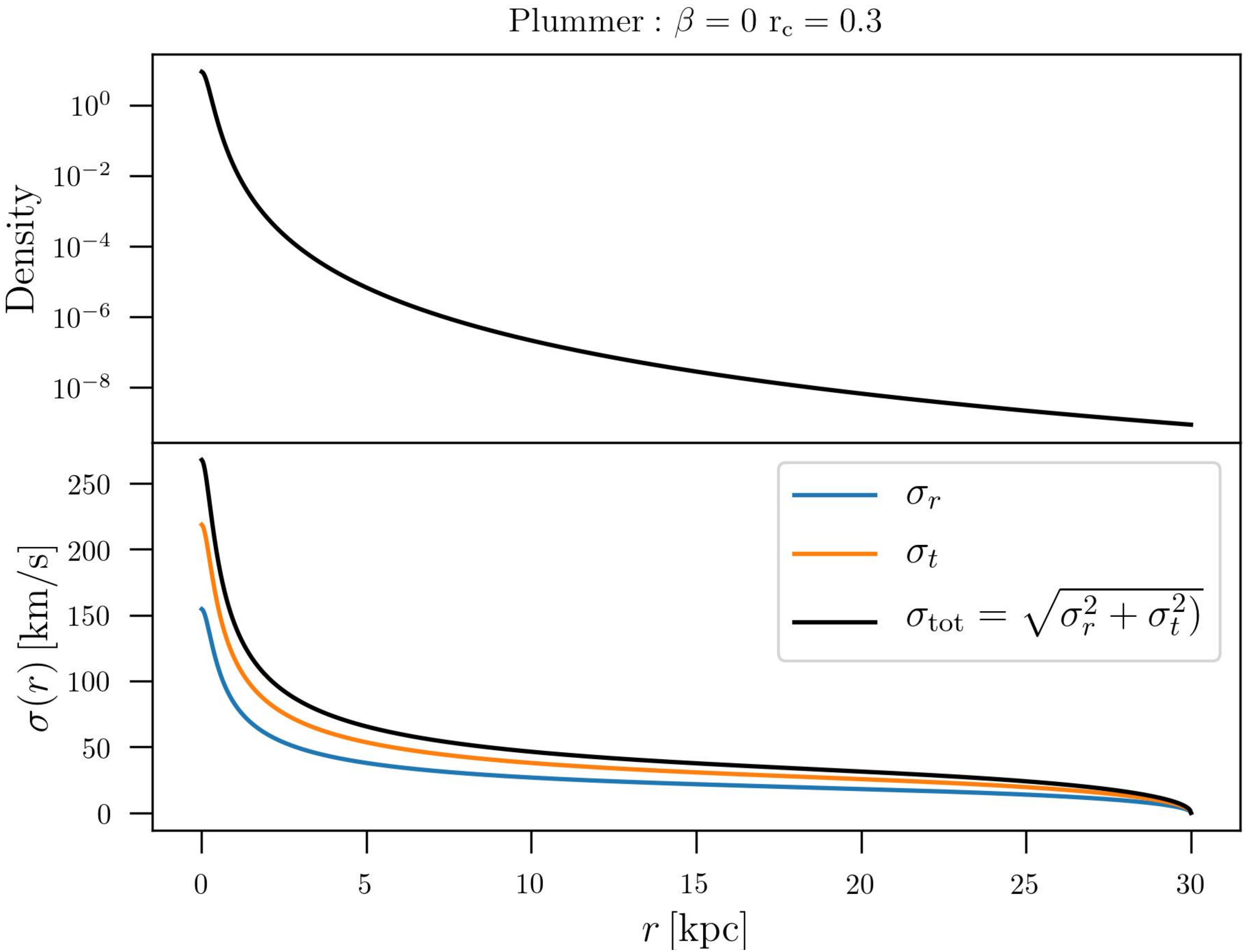
If the system is ergodic (isotropic in velocities) $\beta = 0$

$$\sigma_r^2(r) = \frac{1}{\nu(r)}\int_r^\infty \mathrm{d}r'\nu(r')\frac{\partial\Phi}{\partial r'} = \frac{G}{\nu(r)}\int_r^\infty \mathrm{d}r'\frac{1}{r'^2}\nu(r')M(r')$$

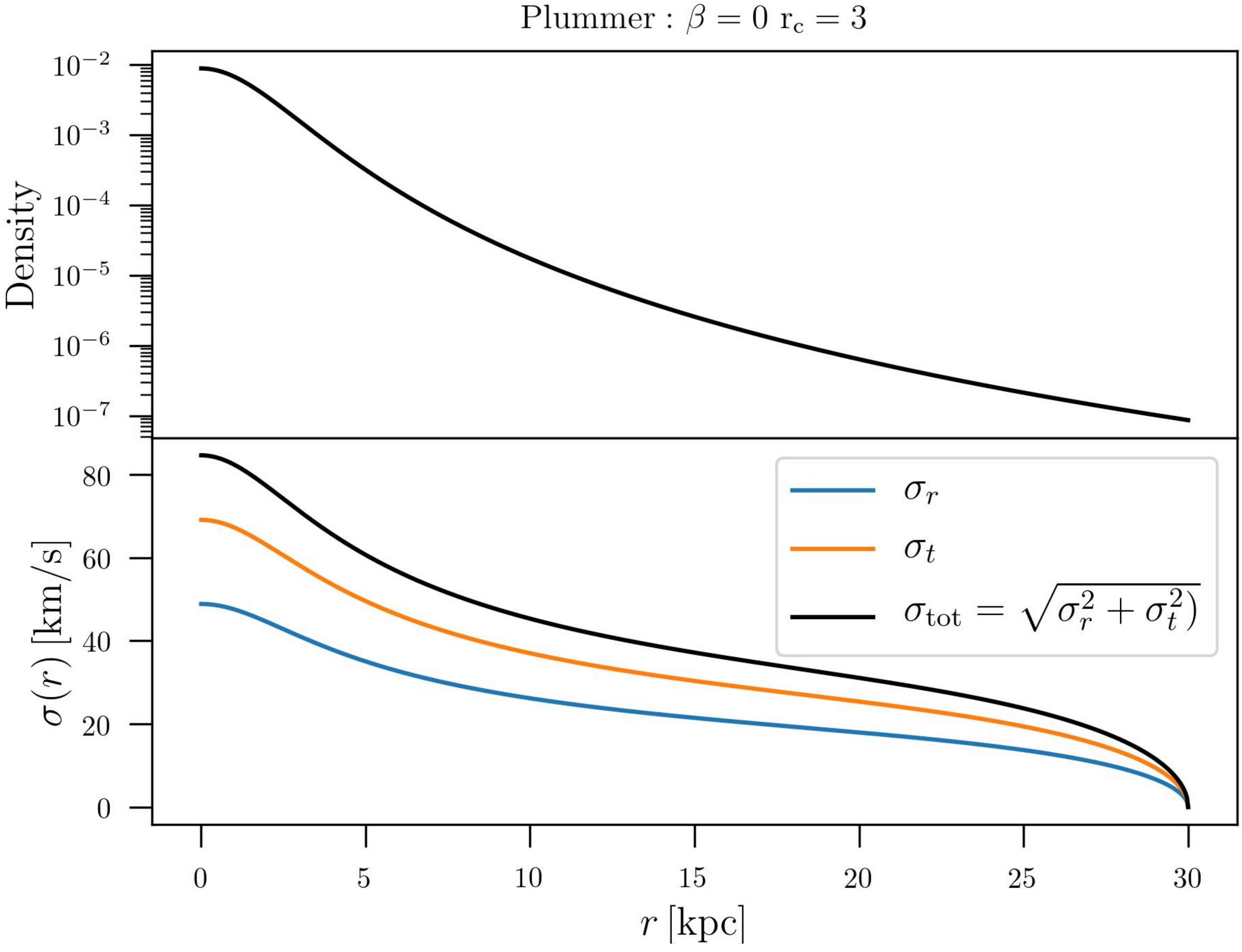
Play with the core radius R_c



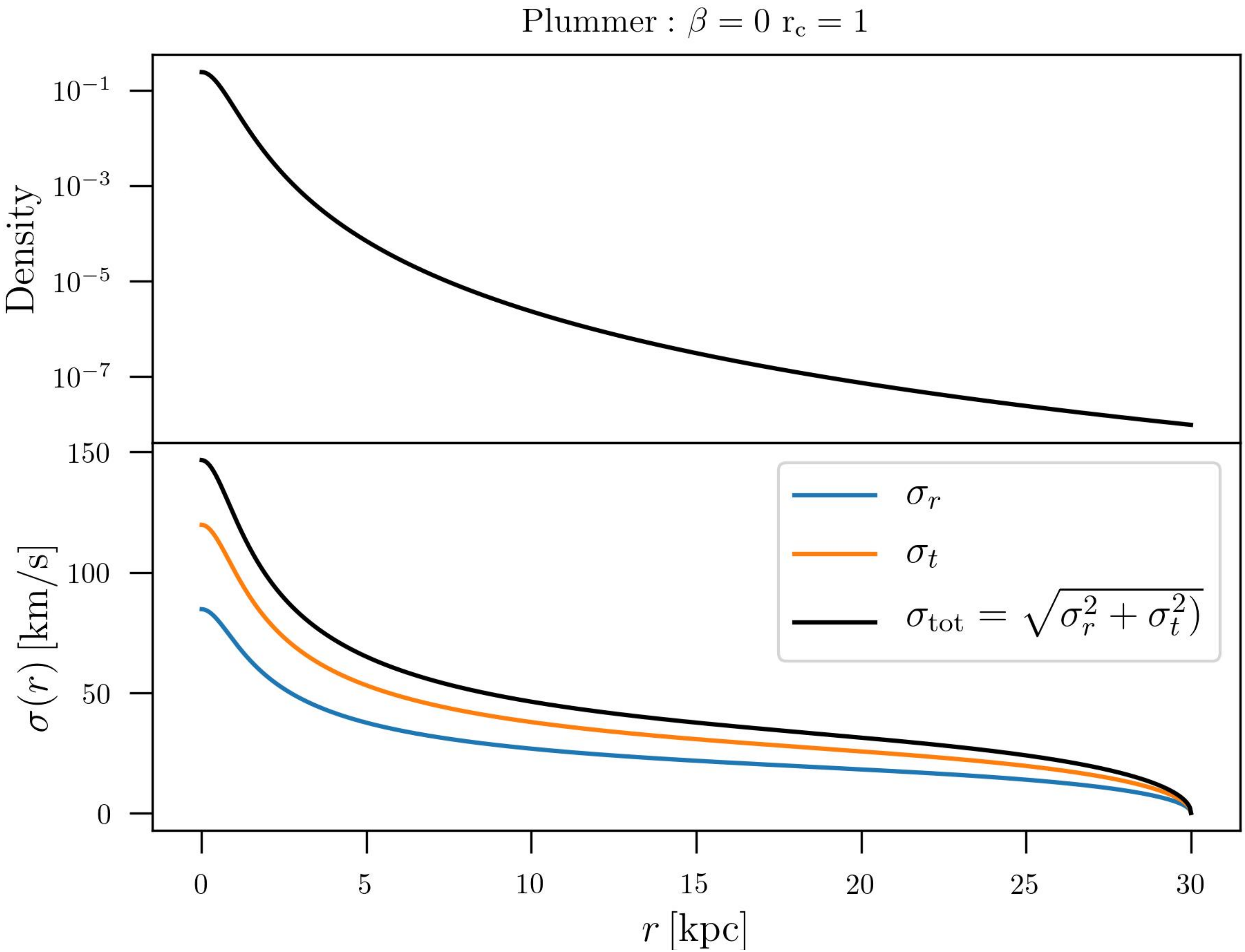
Play with the core radius R_c



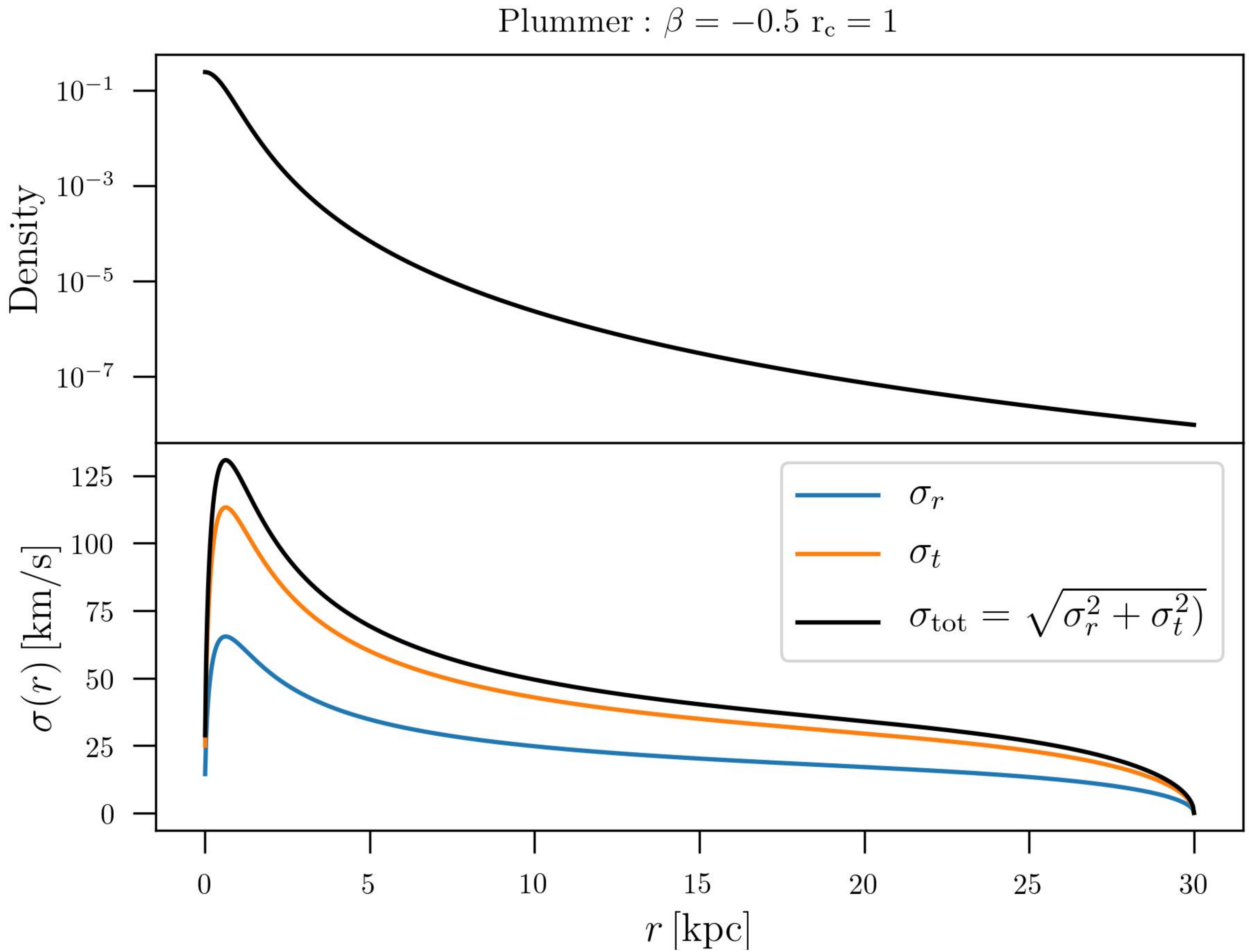
Play with the core radius R_c



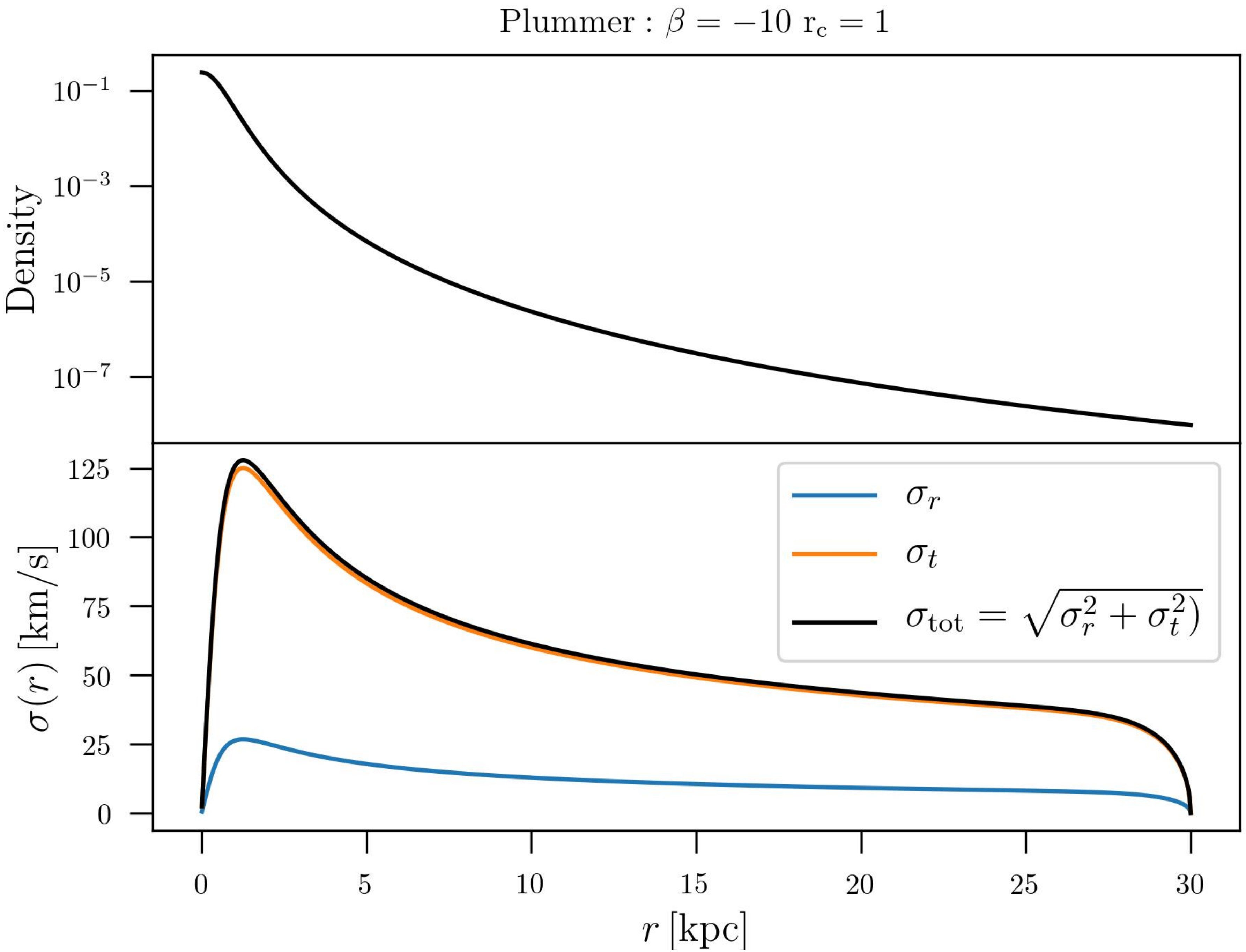
Play with the core radius R_c



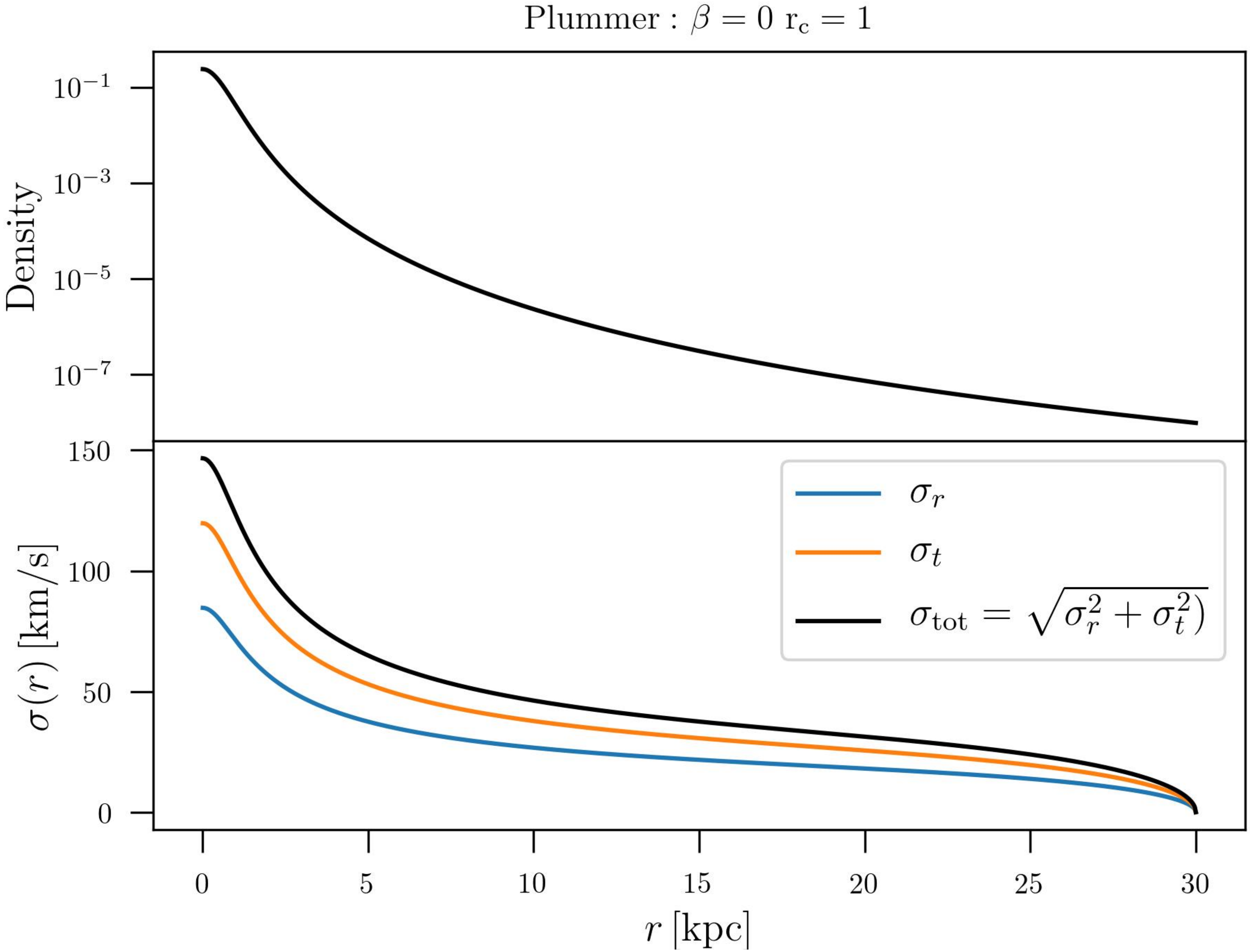
Play with the anisotropy parameter



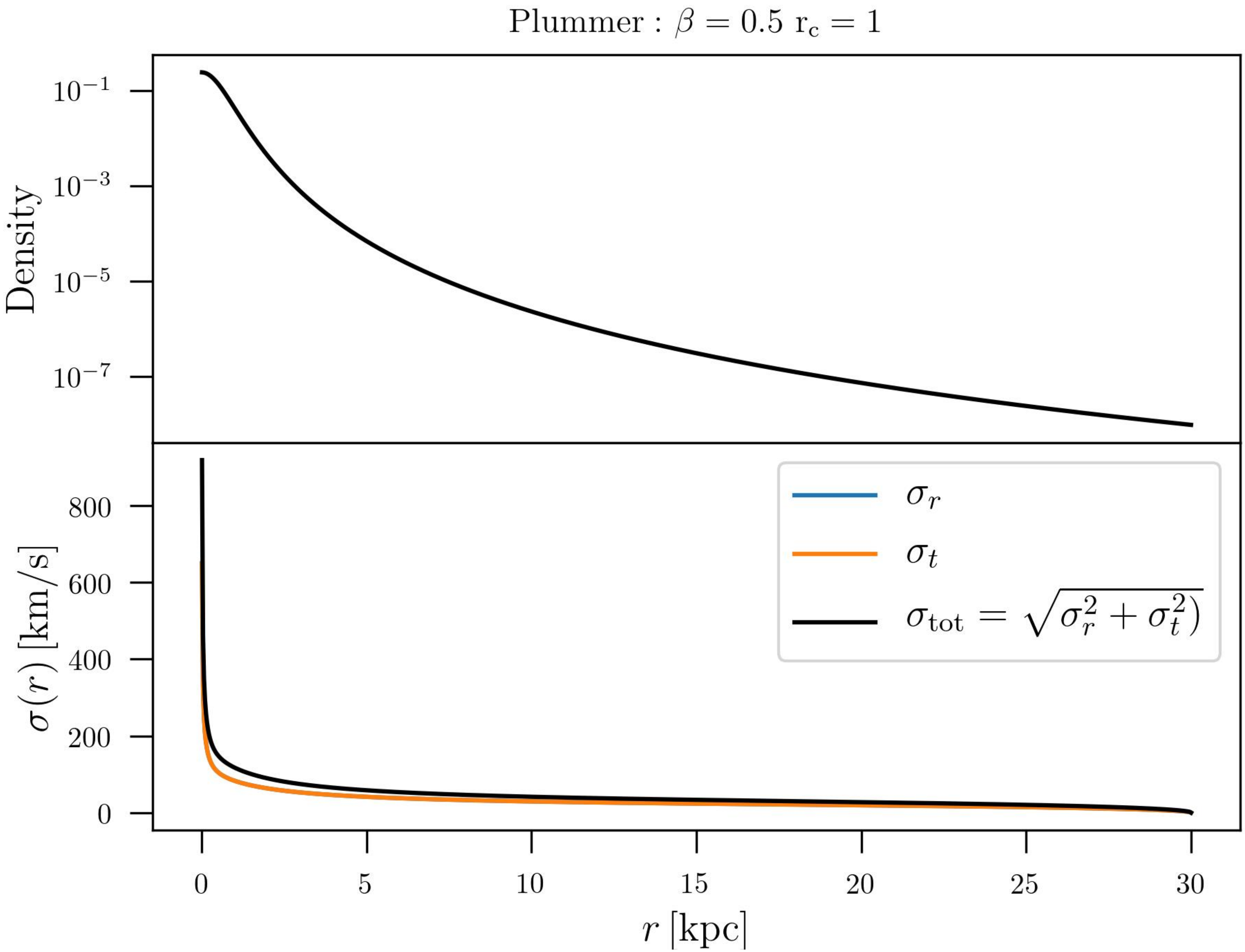
Play with the anisotropy parameter



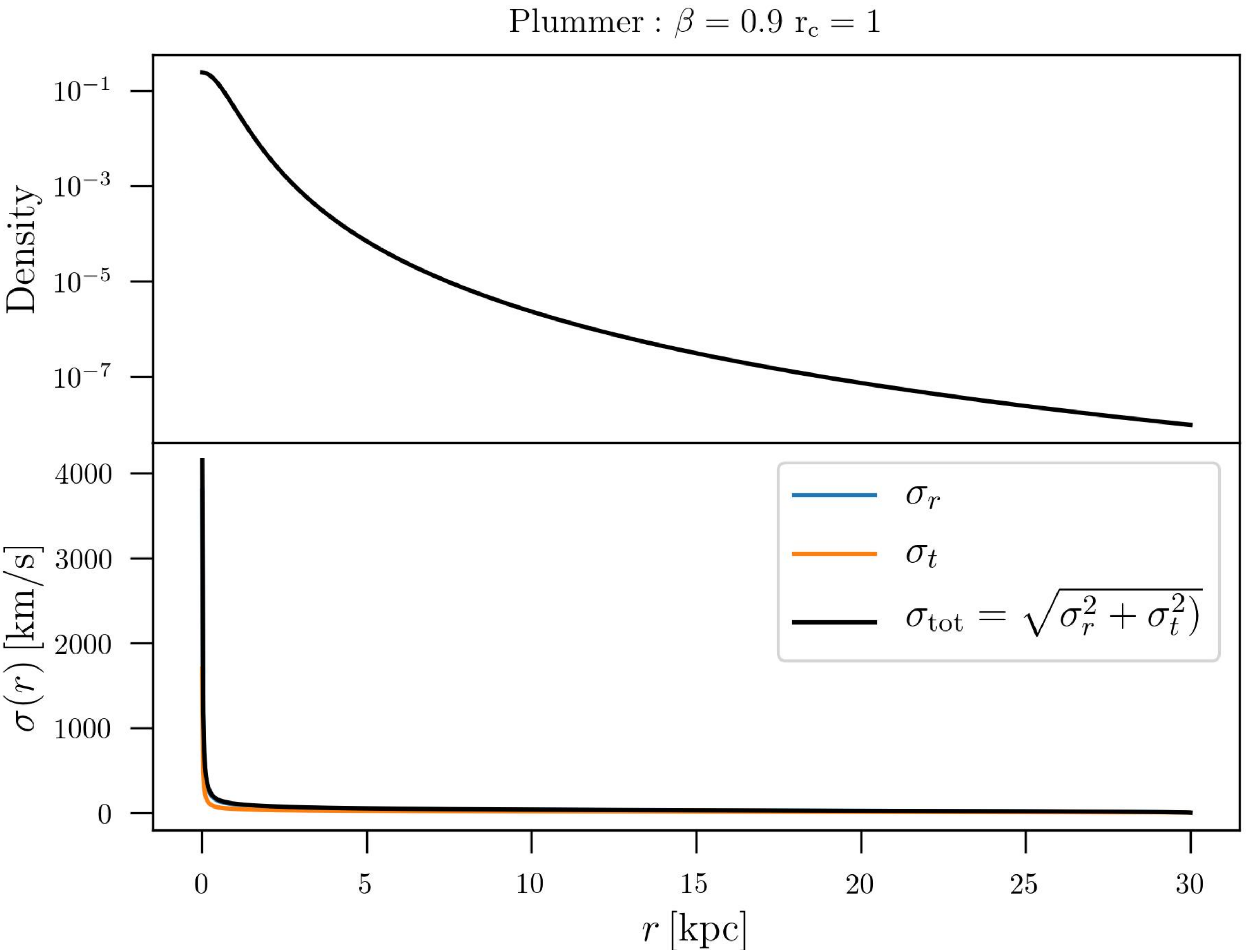
Play with the anisotropy parameter



Play with the anisotropy parameter

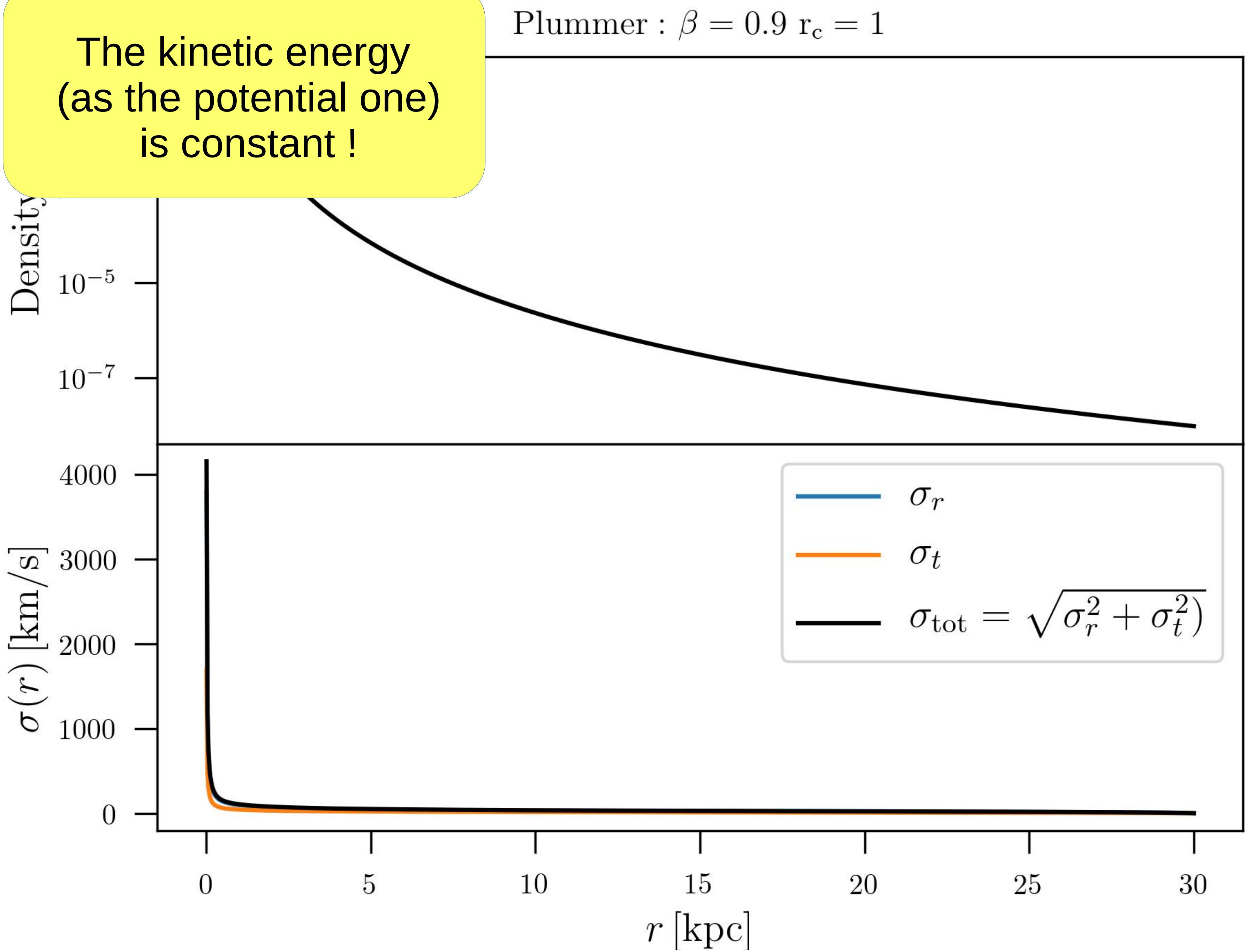


Play with the anisotropy parameter



Play with the anisotropy parameter

The kinetic energy
(as the potential one)
is constant !



Equilibria of collisionless systems

Jeans Equations
for cylindrical systems

The Jeans equations for axisymmetric systems

Canonical momenta

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2\dot{\phi} = Rv_\phi \\ p_z = \dot{z} = v_z \end{cases}$$

The static Collisionless Boltzmann Equation, for axisymmetric systems

$$\cancel{\frac{\partial f}{\partial t}} + p_R\frac{\partial f}{\partial R} + \frac{p_\phi}{R^2}\cancel{\frac{\partial f}{\partial \phi}} + p_z\frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3}\right)\frac{\partial f}{\partial p_R} - \cancel{\frac{\partial \Phi}{\partial \phi}}\frac{\partial f}{\partial p_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial p_z} = 0$$

Zeroth order moment of the Jeans Equations if $f = f(H, L_z) \Rightarrow \overline{v_R^2} = \overline{v_z^2}, \overline{v_R} = \overline{v_z} = 0$

$\overline{v_r^2} = \sigma_r^2 \; \overline{v_z^2} = \sigma_z^2$

$$0 = 0$$

$$0 = 0$$

$$0 = 0$$

Canonical momenta

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2\dot{\phi} = R v_\phi \\ p_z = \dot{z} = v_z \end{cases}$$

The static Collisionless Boltzmann Equation, for axisymmetric systems

$$\cancel{\frac{\partial f}{\partial t}} + p_R \frac{\partial f}{\partial R} + \cancel{\frac{p_\phi}{R^2}\frac{\partial f}{\partial \phi}} + p_z \frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3}\right)\frac{\partial f}{\partial p_R} - \cancel{\frac{\partial \Phi}{\partial \phi}\frac{\partial f}{\partial p_\phi}} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial p_z} = 0$$

First order moment of the Jeans Equations

$$\frac{\partial}{\partial R}\left(\nu \overline{v_R^2}\right) + \frac{\partial}{\partial z}\left(\nu \overline{v_R v_z}\right) + \nu\left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R}\right) = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\nu \overline{v_R v_z}\right) + \frac{\partial}{\partial z}\left(\nu \overline{v_z^2}\right) + \nu \frac{\partial \Phi}{\partial z} = 0$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \nu \overline{v_R v_\phi}\right) + \frac{\partial}{\partial z}\left(\nu \overline{v_z v_\phi}\right) = 0$$

The Jeans equations for axisymmetric systems

Canonical momenta

$$\begin{cases} p_R = \dot{R} = v_R \\ p_\phi = R^2\dot{\phi} = R v_\phi \\ p_z = \dot{z} = v_z \end{cases}$$

The static Collisionless Boltzmann Equation, for axisymmetric systems

$$\cancel{\frac{\partial f}{\partial t}} + p_R \frac{\partial f}{\partial R} + \cancel{\frac{p_\phi}{R^2}\frac{\partial f}{\partial \phi}} + p_z \frac{\partial f}{\partial z} - \left(\frac{\partial \Phi}{\partial R} - \frac{p_\phi^2}{R^3} \right) \frac{\partial f}{\partial p_R} - \cancel{\frac{\partial \Phi}{\partial \phi}}\frac{\partial f}{\partial p_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial p_z} = 0$$

First order moment of the Jeans Equations

if $f = f(H, L_z) \Rightarrow \overline{v_R^2} = \overline{v_z^2}, \overline{v_R} = \overline{v_z} = 0$

$\overline{v_r^2} = \sigma_r^2 \; \overline{v_z^2} = \sigma_z^2$

$$\frac{\partial}{\partial R}\left(\nu \overline{v_R^2}\right) + \nu \left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R} \right) = 0$$

$$\frac{\partial}{\partial z}\left(\nu \overline{v_z^2}\right) + \nu \frac{\partial \Phi}{\partial z} = 0 \qquad \Rightarrow \qquad \boxed{\overline{v_R^2}(R,z) = \overline{v_z^2}(R,z) = \frac{1}{\nu(R,z)} \int_z^\infty \mathrm{d}z' \nu(R,z') \frac{\partial \Phi}{\partial z'}}$$

$$0 = 0 \qquad \Rightarrow \qquad \boxed{\overline{v_\phi^2}(R,z) = \overline{v_R^2} + \frac{R}{\nu(R,z)} \frac{\partial}{\partial R}\left(\nu \overline{v_R^2}\right) + R \frac{\partial \Phi}{\partial R}}$$

Jeans equations for axisymmetric systems

$$\boxed{\overline{v_z^2} = \frac{1}{\nu}\int_z^{\infty} dz' \, \nu(R,z') \, \frac{\partial \phi}{\partial z'}}$$

Note $\quad \overline{v_z^2} = \sigma_z^2 = \overline{v_R^2} = \sigma_R^2 \quad$ as $\quad f = f(H, L_z)$

$$\boxed{\overline{v_\phi^2}(R,z) = \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right) + R\,\frac{\partial \phi}{\partial R}}$$

Interpretation

$$\boxed{\overline{v_\phi^2}(R,z) = \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right) + R\frac{\partial\phi}{\partial R}}$$

In the plane $z=0$

- $R\dfrac{\partial\phi}{\partial R} = V_c^2$
- $\overline{V_\phi^2} = \sigma_\phi^2 + \bar{V}_\phi^2$

$$\boxed{\bar{V}_\phi^2 = V_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)}$$

1 Equation, 2 Unknowns $\bar{V}_\phi$ σ_ϕ

⚠ This equation involves different energies ⚠

Interpretation

$$\boxed{\overline{V_\phi}^2 = V_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right)}$$

1. if $\sigma_\phi = \sigma_R = 0$ $\quad$ ($\Rightarrow \sigma_z = 0$) as $\sigma_R = \sigma_z$ $\quad$! ok if $\nu \sim \delta(z)$ = razor thin disk

$$\boxed{\overline{V_\phi}^2 = V_c^2}$$

The mean azimuthal velocity is the circular velocity

The disk is "super cold"

$\sigma_R = \sigma_z = \sigma_\phi = 0$

2. if $\sigma_R = 0,\ \sigma_\phi \neq 0$ $\quad$ ($\Rightarrow \sigma_z = 0$) $\quad$! ok if $\nu \sim \delta(z)$ = razor thin disk

$$\boxed{\overline{V_\phi}^2 = V_c^2 - \sigma_\phi^2}$$

But $\sigma_r = 0 \Rightarrow$ only circular orbits

1. $\overline{V_\phi}^2 = V_c^2 \Rightarrow \sigma_\phi = 0$ ⚠

2. $\overline{V_\phi}^2 = 0 \Rightarrow$ counter rotating disk with

$\overline{V_\phi} = \frac{1}{2}\left(V_c - V_c\right) = 0$

$\sigma_\phi^2 = \frac{1}{2}\left(V_c^2 + V_c^2\right) = V_c^2$

Interpretation

$$\boxed{\overline{V}_\phi^{\,2} = V_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right)}$$

3. if $\sigma_R = \sigma_\phi \neq 0$ ("Ergodic")

$$\overline{V}_\phi^{\,2} = R\frac{\partial \phi}{\partial R} + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right)$$

$$\frac{1}{R}\overline{V}_\phi^{\,2} = \frac{\partial \phi}{\partial R} + \frac{1}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right)$$

$$\boxed{\underbrace{\frac{1}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right)}_{\sim \frac{\tilde{\nabla} P}{\rho}\ \text{"pressure" force}} = \underbrace{-\frac{\partial \phi}{\partial R}}_{\vec{F}_{grav}\ \text{gravit. force}} + \underbrace{\frac{\overline{V}_\phi^{\,2}}{R}}_{\text{centrifugal force}}}$$

Equilibrium in the rotating frame $\Omega = \frac{\bar{V}_\phi}{R}$

$F_c = \Omega^2 R$ $\quad \Omega = \frac{V}{R}$

$= \frac{V^2}{R}$

Interpretation

$$\boxed{\overline{V_\phi}^2 = V_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)}$$

4. if $\sigma_\phi = 0,\ \sigma_r \neq 0$ (radial orbits)

$$0 = V_c^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)$$

$$\frac{1}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right) + \frac{\sigma_R^2}{R} = -\frac{\partial\phi}{\partial R}$$

Nearly identical to the spherical case.

$$\frac{1}{\nu}\frac{\partial}{\partial r}\left(\nu\sigma_r^2\right) + \frac{2\sigma_r^2}{r} = \frac{\partial\phi}{\partial r}$$

How to close the equation, i.e., choose σ_ϕ

- Epicycle frequencies and energy equipartition

$$\begin{cases} \ddot{x} = -\kappa^2 x \\ \ddot{y} = -\kappa^2 y \end{cases}$$

oscillations around the guiding center (circular motion)

$$\begin{cases} \dot{x}(t) = -X\kappa \sin(\kappa t + \alpha) \\ \dot{y}(t) = -Y\kappa \cos(\kappa t + \alpha) \end{cases}$$

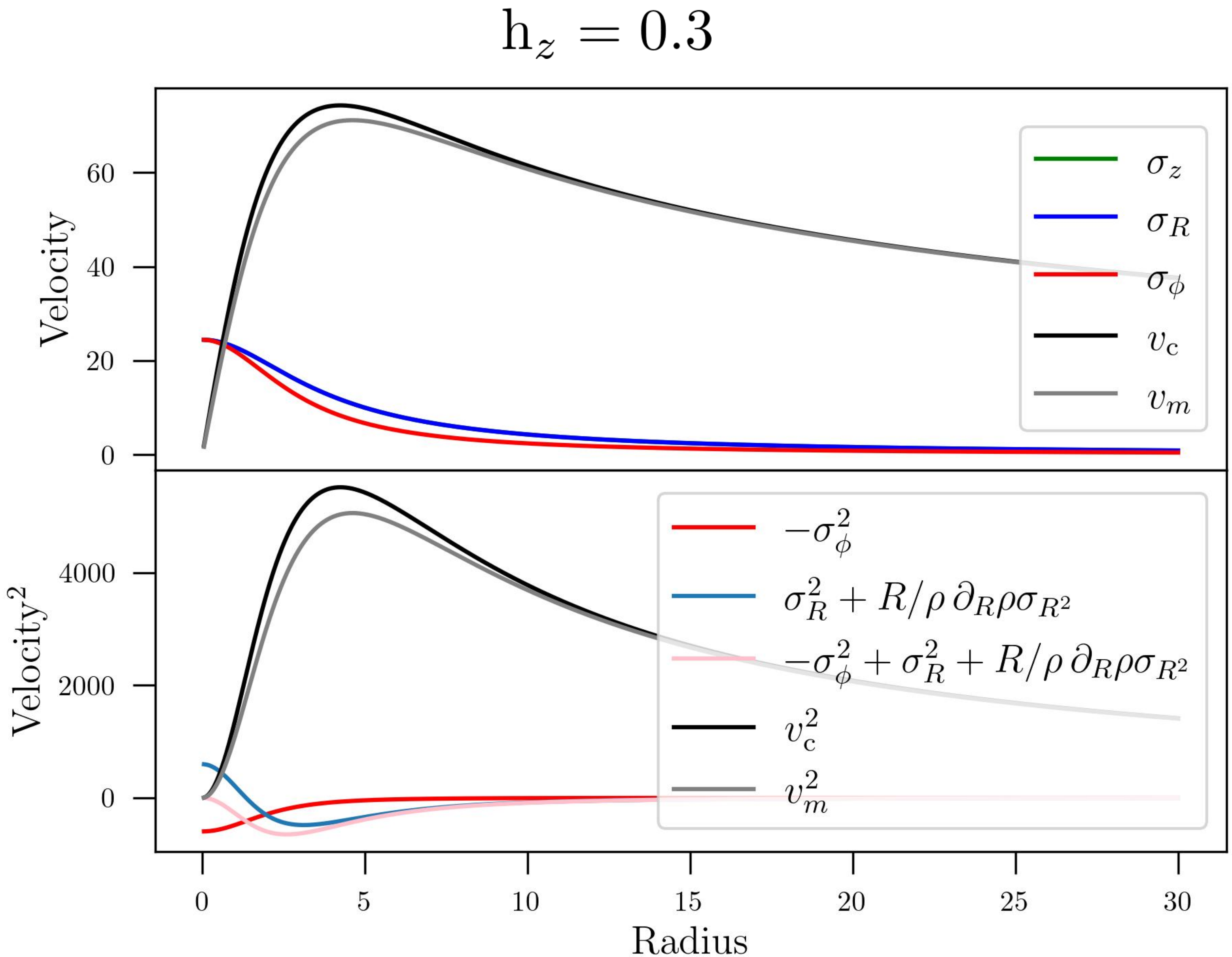
- Velocity disp. generated from those oscillations

$$\begin{cases} \sigma_\phi^2 = \overline{(v_\phi - v_c)^2} = \dfrac{\kappa^4}{8\Omega^2} X^2 \\ \sigma_R^2 = \overline{v_R^2} = \dfrac{\kappa^2}{2} X^2 \end{cases}$$

$$\boxed{\frac{\sigma_\phi^2}{\sigma_R^2} = \frac{\kappa^2}{4\Omega^2}}$$

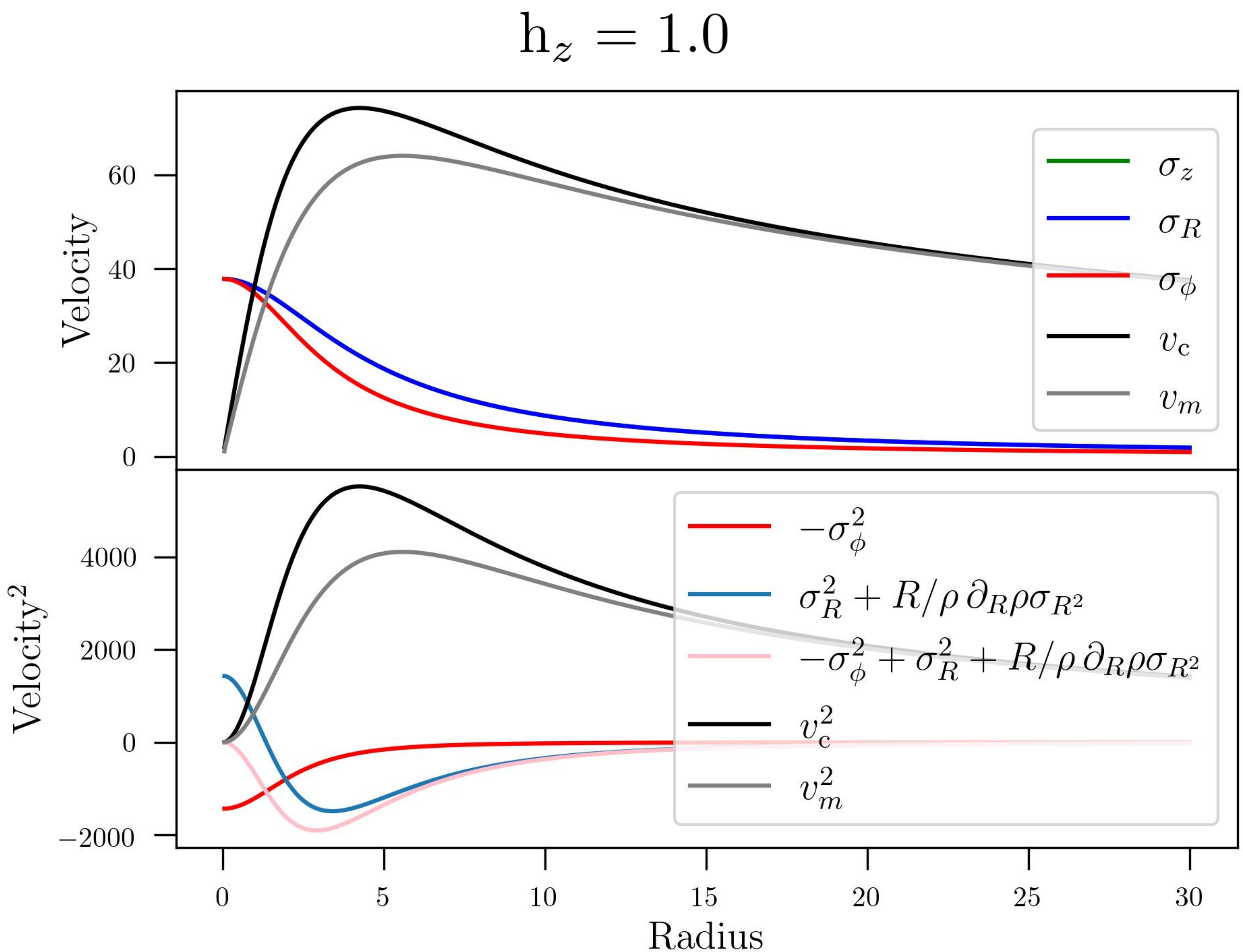
$$\boxed{\overline{v_\phi}^2 = v_c^2 + \left(1 - \frac{\kappa^2}{4\Omega^2}\right)\sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\,\sigma_R^2\right)}$$

Jeans Moments and rotation curve for a Miyamoto-Nagai disk



$$\sigma_z^2 = \frac{1}{\nu}\int_z^\infty \mathrm{d}z'\nu\frac{\partial\Phi}{\partial z'} \qquad \sigma_R^2 = \sigma_z^2 \qquad \frac{\sigma_\phi^2}{\sigma_R^2} = \frac{\kappa^2}{4\Omega^2} \qquad \overline{v_\phi}^2 = v_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)$$

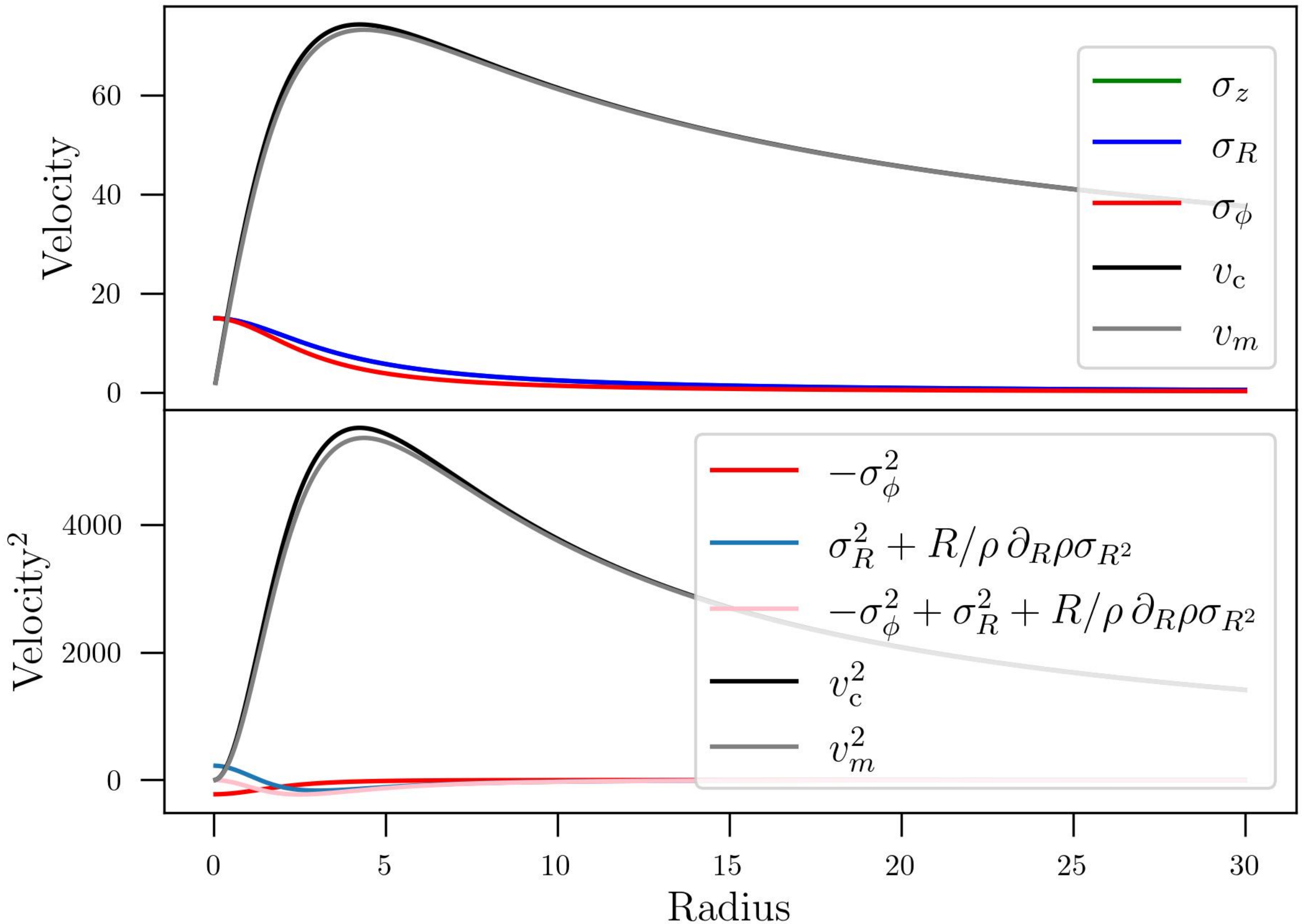
Jeans Moments and rotation curve for a Miyamoto-Nagai disk



$$\sigma_z^2 = \frac{1}{\nu}\int_z^\infty \mathrm{d}z'\,\nu\frac{\partial\Phi}{\partial z'} \qquad \sigma_R^2 = \sigma_z^2 \qquad \frac{\sigma_\phi^2}{\sigma_R^2} = \frac{\kappa^2}{4\Omega^2} \qquad \overline{v_\phi}^2 = v_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu\sigma_R^2\right)$$

Jeans Moments and rotation curve for a Miyamoto-Nagai disk

$\mathrm{h}_z = 0.1$



$$\sigma_z^2 = \frac{1}{\nu}\int_z^\infty \mathrm{d}z' \nu \frac{\partial \Phi}{\partial z'} \qquad \sigma_R^2 = \sigma_z^2 \qquad \frac{\sigma_\phi^2}{\sigma_R^2} = \frac{\kappa^2}{4\Omega^2} \qquad \overline{v_\phi}^2 = v_c^2 - \sigma_\phi^2 + \sigma_R^2 + \frac{R}{\nu}\frac{\partial}{\partial R}\left(\nu \sigma_R^2\right)$$

The End