Exercise 8.1. Compute the flows of the following vector fields.

(a) On the plane $\mathbb{R}^2$, the "angular" vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.

Solution. The integral curves are of the form $\gamma(t) = \begin{pmatrix} r\cos(t-t_0) \\ r\sin(t-t_0) \end{pmatrix}$, with $t_0 \in \mathbb{R}$ and $r \geq 0$. We can rewrite them as

$$\begin{aligned}\gamma(t) &= \begin{pmatrix} r\cos(t)\cos(t_0) + r\sin(t)\sin(t_0) \\ r\sin(t)\cos(t_0) - r\cos(t)\sin(t_0) \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} r\cos(t_0) \\ -r\sin(t_0) \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\end{aligned}$$

where $(x_0, y_0) = \gamma(0)$. Thus the flow is $\Phi_X^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, defined for all points $(x_0, y_0) \in \mathbb{R}^2$ and all $t \in \mathbb{R}$. □

(b) A constant vector field X on the torus $\mathbb{T}^n$.

Solution. Note first that we have an identification $\mathrm{T}_{[p]}\mathbb{T}^n \equiv \mathbb{R}^n$ for all points $[p] = \pi(p) \in \mathbb{T}^n$, where $p \in \mathbb{R}^n$ and $\pi : \mathbb{R}^n \to \mathbb{T}^n$ is the quotient map. This identification is the linear transformation $\mathrm{T}_p\pi$, which is an isomorphism from $\mathrm{T}_p\mathbb{R}^n \equiv \mathbb{R}^n$ to $\mathrm{T}_{[p]}\mathbb{T}^n$. This identification $\mathrm{T}_p\pi : \mathbb{R}^n \to \mathrm{T}_p\mathbb{T}^n$ is independent of which preimage we choose for $[p]$, since if p' is another preimage and τ is the translation of $\mathbb{R}^n$ that maps $p \mapsto p'$, then $\pi(x) = \pi \circ \tau$, and therefore

$$\mathrm{T}_p\pi = \mathrm{T}_{p'}\pi \circ \mathrm{T}_p\tau \equiv \mathrm{T}_{p'}\pi$$

since $\mathrm{T}_p\tau \equiv \mathrm{id}_{\mathbb{R}^n}$.

Thus we can talk about a constant vector field X on $\mathbb{T}^n$. This means that

$$X_{[p]} = a \quad \text{for all } p \in \mathbb{R}^n$$

for some fixed $a \in \mathbb{R}^n$.

Let $\widehat{X} = \pi^* X$ be the vector field on $\mathbb{R}^n$ given by the similar formula $\widehat{X}_p = a$ for all $p \in \mathbb{R}^n$. Note that $\widehat{X}$ is π-related to X, where $\pi : \mathbb{R}^n \to \mathbb{T}^n$ is the quotient map. Therefore $\pi \circ \gamma$ is an integral curve of X if γ is an integral curve of $\widehat{X}$.

For any point $p \in \mathbb{R}^n$, the maximal integral curve of $\widehat{X}$ starting at the point p is $\gamma_{\widehat{X},p}(t) = p + at$. Therefore the curve

$$\gamma_{X,[p]}(t) := \pi(\gamma_{\widehat{X},p}(t)) = [p + ta]$$

is an integral curve of X. It has initial condition $\gamma_{X,[p]}(0) = [p]$ and it is maximal because it is defined for all t.

Therefore the flow of X is $\Phi_X^t[p] = [p + ta]$, which is defined for all points $[p] \in \mathbb{T}^n$ and all $t \in \mathbb{R}$. □

Exercise 8.2. Let X be a $\mathcal{C}^k$ tangent vector field on a manifold M, with $k \geq 1$.

(a) For a point $p \in M$ and numbers $s, t \in \mathbb{R}$, show that the equation $\Phi_X^{(s+t)}(p) = \Phi_X^t(\Phi_X^s(p))$ holds if the right-hand side is defined.

Solution. Since we are only considering one vector field X, we may omit the subindex X and thus write $\Phi := \Phi_X$, $I_p := I_{X,p}$ and $\gamma_p := \gamma_{X,p}$.

The right-hand side is defined if and only if $s \in I_p$ (so that $q := \Phi^s(p)$ is defined) and $t \in I_q$ (so that $\Phi^t(q) = \Phi^t_X(\Phi^s(p))$ is defined). We assume this is the case.

The function $\tau \mapsto \gamma_p(\tau+s)$, defined for $\tau \in I_p - s$, is the curve γ_q, because it is a maximal integral curve of X that visits at time $\tau = 0$ the point $\gamma_p(s) = q$. In particular $I_q = I_p - s$. Thus since $t \in I_q$, it follows that $t+s \in I_p$, and that

$$\Phi^t(\Phi^s(p)) = \Phi^t(q) = \gamma_q(t) = \gamma_p(t+s) = \Phi^{t+s}(p).$$

□

(b) We say that X is **complete** if its flow Φ_X is defined over $M \times \mathbb{R}$. Show that a compactly supported vector field is complete. In particular, on a compact manifold, every vector field is complete.

Solution. We will first show that the flow Φ_X is defined on a set $M \times (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Once this is established, we see that $\Phi^t_X(p)$ is defined for any $(p,t) \in M \times \mathbb{R}$ by decomposing $t = \sum_i t_i$ with $|t_i| < \varepsilon$ and applying the last formula $\Phi^t_X(p) = \Phi^{t_0}_X(\Phi^{t_1}_X \dots (p))$. This shows that X is complete.

Suppose X vanishes outside a compact set $K \subseteq M$. The domain of the flow Φ_X is a set $\mathrm{Dom}(\Phi_X) \subseteq M \times \mathbb{R}$ that contains the set $K \times \{0_{\mathbb{R}}\}$. In addition, $\mathrm{Dom}(\Phi_X)$ is open (this is part of the theorem of differentiability of the flow Φ_X). Since K is compact, by the *tube lemma* the open set $\mathrm{Dom}(\Phi_X)$ also contains a "tube neighborhood" $K \times (-\varepsilon, \varepsilon)$ of the set $K \times \{0_{\mathbb{R}}\}$, for some number $\varepsilon > 0$. But the domain $\mathrm{Dom}(\Phi_X)$ also contains the set $(M \setminus K) \times \mathbb{R}$, because for points $p \in M \setminus K$, since the vector field X vanishes at p, the maximal solution is the constant curve $\gamma_{X,p}(t) = p$, which is defined for all $t \in \mathbb{R}$. This shows that Φ_X is defined on $M \times (-\varepsilon, \varepsilon)$, and therefore Φ_X is complete, as explained above. □

(c) If X is complete, show that the map Φ^t_X is a diffeomorphism $M \to M$.

Solution. Φ^t_X is a diffeomorphism with inverse Φ^{-t}_X since $\Phi^t_X \circ \Phi^{-t}_X = \Phi^{t-t} = \Phi^0 = \mathrm{id}_M$ and similarly $\Phi^{-t}_X \circ \Phi^t_X = \mathrm{id}_M$. □

Exercise 8.3. If X is a complete $\mathcal{C}^k$ vector field with $(k \geq 1)$ and $h \in \mathcal{C}^{k+1}(M, \mathbb{R})$.

(a) Show that the function $X(h) : M \to \mathbb{R}$ that sends $p \mapsto X_p(h)$ is $\mathcal{C}^k$.

Solution. Take a chart (U, φ) and write $X|_U = \sum_i X^i \frac{\partial}{\partial \varphi^i}$. Then $X|_p(h) = \sum_i X^i|_p \frac{\partial}{\partial \varphi^i}|_p h$. Thus to see that the function $X(h)$ is $\mathcal{C}^k$, it suffices to check that the functions X^i and $\frac{\partial}{\partial \varphi^i} h$ are $\mathcal{C}^k$. And indeed: the fact that X is $\mathcal{C}^k$ means that the functions X^i are $\mathcal{C}^k$, and the fact that h is $\mathcal{C}^{k+1}$ implies that its first-order derivatives $\frac{\partial h}{\partial \varphi^i}$ are $\mathcal{C}^k$. □

(b) Show that $X(h) = \frac{\partial}{\partial t}\big|_{t=0} h_t$, where $h_t := (\Phi^t_X)^*(h) = h \circ \Phi^t_X$.
Also show that $X(h_t) = (\Phi^t_X)^*(X(h))$.

Solution. Writing $\Phi := \Phi_X$, we have

$$\frac{\partial}{\partial t}\bigg|_{t=0} h_t(p) = \frac{\partial}{\partial t}\bigg|_{t=0} h\left(\Phi^t(p)\right) = \mathrm{T}_{\Phi^0(p)} h \left(\frac{\partial}{\partial t}\bigg|_{t=0} \Phi^t(p) \right) = \mathrm{T}_p h(X_p) = X_p(h).$$

□

Exercise 8.4. Let $f : M \to N$ be a smooth map. A vector field $X \in \mathfrak{X}(M)$ is **f-related** to a vector field $Y \in \mathfrak{X}(N)$ if $\mathrm{T}_p f(X_p) = Y_{f(p)}$ for all $p \in M$.

(a) X is f-related to Y if and only if $X_p(h \circ f) = Y_{f(p)}(h)$ for all functions $h \in \mathcal{C}^\infty(N, \mathbb{R})$ and all points $p \in M$.

Solution. By definition of $\mathrm{T}_p f$ we have $(\mathrm{T}_p f(X_p))(h) = X_p(h \circ f)$ for all functions $h \in \mathcal{C}^\infty(M)$. Thus

$$\begin{aligned} X \text{ is } f\text{-related to } Y \text{ at } p &\iff Y_{f(p)} = \mathrm{T}_p f(X_p) \\ &\iff Y_{f(p)}(h) = (\mathrm{T}_p f(X_p))(h) \quad \forall h \in \mathcal{C}^\infty(N) \\ &\iff Y_{f(p)}(h) = X_p(h \circ f) \quad \forall h \in \mathcal{C}^\infty(N) \end{aligned}$$

□

(b) If X is f-related to Y and γ is an integral curve of X, show that $f \circ \gamma$ is an integral curve of Y.

Solution. We just need to verify that

$$(f \circ \gamma)'(t) = \mathrm{T}_{\gamma(t)} f(\gamma'(t)) = \mathrm{T}_{\gamma(t)} f(X_{\gamma(t)}) = Y_{f(\gamma(t))} = Y_{f \circ \gamma(t)}$$

for all t in the domain of γ. □

(c) If f is a local diffeo, for every vector field $Y \in \mathfrak{X}(N)$ there exists a unique $X \in \mathfrak{X}(M)$ that is f-related to Y. We denote $f^*Y := X$.
Thus if f is a diffeo, f-relatedness is a bijection from $\mathfrak{X}(M)$ to $\mathfrak{X}(N)$. In this case, if X is f-related to Y, we write $X = f^*Y$ and $Y = f_*X$.

Solution. Assume $f : M \to N$ is a local diffeo. Thus for every point $p \in M$, the linear transformation $\mathrm{T}_p f : M \to N$ is invertible.

Let $Y \in \mathfrak{X}(N)$. A vector field X on M is f-related to Y iff for each point $p \in M$ we have $\mathrm{T}_p f(X|_p) = Y_p$, or, equivalently, $X|_p = (\mathrm{T}_p f)^{-1}(Y_{f(p)})$. Thus there is a unique vector field that is f-related to Y, and it is the function $p \mapsto (\mathrm{T}_p f)^{-1}(Y_{f(p)})$. □

(d) If f is a closed embedding, show that every vector field $X \in \mathfrak{X}(M)$ is f-related to some vector field $Y \in \mathfrak{X}(N)$.

Hint: Construct Y locally, then use partitions of unity.

What happens if f is just an immersion? In this case, find and prove a local version of the fact.

Solution. The local version is the following.

Lemma. Let $f : M \to N$ be a smooth immersion, and let $X \in \mathfrak{X}(M)$. Then for each point $p_0 \in M$ there exist open neighborhoods $U \subseteq M$ and $V \subseteq N$ of p_0 and $f(p_0)$ resp., and a vector field $Y \in \mathfrak{X}(V)$ such that $X|_U$ is f-related to Y.

Proof. By the constant rank theorem, there exist charts $\varphi : U \to \widetilde{U}$ and $\psi : V \to \widetilde{V}$ of M and N centered at p_0 and $f(p_0)$ such that the local expression $\widetilde{f} = \psi \circ f \circ \varphi^{-1}$ of f is given by $\widetilde{f}(x^0, \dots, x^{m-1}) = (x^0, \dots, x^{m-1}, 0, \dots, 0)$. Moreover, we can assume that $\widetilde{V} = \widetilde{U} \times \widetilde{W}$ for some open set $\widetilde{W} \subseteq \mathbb{R}^{n-m}$ that contains the origin.

Note that each coordinate vector field $\frac{\partial}{\partial \varphi^i} \in \mathfrak{X}(U)$ of the chart φ is f-related to the corresponding coordinate vector field $\frac{\partial}{\partial \psi^i} \in \mathfrak{X}(V)$ of the chart ψ. That is, for each $p \in U$ we have $\mathrm{T}_p f\left(\left.\frac{\partial}{\partial \varphi^i}\right|_p \right) = \left.\frac{\partial}{\partial \psi^i}\right|_{f(p)}$.

Let $\pi : \widetilde{V} \to \widetilde{U}$ be the projection map $(x^0, \dots, x^{m-1}) \mapsto (x^0, \dots, x^{n-1})$, and let $\rho = \varphi^{-1} \circ \pi \circ \psi : U \to W$. Note that ρ is a retraction of $f|_U^V$.

Let X^i be the components of X w.r.t. the chart φ. Thus $X|_U = \sum_{0 \le i < n} X^i \frac{\partial}{\partial \varphi^i}$ We construct a vector field $Y \in \mathfrak{X}(V)$ whose components w.r.t. the chart ψ are

$$Y^i = \begin{cases} X^i \circ \rho & \text{if } i < n, \\ 0 & \text{if } i \ge n. \end{cases}$$

Thus $Y|_q = \sum_{0 \le i < n} X^i(\rho(q)) \left.\frac{\partial}{\partial \psi^i}\right|_q$ for each point $q \in V$.

In particular, at a point $q = f(p)$ we have $\rho(q) = p$, therefore

$$\begin{aligned} Y|_q &= \sum_{0\le i<n} X^i(p) \left.\frac{\partial}{\partial \psi^i}\right|_q \\ &= \sum_{0\le i<n} X^i(p)\ \mathrm{T}_p f\left(\left.\frac{\partial}{\partial \varphi^i}\right|_p\right) \\ &= \mathrm{T}_p f\left(\sum_{0\le i<n} X^i(p) \left.\frac{\partial}{\partial \varphi^i}\right|_p\right) = \mathrm{T}_p f(X_p). \end{aligned}$$

This shows that Y is $f|_U^V$-related to X. □

Now we can prove the global version.

Let $f : M \to N$ be a closed embedding, and let $X \in \mathfrak{X}(M)$. We shall construct a vector field $Y \in \mathfrak{X}(N)$ such that X is f-related to Y.

The closed set $f(M)$ can be covered by open sets $(V_k)_{k\ge 1}$ where there is a vector field $Y_k \in \mathfrak{X}(V_k)$ that is f-related to X.

We also define the open set $V_0 = N \setminus f(M)$ and we put any vector field Y_0 on V_0, for example $Y_0 \equiv 0$. Note that X is f-related to Y_0 trivially. The open sets $(V_k)_{k\ge 0}$ form an open cover of N. Let $(\eta_k)_{k\ge 0}$ be a partition of unity subordinate to this cover, and consider the vector field $Y = \sum_k \eta_k Y_k \in \mathfrak{X}(N)$. We claim that X is f-related to Y. Indeed, for each point $p \in M$ we have

$$Y_{f(p)} = \sum_k \eta_k(f(p))\, Y_k|_{f(p)} = \sum_k \eta_k(f(p))\ \mathrm{T}_p f(X|_p) = \mathrm{T}_p f(X|_p)$$

because $\sum_k \eta_k(f(p)) = 1$. □

(e) A vector field $X \in \mathfrak{X}(M)$ is **tangent** to a smooth embedded submanifold $S \subseteq M$ if $X_p \in T_pS$ for all points $p \in S$. If this happens and in addition S is closed, show that every integral curve of X that visits S is contained in S.

Solution. Let $\iota : S \to M$ be the inclusion map, and let $Y = X|_S \in \mathfrak{X}(S)$. Note that Y is ι-related to X.

Let $\gamma : I \to M$ be an integral curve of X that visits S. The set $I' = \gamma^{-1}(S) = \{t \in I : \gamma(t) \in S\}$ is nonempty and closed (because S is closed). We want to prove that $I' = I$, and for this it suffices to show that I' is open. Let $t_0 \in I'$. This means that $\gamma(t_0) \in S$. Let $\beta : J \to S$ be an integral curve of Y that coincides with γ at the instant t_0, where $J \subseteq \mathbb{R}$ is an open interval containing t_0. Since Y is ι related to X, the curve $\iota \circ \beta$ is an integral curve of X that coincides with γ at t_0, thus it coincides with γ in the interval $I'' = I \cap J$, which is a neighborhood of t_0. This implies that $\gamma(t) \in S$ for all $t \in I''$. Therefore $I'' \subseteq I'$, which proves that I' is open, as intended. □

Exercise 8.5. If X is a smooth vector field on a manifold M and $p \in M$ is a point where $X_p \neq 0$, then there exists a chart (U, ϕ) of M defined at p such $X|_U = \frac{\partial}{\partial \phi^0}$. ***Hint:*** It is easier to construct the inverse $\psi = \phi^{-1}$. Use a function of the form $\psi(x) = \Phi_X^{x^0}(f(x^1, \dots, x^{n-1}))$, where $f : U \to M$ is a suitable function defined on an open set $U \subseteq \mathbb{R}^{n-1}$.

Solution. Let (V, η) be a chart centered at p, i.e. such that $\eta(p) = 0$. Denote X^i the components of X with respect to the chart η. Since $X|_p \neq 0$, we may assume w.l.o.g that $X^0 \neq 0$ at p, which means that the vectors $X|_p$, $\left.\frac{\partial}{\partial \eta^1}\right|_p, \dots, \left.\frac{\partial}{\partial \eta^{n-1}}\right|_p$ are linearly independent.

Consider the map $\iota : \mathbb{R}^{n-1} \to \mathbb{R}^n : (x^1, \dots, x^{n-1}) \mapsto (0, x^1, \dots, x^{n-1})$, and let $W = \iota^{-1}(\eta(V))$, so that we can define the map $f = \eta^{-1} \circ \iota : W \to V$.

Define the map $\psi(x^0, x^1, \dots, x^{n-1}) = \Phi_X^{x^0}(f(x^1, \dots, x^{n-1}))$ at all points where the right hand side is defined. The domain of ψ is an open set which includes the slice

$\{0_{\mathbb{R}}\} \times W$. The partial derivative of ψ with respect to x^0 is $\frac{\partial \psi(x^0,\ldots,x^{n-1})}{\partial x^0} = X$. Its other partial derivatives at the point $x = 0$ are

$$\left.\frac{\partial \psi(x^0,\ldots,x^{n-1})}{\partial x^i}\right|_{x=0} = \left.\frac{\partial f(x^1,\ldots,x^{n-1})}{\partial x^i}\right|_{x=0} = \left.\frac{\partial}{\partial \eta^i}\right|_p$$

for $i \neq 0$. Since the vectors X_p, $\left.\frac{\partial}{\partial \eta^1}\right|_p, \ldots, \left.\frac{\partial}{\partial \eta^{n-1}}\right|_p$ are linearly independent, we conclude that $\mathrm{T}_0\psi$ is an isomorphism. Thus there is a neighborhood Z of 0 in $\mathbb{R}^n$ such that the map $\psi|_Z : Z \to \psi(Z) \subseteq M$ is a diffeomorphism. Hence $\psi|_Z$ is a local parametrization of M. □