A quantum game... the Mermin-Peres magic square

Magic squares are abundant in mathematics and usually involve constraints satisfied by all the columns and rows in a grid of numbers. Sometimes, designing them can become arduous, if not impossible.

Consider the following example. Suppose we want to construct a 3×3 magic square with numbers in $\mathcal{S} = \{-1, 1\}$ such that their product 1 for each row, and -1 for each column.

Question 1: *Let*

$$M = \left[\begin{array}{c|c|c} 1 & 1 & 1 \\ \hline -1 & 1 & -1 \\ \hline 1 & -1 & -1 \end{array}\right] \tag{1}$$

Does M satisfy the above constraints on rows and columns? Show that in fact no 3×3 magic square exists with the above constraints. Hint: consider $L_i = \prod_j M_{i,j}$ and $C_j = \prod_i M_{i,j}$. Compute $\prod_i L_i$ and $\prod_j C_j$. What do you conclude?

Despite this issue, Alice and Bob are being challenged by Eve with the **magic square game**. The setting of the game is the following:

1. At first, Alice and Bob can discuss together as long as they want and plan a strategy.
2. Then they are isolated in two separate rooms with no communication channel.
3. Eve draws a random number $i \in \{1, 2, 3\}$ uniformly and sends it only to Alice. Alice has to fill row i with three numbers a_{i1}, a_{i2}, a_{i3} which multiply to 1, and secretly send the numbers to Eve. Bob does not have access to this information.
4. Eve draws a random number $j \in \{1, 2, 3\}$ uniformly and sends it only to Bob. He then has fill column j with three numbers b_{1j}, b_{2j}, b_{3j} which multiply to -1, and secretly send the three numbers to Eve. Alice does not have this information.
5. Row i and column j intersect at the "matrix-element" ij. Eve declares that Alice and Bob win the game if $a_{ij} = b_{ij}$, in other words if the choices of Alice and Bob are compatible. Otherwise Eve declares that they loose the game.

Question 2: *Explain why it is not possible for Alice and Bob to design a strategy that always wins the game. Design a simple strategy such that Alice and Bob win the game with maximal probability (no formal proof asked).*

Fortunately, Alice and Bob know we don't live in a classical world. Indeed they took quantum science classes in university and they are even able to build simple quantum devices! Therefore, before being isolated they prepare a quantum strategy (this strategy satisfies the no-communication requirement after they are separated).

Alice and Bob prepare 2 maximally entangled qubits (EPR or Bell pairs) in the state:

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}} |B_{00}\rangle_{A,B} \otimes \frac{1}{\sqrt{2}} |B_{00}\rangle_{A,B} \tag{2}$$

$$= \frac{1}{\sqrt{2}} \Big(|00\rangle_{A_1,B_1} + |11\rangle_{A_1,B_1} \Big) \otimes \frac{1}{\sqrt{2}} \Big(|00\rangle_{A_2,B_2} + |11\rangle_{A_2,B_2} \Big) \tag{3}$$

Note that they prepare a big enough reservoir of such states so they can play many rounds of the game (but to fix ideas we think of one round in our discussion).

Therefore they share four qubits in total. When they are separated, Alice brings qubits A_1 and A_2 in her room, while Bob gets B_1 and B_2. Before going further into the game, they also agreed on a mysterious set of 9 observables and fill the magic square with these observables:

$$Q = \begin{bmatrix} \sigma_x \otimes \sigma_x & \sigma_x \otimes \mathbf{I} & \mathbf{I} \otimes \sigma_x \\ \sigma_y \otimes \sigma_y & -\sigma_x \otimes \sigma_z & -\sigma_z \otimes \sigma_x \\ \sigma_z \otimes \sigma_z & \mathbf{I} \otimes \sigma_z & \sigma_z \otimes \mathbf{I} \end{bmatrix} \tag{4}$$

To proceed further it is useful to keep in mind the following aspects of the measurement postulate:

1. an observable is a measurable quantity described an hermitian matrix;
2. the measurement apparatus projects the state (or wave function) on one of the eigen-basis vectors $|v\rangle$;
3. the value of the observable is given by the eigenvalue associated with the eigenvector;
4. simultaneous measurements of many observables are only possible for commuting observables since they must have a common eigenbasis.
5. the Born rule states that $\mathcal{P}(|\Psi\rangle \to |v\rangle) = |\langle v|\Psi\rangle|^2$. If an eigenvalue is degenerate the probability of measuring this eigenvalue is the sum of these probabilities over corresponding eigenvectors.

A remark that you might find useful later on is that when eigenvalues are degenerate the corresponding eigenvectors and therefore eigenbasis are not unique.

Question 3: *Check for instance the observable $Q_{1,2} = \sigma_x \otimes \boldsymbol{I}$. What are the possible eigenvalues of this observable? Give two sets of possible eigen-basis vectors.*

The quantum strategy of Alice and Bob is the following. First they prepare and share the state $|\Psi\rangle_{AB}$. After being given a row i, Alice makes the measurement described by the observables $Q_{i,1}, Q_{i,2}, Q_{i,3}$, stores the results in a_{i1}, a_{i2}, a_{i3}, and sends these three numbers to Eve. Bob proceeds similarly, he stores the results of a simultaneous measurement of the observables $Q_{1,j}, Q_{2,j}, Q_{3,j}$ in b_{1j}, b_{2j}, b_{3j}, and

sends the three numbers to Eve. It turns out this is always a winning strategy. We guide you through the theory and then you will implement the game on the IBM Q NISQ devices!

Question 4: *Check the following properties of the quantum magic square Q:*

1. *Check that observables in rows and columns commute. Therefore simultaneous measurements (by Alice) in a given row and simultaneous measurements (by Bob) in a given column are allowed.*

2. *Check also that if observables do not belong to the same row or column they do not necessarily commute. Note that the strategy of Alice and Bob does not require simultaneous such measurements!*

3. *Calculate the products $\prod_j Q_{i,j}$ for each j, and the products $\prod_i Q_{i,j}$ for each i. What do you observe? Compare with the classical magic square where we used only numbers in $\mathcal{S} = \{-1, +1\}$.*

In order to better understand the previous observations, remember that if two operators commute, then they can be diagonalized in a common basis. For instance, in the basis $\mathcal{B}_1^A = \{|{++}\rangle, |{+-}\rangle, |{-+}\rangle, |{--}\rangle\}$, the operators $Q_{1,1} = \sigma_x \otimes \sigma_x, Q_{1,2} = \sigma_x \otimes \mathbf{I}, Q_{1,3} = \mathbf{I} \otimes \sigma_x$ can be diagonalized as they can be written:

$$Q_{1,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad Q_{1,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad Q_{1,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5}$$

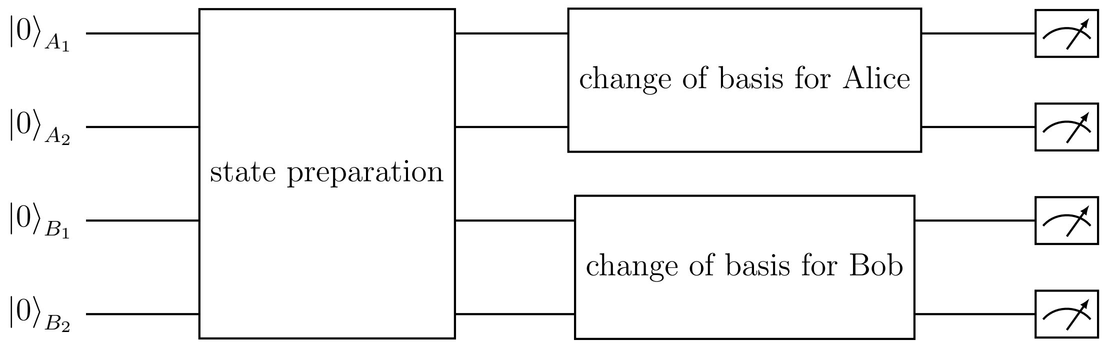
Question 5: *For each 4 possible outcomes in $\mathcal{B}_1^A$ for Alice when she gets row $i = 1$, specify what result is stored in $[a_{i1}, a_{i2}, a_{i3}]$, and check that the product equals 1.*

Question 6: *Now let's see what happens on Bob's side when he gets column $j = 2$. Find the common basis $\mathcal{B}_2^B$ for the operators $Q_{1,2}, Q_{2,2}, Q_{3,2}$, and for each 4 possible outcomes in $\mathcal{B}_2^B$, explain what result is stored in $[b_{1j}, b_{2j}, b_{3j}]$ and check the product $b_{1j}b_{2j}b_{3j}$.*

Question 7: *Let's keep assuming that Alice got $i = 1$, and Bob got $j = 2$. In order to win the game they need to have measured $a_{12} = b_{12}$. Calculate the probability $P(a_{12} = b_{12}) = P(a_{12} = 1, b_{12} = 1) + P(a_{12} = -1, b_{12} = -1)$ using the previous input wave function $|\psi\rangle_{A,B}$ mentioned earlier. What do you conclude?*

Experiments

Question 8: *In order to test these results experimentally, you will use an IBM-Q NISQ device and replicate the possible outcomes for the specific case $i = 1$ and $j = 2$. Design* *the appropriate quantum circuit. It ought to have the following form (note that IBM-Q only allows you to make measurements in the computational basis* $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ *so you will have to find a way to circumvent this issue using a basis change). Your circuit ought to have the following form:*



Now, run your circuit on the IBM "simulator" machine. Write down the outputs and discuss the results: which outcome corresponds to which results a_{i1}, a_{i2}, a_{i3} *and* b_{1j}, b_{2j}, b_{3j}*? Do Alice and Bob respect the rules of the game? Do they win the game all the time?*

Question 9: *If you are satisfied with your quantum circuit, you can now run it on a real quantum machine! (Depending on the resources availability, launch queue may take up to a few minutes). Do you observe any difference with the results in Question 11? Why?*

Question 10: *Assume now that Alice is given* $i = 3$ *and Bob* $j = 1$.

1. *What is the common basis* $\mathcal{B}_3^A$ *for Alice's observables? What is the common basis* $\mathcal{B}_1^B$ *for Bob's observables?*

2. *Propose a quantum circuit as in question 8, run it on "simulator", and write down the outputs with the corresponding outcomes* a_{i1}, a_{i2}, a_{i3} *and* b_{1j}, b_{2j}, b_{3j}. *Run the circuit on a real quantum machine.*

Question 11: *Assume now that Alice is given* $i = 2$ *and Bob* $j = 3$.

1. *What is the common basis* $\mathcal{B}_2^A$ *for Alice's observables? What is the common basis* $\mathcal{B}_3^B$ *for Bob's observables?*

2. *Propose a quantum circuit as in question 8, run it on "simulator", and write down the outputs with the corresponding outcomes* a_{i1}, a_{i2}, a_{i3} *and* b_{1j}, b_{2j}, b_{3j}. *Run the circuit on a real quantum machine.*

Bonus Question. *Write a Qiskit code that deals with all possible rows and columns given to Alice and Bob (there are thus 9 possible circuits). The input should be the row* i *and the column* j. *The output should be a histogram with the answers of Alice and Bob. Run it on the simulator and then on a real quantum machine.*