Stability of collisionless systems

2nd part

Outlines

Linear response theory

      - in fluid systems
      - in stellar systems

The Jeans instability

      - in fluid systems
      - in stellar systems

Entropy of a collisionless system of N particles

By analogy : $\quad P \rightarrow \mathscr{g}(\vec{q}, \vec{p})$

$$S := -N \int d^3\vec{q}\, d^3\vec{p}\; \mathscr{g}(\vec{q},\vec{p})\; \ln\left(\mathscr{g}(\vec{q},\vec{p})\right) \qquad\qquad \mathscr{g} : DF$$

Can we maximize S for a system of a given mass M and energy E ?

Yes : The solution is the isothermal sphere !

But ! The isothermal sphere is unphysical
$$\left\{ \begin{array}{l} M = \infty \\[2mm] E = \infty \end{array} \right. \qquad \triangle$$

Where is the problem ?

Demonstration   Spherical system $\Big\}$ mass   M

total energy   E

Idea: split the system into : ① inner part

② outer part

=   ① r_1 +   ② r_2
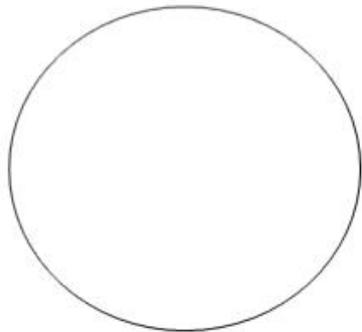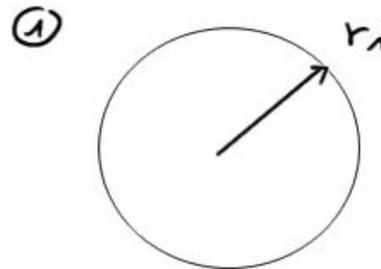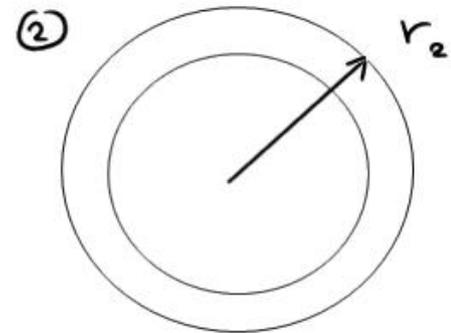
M M_1 $M_2 << M$

Virial Equation

$2K + W = 0$

$E = K + W = \dfrac{W}{2}$

$E_1 = \dfrac{W}{2} \cong -\dfrac{GM_1^2}{2r_1}$

↳ virial

$E_2 \cong -\dfrac{GM_1 M_2}{2r_2}$ the pot. is dominated by M_1

4

If we shrink ①
_____

$$r_1 \longrightarrow (1 - \varepsilon)\, r_1 \qquad\qquad (\varepsilon > 0 \text{ but } \ll 1)$$

$$E_1 \longrightarrow -\frac{GM_1^2}{2\, r_1(1-\varepsilon)}$$

$$\Delta E = E_1 - E_1' \cong \varepsilon\, \frac{GM_1^2}{r_1} \qquad (> 0 \text{ excess of grav. energy})$$

This excess of energy is deposited into ② which will extend $r_2' > r_2$

$$E_2' = E_2 + \Delta E = -\frac{GM_1 M_2}{r_2'} \qquad \boxed{\frac{1}{r_2'} \sim E_2 + \Delta E}$$

At the virial equilibrium,
the envelope velocity will be

$$\boxed{\sigma_2'^{\,2} \sim \frac{GM_1}{r_2'}}$$

5

The points in ② will cover a region of the phase space with a volume of about :

$$V_② = \sigma_2'^3 \cdot r_2'^3 \cong \left(\frac{GM_1}{r_2'}\right)^{3/2} \cdot r_2'^3 = \left(GM_1 r_2'\right)^{3/2}$$

as

$$\rho_② \sim \frac{1}{V_②} \Rightarrow \rho_② \sim r_2'^{-3/2}$$

Entropy of ②

$$S_② = -N_2 \underbrace{\int d^3r \, d^3v \, \rho_②}_{\cong 1} \ln(\rho_②)$$

$$-\frac{GM_1 M_2}{2 r_2} \qquad \varepsilon \frac{GM_1^2}{r_1}$$

$$= -N_2 \frac{3}{2} \ln\left(\frac{1}{r_2'}\right) = -N_2 \frac{3}{2} \ln\left(E_2 + \Delta E\right)$$

if $\quad \Delta E = |E_2|$

i.e $\quad \varepsilon = \frac{1}{2} \frac{M_2}{M_1} \frac{r_1}{r_2} \qquad \Rightarrow \qquad S_② \to \infty \qquad \text{!!!} \qquad S_① + S_② \to \infty$

We can always collapse a portion of a system and release gravitational energy to the outer part (diffuse envelope) in such a way that the entropy increase (no bounds)

6

Conclusions

Stellar systems may increase they entropy, but never reach a maximum, as this maximum does not exist.

==Finite stellar systems cannot reach a thermodynamical equilibrium.==

Moreover

- The entropy is defined locally, while ==the granty is a long-range force==
- When defining the entropy, we consider the conservation of energy ==but not angular momentum==

Our goal   Study the stability of systems
            at equilibrium

Method   :   perturbation theory

perturbation   →   response

Types of responses

- Exponnential growth of the perturbation
- Oscillation of the perturbation
- Die of the perturbation

Stability of collisionless systems

Linear response theory

Linear response theory

Study the perturbation of system at the equilibrium

Perturbation: external gravitational field $-\varepsilon \, \vec{\nabla} \phi_e$: $|\vec{\nabla} \phi_e| \sim |\vec{\nabla} \phi_o|$

Linear response theory

Study the perturbation of system at the equilibrium

Perturbation: external gravitational field $-\varepsilon \vec{\nabla} \phi_e$: $|\vec{\nabla} \phi_e| \sim |\vec{\nabla} \phi_o|$

Steller / system
at equilibrium

$\rho_{s_o}(x)$

$\phi_{s_o}(x)$

$\nabla^2 \phi_{s_o} = 4\pi G \rho_{s_o}$

Linear response theory

Study the perturbation of system at the equilibrium

Perturbation: external gravitational field $-\varepsilon \vec{\nabla}\phi_e$: $|\vec{\nabla}\phi_e| \sim |\vec{\nabla}\phi_o|$

Steller/system
at equilibrium

external
perturbation

$\rho_{s_o}(x)$

$\varepsilon \rho_e(x)$

$\phi_{s_o}(x)$

$\varepsilon \phi_e(x)$

$\nabla^2\phi_{s_o} = 4\pi G \rho_{s_o}$

$\nabla^2\phi_e = 4\pi G \rho_e$

Linear response theory

Study the perturbation of system at the equilibrium

Perturbation: external gravitational field $-\varepsilon \vec{\nabla}\phi_e$: $|\vec{\nabla}\phi_e| \sim |\vec{\nabla}\phi_o|$

---

stellar/system at equilibrium	external perturbation	stellar/fluid responses linearly

$\rho_{s_o}(x)$	$\varepsilon \rho_e(x)$	$\rho_s = \rho_{s_o} + \varepsilon \rho_{s_1}$
$\phi_{s_o}(x)$	$\varepsilon \phi_e(x)$	$\phi_s = \phi_{s_o} + \varepsilon \phi_{s_1}$
$\nabla^2 \phi_{s_o} = 4\pi G \rho_{s_o}$	$\nabla^2 \phi_e = 4\pi G \rho_e$	$\nabla^2 \phi_s = 4\pi G \rho_s$
		$\nabla^2 \phi_{s_1} = 4\pi G \rho_{s_1}$

13

Linear response theory

Study the perturbation of system at the equilibrium

Perturbation: external gravitational field $-\varepsilon \vec{\nabla}\phi_e$: $|\vec{\nabla}\phi_e| \sim |\vec{\nabla}\phi_o|$


| stellar/system at equilibrium	external perturbation	stellar/fluid responses linearly	total stellar/fluid linear response

$\rho_{s_o}(x)$ $\quad+\quad$ $\varepsilon\rho_e(x)$ $\quad\longrightarrow\quad$ $\rho_s = \rho_{s_o} + \varepsilon\rho_{s_1}$ $\quad=0\quad$ $\rho_{tot} = \rho_{s_o} + \varepsilon\rho_{s_1} + \varepsilon\rho_e$

$$= \rho_{s_o} + \varepsilon\rho_1$$

$\phi_{s_o}(x)$ $\qquad\qquad$ $\varepsilon\phi_e(x)$ $\qquad\qquad$ $\phi_s = \phi_{s_o} + \varepsilon\phi_{s_1}$ $\qquad$ $\phi_{tot} = \phi_{s_o} + \varepsilon\phi_{s_1} + \varepsilon\phi_e$

$$= \phi_{s_o} + \varepsilon\phi_1$$

$\nabla^2\phi_{s_o} = 4\pi G \rho_{s_o}$ $\qquad$ $\nabla^2\phi_e = 4\pi G \rho_e$ $\qquad$ $\nabla^2\phi_s = 4\pi G \rho_s$ $\qquad$ $\nabla^2\phi_1 = 4\pi G \rho_1$

$$\nabla^2\phi_{s_1} = 4\pi G \rho_{s_1} \qquad\qquad \nabla^2\phi_{tot} = 4\pi G \rho_{tot}$$

Linear response theory

Study the perturbation of system at the equilibrium

Perturbation: external gravitational field $-\varepsilon \vec{\nabla}\phi_e$: $|\vec{\nabla}\phi_e| \sim |\vec{\nabla}\phi_o|$

---

stellar/system at equilibrium

$\rho_{s_o}(x)$

$\phi_{s_o}(x)$

$\nabla^2 \phi_{s_o} = 4\pi G \rho_{s_o}$

$+$

external perturbation

$\varepsilon \rho_e(x)$

$\varepsilon \phi_e(x)$

$\nabla^2 \phi_e = 4\pi G \rho_e$

$\longrightarrow$

stellar/fluid responses linearly
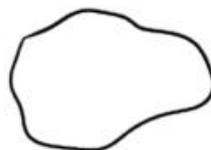
$\rho_s = \rho_{s_o} + \varepsilon \rho_{s_1}$

$\phi_s = \phi_{s_o} + \varepsilon \phi_{s_1}$

$\nabla^2 \phi_s = 4\pi G \rho_s$

$\nabla^2 \phi_{s_1} = 4\pi G \rho_{s_1}$

$=0$

total stellar/fluid linear response

$\rho_{tot} = \rho_{s_o} + \varepsilon \rho_{s_1} + \varepsilon \rho_e$

$\quad = \rho_{s_o} + \varepsilon \rho_1$

$\phi_{tot} = \phi_{s_o} + \varepsilon \phi_{s_1} + \varepsilon \phi_e$

$\quad = \phi_{s_o} + \varepsilon \phi_1$
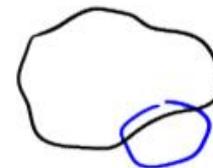
$\nabla^2 \phi_1 = 4\pi G \rho_1$

$\nabla^2 \phi_{tot} = 4\pi G \rho_{tot}$

Conventions: $\rho_{s_o} = \rho_0$ $\phi_{s_o} = \phi_0$ $\rho_{tot} = \rho$ $\phi_{tot} = \phi$

Lineonized equations for a self-gravitating system

Self-gravitating fluid $\rho(\vec{x},t)$ $\phi(\vec{x},t)$ $p(\vec{x},t)$ $\vec{v}(\vec{x},t)$

6 Equ.
6 Unkn.

① Continuity Equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$$

② Euler Equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\phi$$

③ The Poisson Equation

$$\nabla^2 \phi = 4\pi G \rho$$

④ Equation of state (barotropic)

$$p(\vec{x},t) = p(\rho(x,t))$$

Self-gravitating stellar system $f(\vec{x},\vec{v},t)$, $\rho(\vec{x},t)$, $\phi(\vec{x},t)$

3 Equ
3 Unkn.

① The collisionless Boltzmann Equation

$$\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{x}} - \frac{\partial \phi}{\partial \vec{x}}\frac{\partial f}{\partial \vec{v}} = 0 \qquad = \frac{\partial f}{\partial t} + [f, H] = 0$$

② The Poisson Equation

$$\nabla^2 \phi = 4\pi G \rho = 4\pi G \int d^3v \, f(\vec{x},\vec{v},t)$$

Linearizing equations

Goal : describe the response of the system (without the perturbation)

$$\rho_s(\bar{x},t) = \rho_{s_0}(\bar{x}) + \varepsilon \, \rho_{s_1}(\hat{x},t)$$

$\phi_s \quad \bar{v}_s \quad p_s \quad \rho_s$

① Equations for $\rho_s(\bar{x},t)$

② Equations for $\rho_{s_0}(\bar{x})$ (system at equilibrium)

③ Keep only first order terms $(\varepsilon^2 \rightarrow 0)$

linear differential equations for ρ_{s_1} $\phi_{s_1} \quad \bar{v}_{s_1} \quad p_{s_1} \quad \rho_{s_1}$

Linearized Equations for a self-gravitating fluid

Equations for a self-gravitating fluid

System of density $\rho_s(\vec{x}, t)$ $\phi_s(\vec{x}, t)$ $\vec{v}_s(x, t)$ $P_s(\vec{x}, t)$

① Continuity Equation

$$\frac{\partial \rho_s}{\partial t} + \vec{\nabla}(\rho_s \vec{v}_s) = 0$$

total potential $\phi = \phi_s + \varepsilon \phi_e$

② Euler Equation $\}$

$$\frac{\partial}{\partial t}\vec{v}_s + (\vec{v}_s \cdot \vec{\nabla})\vec{v}_s = -\frac{\vec{\nabla} P_s}{\rho_s} - \vec{\nabla}\phi$$

③ The Poisson Equation

$$\nabla^2 \phi_s = 4\pi G \rho_s$$

④ Equation of state

$$P_s = P(\rho_s)$$

18

Definition : $\boxed{\text{specific enthalpy}}$

$$h(\rho_s) = \int_0^{\rho_s} \frac{1}{\rho'} \frac{\partial P_s}{\partial \rho_s}(\rho') \, d\rho'$$

for a barotropic
EOS $\quad P = P(\rho)$

Euler equation
becomes

$$\boxed{\vec{\nabla} h_s = \frac{1}{\rho_s} \vec{\nabla} P_s}$$

$$\frac{\partial}{\partial t} \vec{v}_s + \left(\vec{v}_s \cdot \vec{\nabla} \right) \vec{v}_s = - \frac{\vec{\nabla} P_s}{\rho_s} - \vec{\nabla} \phi$$

$$\frac{\partial}{\partial t} \vec{v}_s + \left(\vec{v}_s \cdot \vec{\nabla} \right) \vec{v}_s = - \vec{\nabla}(h_s + \phi)$$

Definition : $\boxed{\text{sound speed}}$

$$\boxed{v_s^2 = \frac{\partial P(\rho)}{\partial \rho}\bigg|_{\rho_0}}$$

for the
unperturbed
system

$\triangle$

19

More on enthalpy

$$h := u + \frac{P}{\rho}$$

$\underbrace{}$ spec. internal energy $\underbrace{\phantom{\frac{P}{\rho}}}$ mechanical energy

with $du = T\,dS + \frac{P}{\rho^2}\,d\rho$

$$dh = du + \frac{dP}{\rho} - \frac{P}{\rho^2}\,d\rho$$

$$dh = T\,dS + \frac{dP}{\rho}$$

For $S = cte$ and $P = P(\rho)$ (barotropic EOS)

$$dh = \frac{dP}{\rho} = \frac{1}{\rho}\frac{\partial P}{\partial \rho}\,d\rho$$

$$h(\rho) = \int_0^\rho \frac{1}{\rho'}\frac{\partial P}{\partial \rho}(\rho')\,d\rho'$$

$\}$ using (in red)

$$\vec{\nabla} h(\rho) = \frac{\partial h}{\partial \rho}\vec{\nabla}\rho = \frac{1}{\rho}\frac{\partial P}{\partial \rho}(\rho)\vec{\nabla}\rho = \frac{1}{\rho}\vec{\nabla}P$$

Isolated fluid at equilibrium $\rho_{so}(\bar{x})$ $\bar{v}_{so}(x)$ $h_{so}(\bar{x})$ $\phi_{so}(x)$

solutions   of

① Continuity Equation

$$\frac{\partial \rho_{so}}{\partial t} + \hat{\nabla}(\rho_{so}\hat{v}_{so}) = 0$$

② Euler Equation

$$\frac{\partial}{\partial t}\hat{v}_{so} + (\bar{v}_{so}\cdot\hat{\nabla})\hat{v}_{so} = -\bar{\nabla}(h_{so} + \phi_{so})$$

③ The Poisson Equation

$$\nabla^2\phi_{so} = 4\pi G\,\rho_{so}$$

④ Equation of state

$$P_o = P(\rho_{so})$$

The response of the system to a weak perturbation $\qquad -\varepsilon \vec{\nabla} \phi_e$

$$\rho_{so}(\vec{x}) \quad \longrightarrow \quad \rho_s(\vec{x},t) = \rho_{so}(\vec{x}) + \varepsilon \rho_{s1}(\vec{x},t)$$

$$h_{so}(\vec{x}) \quad \longrightarrow \quad h_s(\vec{x},t) = h_{so}(\vec{x}) + \varepsilon h_{s1}(\vec{x},t)$$

$$\vec{v}_{so}(\vec{x}) \quad \longrightarrow \quad \vec{v}_s(\vec{x},t) = \vec{v}_{so}(\vec{x}) + \varepsilon \vec{v}_{s1}(\vec{x},t)$$

$$\phi_{so}(x) \quad \longrightarrow \quad \phi_s(x,t) = \phi_{so}(\vec{x}) + \varepsilon \phi_{s1}(\vec{x},t)$$

$$\Vert$$

$$\phi_{so}(x)_{tot} \quad \longrightarrow \quad \phi(x,t) = \phi_{so}(\vec{x}) + \varepsilon \phi_1(\vec{x},t)$$

system only, without the perturbation ⚠

} total pot. include the perturbation

$$= \quad \phi_{so}(\vec{x}) + \varepsilon \phi_{s1}(\vec{x},t) + \varepsilon \underbrace{\phi_e(\vec{x},t)}_{\text{contrib. of the ext. perturb.}}$$

$\longrightarrow$ insert those equations into the equations for a

self-gravitating fluid $\rho_s(\vec{x},t)$ $\phi_s(\vec{x},t)$ $\vec{v}_s(x,t)$ $h_s(\vec{x},t)$

$\longrightarrow$ use equations for the unperturbed fluid

$\longrightarrow$ keep only first order terms $\quad (\varepsilon^2 \to 0)$

22

We get a set of linear differential equations

→ time evolution of the perturbation

---

① "Continuity Equation"

$$\frac{\partial}{\partial t} \rho_{s1} + \vec{\nabla}(\rho_{s0}\, \vec{v}_{s1}) + \vec{\nabla}(\rho_{s1}\, \vec{v}_{s0}) = 0$$

$\rho_{s1}\ \ \vec{v}_{s1}\ \ \phi_{s1}\ \ h_{s1}$

6 unknowns

6 equations

② "Euler Equation"

$$\frac{\partial}{\partial t}\vec{v}_{s1} + \vec{v}_{s0}\cdot(\vec{\nabla}\cdot\vec{v}_{s1}) + \vec{v}_{s1}(\vec{\nabla}\cdot\vec{v}_{s0}) = -\vec{\nabla}\left(h_{s1} + \phi_{s1} + \phi_e\right)$$

$$= -\frac{\vec{\nabla}P_{s1}}{\rho_{s0}} - \vec{\nabla}(\phi_{s1} + \phi_e)$$

③ "Poisson Equation"

$$\vec{\nabla}\phi_{s1} = 4\pi G \rho_{s1}$$

④ "Equation of state"

$$h_{s1} = \left.\frac{d\,P(\rho)}{d\rho}\right|_{\rho_0} \frac{\rho_{s1}}{\rho_{s0}} = v_s^2\, \frac{\rho_{s1}}{\rho_{s0}}$$

23

Linearized Equations for a self-gravitating stellar system

Equations for a self-gravitating stellar system

System of DF $\qquad\qquad f_s(\bar{x}, \bar{v}, t) \longrightarrow f_s(\bar{x}, t) \quad \phi_s(\bar{x}, t)$

(without the perturbation)

① The collisionless Boltzmann Equation

total potential $\phi = \phi_s + \varepsilon f_e$

$$\frac{\partial f_s}{\partial t} + \hat{v} \frac{\partial f_s}{\partial \bar{x}} - \frac{\partial f}{\partial \bar{x}} \frac{\partial f_s}{\partial \bar{v}} = 0 \qquad\qquad = \frac{\partial f_s}{\partial t} + [f_s, u] = 0$$

$$u = \tfrac{1}{2} \bar{v}^2 + \phi(\bar{x}, t)$$

② The Poisson Equation

$$\nabla^2 \phi_s = 4\pi G \rho_s = 4\pi G \int d^3v \, f_s(\bar{x}, \bar{v}, t)$$

Isolated stellar system at equilibrium $\qquad f_o(\bar{x}, \bar{v}) \quad f_o(\bar{x}) \quad \phi_o(\bar{x})$

---

solutions of

① The collisionless Boltzmann Equation

$$\boxed{\frac{\partial f_{so}}{\partial t} + [f_{so}, u_{so}] = 0}$$

$$u_o = \frac{1}{2}\vec{v}^2 + \phi_{sc}(\bar{x})$$

② The Poisson Equation

$$\boxed{\nabla^2 \phi_{so} = 4\pi G \, \rho_{so} = 4\pi G \int d^3v \, f_{so}(\bar{x}, \bar{v}, t)}$$

The response of the system to a weak perturbation $\qquad -\varepsilon \bar{\nabla} \phi_e$

$$f_{s_0}(\bar{x}, \bar{v}, t) \longrightarrow f_s(\bar{x}, \bar{v}, t) = f_{s_0}(\bar{x}, \bar{v}) + \varepsilon f_{s_1}(\bar{x}, \bar{v}, t)$$

$$\rho_{s_0}(\bar{x}, t) \longrightarrow \rho_s(\bar{x}, t) = \rho_{s_0}(\bar{x}) + \varepsilon \rho_{s_1}(\bar{x}, t)$$

$$\phi_{s_0}(x, t) \longrightarrow \phi_s(x, t) = \phi_{s_0}(\bar{x}) + \varepsilon \phi_{s_1}(\bar{x}, t)$$

$$\overset{\text{III}}{\phi_{s_0}(x, t)} \longrightarrow \phi(x, t) = \phi_{s_0}(\bar{x}) + \varepsilon \phi_1(\bar{x}, t)$$

system only, without the perturbation ⚠

total pot. include the perturbation

$$\color{red}{= \phi_{s_0}(\bar{x}) + \varepsilon \phi_{s_1}(\bar{x}, t) + \varepsilon \phi_e(\bar{x}, t)}$$

contrib. of the ext. perturb.

$$H_0(\bar{x}, \bar{v}) \longrightarrow H_{tot}(x, t) = \tfrac{1}{2}\bar{v}^2 + \phi_0(\bar{x}) + \varepsilon \phi_1(\bar{x}, t)$$

include the perturbation

$$= H_0(\bar{x}, \bar{v}) + \varepsilon \phi_1(\bar{x}, t)$$

$\longrightarrow$ insert those equations into the equations for a

self-gravitating stellar system $\quad f_s(\bar{x}, \bar{v}, t) \; \rho_s(\bar{x}, t) \; \phi_s(\bar{x}, t)$

$\longrightarrow$ use equations for the unperturbed stellar system

$\longrightarrow$ keep only first order terms $\quad (\varepsilon^2 \to 0)$

26

We get a set of linear differential equations

$\rightarrow$ time evolution of the perturbation

① "The collisionless Boltzmann Equation"

δ_{s1} f_{s1} ϕ_{s1}

3 unknowns
3 equations

$$\frac{\partial \delta_{s1}}{\partial t} + \left[\delta_{s1}, H_{s0} \right] = \left[\delta_{s0}, \phi_1 \right]$$

$$\frac{\partial \delta_{s1}}{\partial t} + \frac{\partial \delta_{s1}}{\partial x} \bar{v} - \frac{\partial \delta_{s1}}{\partial \bar{v}} \frac{\partial \phi_{s0}}{\partial \bar{x}} = \frac{\partial \delta_{s0}}{\partial \bar{v}} \frac{\partial \phi_1}{\partial \bar{x}}$$

② "The Poisson Equation"

$$\nabla^2 \phi_{s1} = 4 \pi G \rho_{s1} = 4 \pi G \int d^3v \, \delta_{s1}$$

27

Interpretation

Reminder — Variation of $\rho_{s\wedge}$ along the flow (Lagrangian derivative)

$$\frac{\partial \rho_{s\wedge}}{\partial t} + \left[\rho_{s\wedge}, H_{so} \right] = \left[\rho_{so}, \phi_\wedge \right]$$

$$\underbrace{\frac{\partial \rho_{s\wedge}}{\partial t} + \frac{\partial \rho_{s\wedge}}{\partial \vec{x}} \vec{v} - \frac{\partial \rho_{s\wedge}}{\partial \vec{v}} \frac{\partial \phi_{so}}{\partial \vec{x}}}_{\frac{d}{dt} \rho_{s\wedge}} = \frac{\partial \rho_{so}}{\partial \vec{v}} \frac{\partial \phi_\wedge}{\partial \vec{x}}$$

Source

velocities   gravity

$P_v(\vec{x})$ may become big if σ is small

28

Stability of collisionless systems

The Jeans instability

(1902)

The Jeans instability

Homogeneous medium subject to a perturbation

- consider an infinite static homogeneous system at equilibrium
  - ⚠ no polytropic homogeneous system exists

$$\vec{V_0} = 0$$

Euler Equation
$$\cancel{\frac{\partial}{\partial t}\vec{v_0}} + \cancel{\left(\bar{v}\cdot\vec{\partial}\right)\hat{v_0}} = -\cancel{\frac{\vec{\partial}P_0}{\rho_0}} - \vec{\partial}\phi_0$$

$$\Rightarrow \quad \vec{\partial}\phi_0 = 0$$

Poisson Equation
$$\nabla^2\phi_0 = 4\pi G\rho_0 \qquad \Rightarrow \qquad \underline{\rho_0 = 0}$$

Linearized Poisson
$$\nabla^2\phi_s = 4\pi G\rho_s = \nabla^2(\phi_0 + \varepsilon\phi_{s1}) = 4\pi G(\rho_0 + \varepsilon\rho_{s1})$$

$$\boxed{\nabla^2\phi_{s1} = 4\pi G\rho_{s1}} \qquad\qquad \text{Jeans swindle}$$

The response of an homogeneous fluid

Linearized fluid equations $\rho_{sn}(\bar{x},t)$, $\phi_{sn}(\bar{x},t)$, $\vec{v}_{sn}(x,t)$, $h_{sn}(\bar{x},t)$

$$\left\{\begin{array}{l} \text{①} \quad \frac{\partial}{\partial t}\rho_{sn} \;+\; \vec{\nabla}(\rho_o\,\vec{v}_{sn}) \;+\; \vec{\nabla}(\rho_{sn}\,\cancel{\vec{v}_o}) \;=\; 0 \qquad\qquad \boxed{\color{red}\vec{v}_o = 0} \\[2mm] \text{②} \quad \frac{\partial}{\partial t}\vec{v}_{sn} \;+\; \vec{v}_o\cdot(\cancel{\vec{\nabla}\cdot\vec{v}_{sn}}) \;+\; \vec{v}_{sn}(\cancel{\vec{\nabla}\cdot\vec{v}_o}) \;=\; -\,\vec{\nabla}\big(h_{sn} + \phi_{sn} + \phi_e\big) \\[2mm] \text{③} \quad \vec{\nabla}\phi_{sn} \;=\; 4\pi\,G\,\rho_{sn} \\[2mm] \text{④} \quad h_{sn} \;=\; v_s^2\,\dfrac{\rho_{sn}}{\rho_o} \end{array}\right.$$

Solution

$$\frac{\partial}{\partial t}\text{①} \;:\; \frac{\partial^2}{\partial t^2}\rho_{sn} \;+\; \rho_o\,\vec{\nabla}\Big(\frac{\partial}{\partial t}\vec{v}_{sn}\Big) \;=\; 0$$

$$\vec{\nabla}\cdot\text{②} \;:\; \vec{\nabla}\cdot\frac{\partial\vec{v}_{sn}}{\partial t} \;=\; -\,\nabla^2\big(h_{sn} + \phi_{sn} + \phi_e\big)$$

Thus, we have $\big(\ \vec{\nabla}\cdot\text{②}\ \text{into}\ \frac{\partial}{\partial t}\text{①}\ \big)$

$$\frac{\partial^2}{\partial t^2}\rho_{sn} \;-\; \rho_o\,\nabla^2\big(h_{sn} + \phi_{sn} + \phi_e\big) \;=\; 0$$

31

$$\frac{\partial^2}{\partial t^2} \rho_{sn} - \rho_o \nabla^2 (h_{sn} + \phi_{sn} + \phi_e) = 0$$

$$\frac{\partial^2}{\partial t^2} \rho_{sn} - \rho_o \nabla^2 h_{sn} - \rho_o \nabla^2 \phi_{sn} = \rho_o \nabla^2 \phi_e$$

(*) $\quad \vec{\nabla} h_{sn} = \frac{1}{\rho_s} \vec{\nabla} \rho_{sn}$

$\bigg)$ (*)

$$\frac{1}{\rho_o} \frac{\partial^2}{\partial t^2} \rho_{sn} - \vec{\nabla} \cdot \left(\frac{\vec{\nabla} \rho_{sn}}{\rho_s} \right) - \vec{\nabla} \cdot \left(\vec{\nabla} \phi_{sn} \right) = \vec{\nabla} \cdot \left(\vec{\nabla} \phi_e \right)$$

$\underbrace{}$ $\underbrace{}$ $\underbrace{}$ $\underbrace{}$

growth accel. $F_{pressure}$ $F_{grav, self}$ $F_{grav, ext}$ (spec. forces)
of ρ_{sn}

$$\frac{\partial^2}{\partial t^2} \rho_{sn} - \rho_o \nabla^2 h_{sn} - \rho_o \nabla^2 \phi_{sn} = \rho_o \nabla^2 \phi_e$$

④ ③
$\underbrace{}$ $\underbrace{}$ $\underbrace{}$

$\quad - V_s^2 \nabla^2 \rho_{sn} \quad - 4\pi G \rho_o \rho_{sn} \quad 4\pi G \rho_o \rho_e$

$\nabla^2 \phi_e = 4\pi G \rho_e$

$$\frac{\partial^2}{\partial t^2} \rho_{sn} - V_s^2 \nabla^2 \rho_{sn} - 4\pi G \rho_o \rho_{sn} = 4\pi G \rho_o \rho_e$$

$\underbrace{}$ $\underbrace{}$ $\underbrace{}$

pressure   self grav.   ext. grav.

Differential equation for ρ_{sn} only

Evolution of a perturbation in an homogeneous fluid

32

Some simple cases

$$\frac{\partial^2}{\partial t^2} \rho_{s1} - v_s^2 \nabla^2 \rho_{s1} - 4\pi G \rho_0 \rho_{s1} = 4\pi G \rho_0 \rho_e$$

1) no gravity, no ext forces

$$\frac{\partial^2}{\partial t^2} \rho_{s1} - v_s^2 \nabla^2 \rho_{s1} = 0$$

In 1-D $\frac{\partial^2}{\partial t^2} \rho_{s1}(x,t) - v_s^2 \frac{\partial^2}{\partial x^2} \rho_{s1}(x,t) = 0$ (wave equation)

general solution : f : an arbitrary function

$$\rho_{s1}(x,t) = f(x - v_s t) + f(x + v_s t)$$

$\underbrace{\qquad\qquad}$ $\underbrace{\qquad\qquad}$
move at   move at
speed v_s speed $-v_s$

sound wave traveling at sound speed v_s

ρ_{s1}
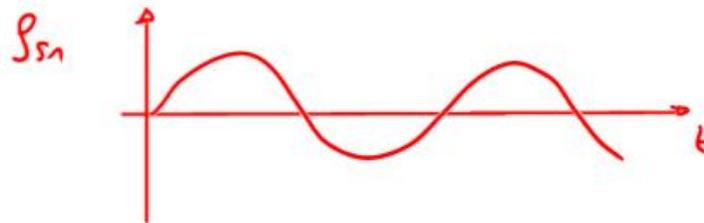
oscillatory
solution

STABLE

33

$$\frac{\partial^2}{\partial t^2} \rho_{sn} - v_s^2 \nabla^2 \rho_{sn} - 4\pi G \rho_0 \rho_{sn} = 4\pi G \rho_0 \rho_e$$

2) no pressure, no external forces

$$\frac{\partial^2}{\partial t^2} \rho_{sn} = 4\pi G \rho_0 \rho_{s2}$$

Solutions :   any linear combinaison of

$$\rho_{sn}^{\textcircled{1}}(x,t) = \tilde{\rho}(\bar{x}) \; e^{\sqrt{4\pi G \rho_0}\, t} \quad \text{and} \quad \rho_{sn}(x,t) = \tilde{\rho}(\bar{x}) \; e^{-\sqrt{4\pi G \rho_0}\, t}$$

$\rho_{sn}^{\textcircled{1}}$ UNSTABLE $\rho_{sn}^{\textcircled{2}}$ DAMPED
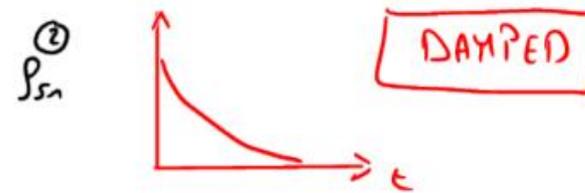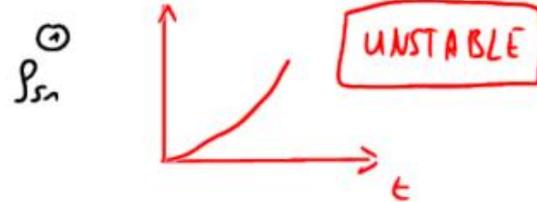
3) no external forces

$$\frac{\partial^2}{\partial t^2} \rho_{sn} - v_s^2 \nabla^2 \rho_{sn} - 4\pi G \rho_0 \rho_{sn} = 0$$

          pressure   self-gravity

          <u>stabilize</u>   <u>destabilize</u>

34

General solutions

$$e^{i(\vec{k}\vec{x} - \omega t)}$$

- $x \; e^{-i\vec{k}\vec{x}} \; + \; \int d^3\vec{x} \quad = 0 \qquad$ spatial Fourier transform

$$\bar{\rho}_{s_\Lambda}(\vec{k},t) = \int d^3x \; e^{-i\vec{k}\vec{x}} \; \rho_{s_\Lambda}(\vec{x},t)$$

$$\frac{\partial^2}{\partial t^2} \bar{\rho}_{s_\Lambda}(\vec{k},t) \; + \; v_s^2 k^2 \, \bar{\rho}_{s_\Lambda}(\vec{k},t) \; - \; 4\pi G \rho_0 \, \bar{\rho}_{s_\Lambda}(\vec{k},t) \; = \; 4\pi G \rho_e \, \bar{\rho}_e(\vec{k},t)$$

Evolution in absence of perturbation (after the)

$$\rho_e(x,t) = 0 \qquad \bar{\rho}_e = 0$$

Solutions are in the form

$$\bar{\rho}_{s_\Lambda}(\vec{k},t) \sim e^{-i\omega t} \qquad \begin{array}{l} \omega = \omega(|\vec{k}|) \in \mathbb{C} \\ k = |\vec{k}| \end{array}$$

Introducing in the previous equation, we get the dispersion relation

$$\omega^2(k) = v_s^2 k^2 \; - \; 4\pi G \rho_0$$

35

Definition : Jeans wave number $\boxed{k_J^2 = \dfrac{4\pi G \rho_o}{v_s^2}}$

$\boxed{\omega^2(k) = v_s^2 k^2 - 4\pi G \rho_o} \longrightarrow \boxed{\omega^2(k) = v_s^2(k^2 - k_J^2)}$

$\rho_{S_n} \sim e^{ikx} e^{-i\omega t}$ $(k \in \mathbb{R}e)$

- If $k > k_J \rightarrow \omega^2 > 0 \rightarrow \omega \in \mathbb{R}e$ $\quad \bar{\rho}_{S_n} \sim e^{-i\omega t}$ $\quad \bar{\rho}_{S_n} \sim$

  A perturbation with a short wavelenght $(k > k_J)$ will see its amplitude oscillate

- If $k < k_J \rightarrow \omega^2 < 0 \rightarrow \omega \in \mathbb{I}m$ $\quad \bar{\rho}_{S_n} \sim e^{-\omega' t}$ $\quad \omega' = i\omega \in \mathbb{R}e$ $\quad \bar{\rho}_{S_n} \sim$

  A perturbation with a long wavelenght $(k < k_J)$ will see its

36

More discussion

$$k_J^2 = \frac{4\pi G \rho_0}{V_s^2} \qquad \omega^2(k) = V_s^2(k^2 - k_J^2)$$

Assume a perturbation with a fixed wave number k.

① $\rho_0 = 0$ (no gravity) $\qquad \Rightarrow k_J = 0 \qquad \qquad \underline{k > k_J}$

$$\omega^2(k) = V_s^2 k^2 \in \mathbb{R}_e \qquad \text{dispersion relation}$$
for a sound wave

$$\bar{\rho}_{s_1} \sim e^{-i\omega t} \qquad \text{oscillation at freqn. } \omega$$

$\boxed{\text{STABLE}}$

② $\rho_0 \neq 0$

$\underline{k > k_J}$ $\qquad \omega$ decreases with k increases

$\boxed{\text{STABLE}}$

$\underline{k = k_J}$ $\qquad \omega = 0$

$\underline{k < k_J}$ $\qquad \omega^2 \in \bar{I}_m$ $\qquad \boxed{\text{UNSTABLE}}$

③ $\rho \gg V_s^2/4\pi G$ (no pressure) $\quad k_J \to \infty$ $\qquad \boxed{\text{UNSTABLE}}$

37

Definitions

Jeans length

$$\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi V_s^2}{G \rho_o}}$$

$\lambda > \lambda_J \quad \Rightarrow \quad$ UNSTABLE

Jeans mass

Mass inside a sphere of

radius $\frac{1}{2} \lambda_J$

$$M_J = \frac{4}{3} \pi \left(\frac{1}{2}\lambda_J\right)^3 \rho_o$$

$$M_J = \frac{\pi^{5/2}}{6} \frac{V_s^3}{G^{3/2} \rho_o^{1/2}}$$

"Minimum mass that can collapse"
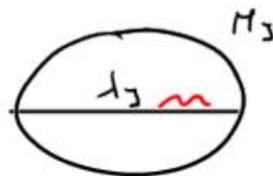
any perturbation with
$\lambda < \lambda_J$ i.e. involving a
mass $< M_J$ will be STABLE

38

The Jeans instability in fluid systems

$$\lambda_{\mathrm{J}} = 1.50 \ \lambda = 0.25$$

$$\lambda_{\mathrm{J}} = 1.50 \;\; \lambda = 0.50$$

$$\lambda_{\mathrm{J}} = 0.25 \ \lambda = 0.25$$

The Jeans instability in fluid systems

$$\lambda_{\mathrm{J}} = 0.25 \ \lambda = 0.50$$
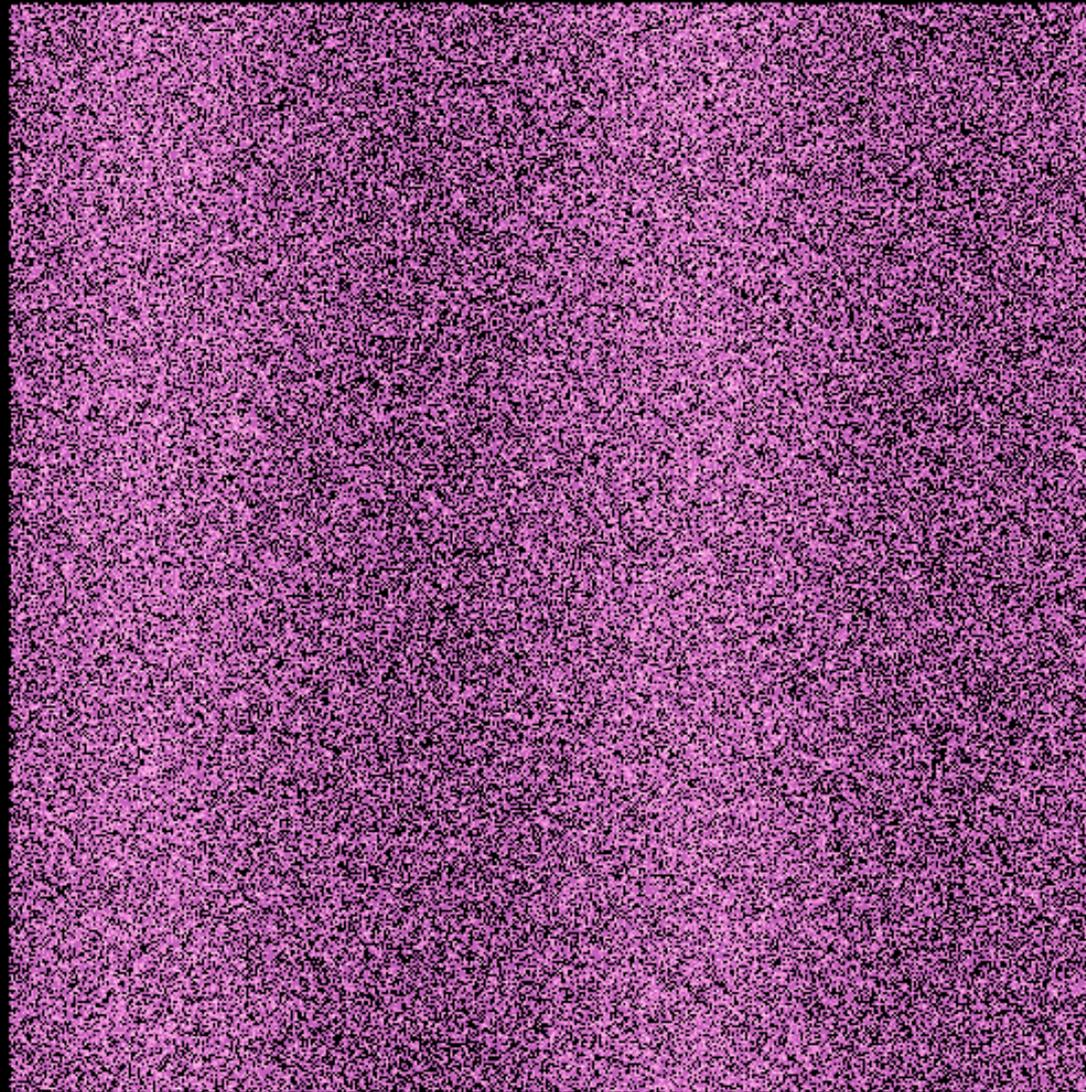
The response of an homogeneous stellar system

Linearized equations

$$\text{①} \quad \frac{\partial \delta_{s_1}}{\partial t} + [\delta_{s_1}, H_0] = [\delta_0, \phi_1]$$

$$\equiv \quad \frac{\partial \delta_{s_1}}{\partial t} + \frac{\partial \delta_{s_1}}{\partial \vec{x}} \vec{v} - \cancel{\frac{\partial \delta_{s_1}}{\partial \vec{v}} \frac{\partial \phi_0}{\partial \vec{x}}} = \frac{\partial \delta_0}{\partial \vec{v}} \frac{\partial \phi_1}{\partial \vec{x}} \equiv \frac{\partial \delta_0}{\partial \vec{v}} \frac{\partial}{\partial \vec{x}} \left(\phi_{s_1} + \phi_e \right)$$

$$\textcolor{red}{\phi_0 = 0}$$

$$\text{②} \quad \nabla^2 \phi_{s_1} = 4\pi G \delta_{s_1} = 4\pi G \int d^3\vec{v} \, \delta_{s_1}$$

Manipulation + spacial Fourier space

$$\bar{\delta}_1(\vec{k}, \vec{v}, t) = i\vec{k} \cdot \frac{\partial \delta_0}{\partial \vec{v}} \int_{-\infty}^{t} dt' \, e^{i\vec{k}\cdot\vec{v}(t'-t)} \left[\bar{\phi}_{s_1}(\vec{k}, t') + \bar{\phi}_e(\vec{k}, t') \right]$$

$$\textcolor{red}{\triangle \quad \text{The DF depends on the past history} \quad \left(\int_{-\infty}^{t} \phi_{s_1} + \phi_e \right)}$$

43

Integration over $d^3\vec{v}$ + Poisson

$$\bar{\rho}_{s_1}(\vec{k},t) = -\frac{4\pi G}{k^2} i \int d^3\vec{v} \; \vec{k} \cdot \frac{\partial f_o}{\partial \vec{v}} \int_{-\infty}^{t} dt' \; e^{i\vec{k}\cdot\vec{v}(t'-t)} \left[\bar{\rho}_{s_1}(k,t') + \bar{\rho}_e(k,t') \right]$$

In temporal Fourier space

This term may diverge if $\vec{k}\cdot\vec{v} = \omega$

$$\tilde{\bar{\rho}}_{s_1}(\vec{k},\omega) = \int dt \; \bar{\rho}_{s_1}(\vec{k},t) e^{-i\omega t}$$

$$\tilde{\bar{\rho}}_{s_1}(\vec{k},\omega) = \left(-\frac{4\pi G}{k^2} \int \frac{d^3\vec{v}}{\vec{k}\cdot\vec{v}-\omega} \; \vec{k} \cdot \frac{\partial f_o}{\partial \vec{v}} \right) \left(\tilde{\bar{\rho}}_{s_1}(\vec{k},\omega) + \tilde{\bar{\rho}}_e(\vec{k},\omega) \right)$$

In absence of perturbation $\;\;\; \rho_e = 0$

we must have:

$$\boxed{ -\frac{4\pi G}{k^2} \int \frac{d^3\vec{v}}{\vec{k}\cdot\vec{v}-\omega} \; \vec{k} \cdot \frac{\partial f_o}{\partial \vec{v}} = 1 }$$

$Im(\omega) > 0$

(instead, we may have a divergence)

This is our dispertion relation

$$\omega = \omega(\vec{k}, f_o)$$

44

Assuming a Maxwellian for the unperturbed DF f_0

$$f_0(\vec{v}) = \frac{f_0}{(2\pi\sigma^2)^{3/2}} e^{-\frac{v^2}{2\sigma^2}}$$

The dispersion relation becomes

$$-\frac{4\pi G}{k^2} \int \frac{d^3\vec{v}}{\vec{k}\cdot\vec{v} - \omega} \vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}} = 1 \quad \Rightarrow \quad \frac{4\pi G f_0}{k^2\sigma^2}\left(1 + \omega' Z(\omega')\right) = 1$$

with
_____

$$\omega' = \sqrt{2}\, k\sigma\,\omega \qquad Z(\omega') = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} ds \frac{e^{-s^2}}{s - \omega'} = i\sqrt{\pi}\, e^{-\omega'^2}\left[1 + \text{erf}(i\omega')\right]$$

$$k = |\vec{k}| \qquad\qquad \text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z dt\, e^{-t^2}$$

45

The dispersion relation is $\qquad Im(w) > 0$

$$\frac{4\pi G \rho_0}{k^2 \sigma^2} \left(1 + w' \, z(w') \right) = 1$$

$$w' = \sqrt{2} \, k \sigma w$$

$$z(w') = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds \, \frac{e^{-s^2}}{s - w'}$$

Defining $\qquad k_J^2 = \dfrac{4\pi G \rho_0}{\sigma^2}$

$$\frac{k^2}{k_J^2} = 1 + w' \cdot z(w')$$

$$\frac{w^2}{\xi(w')} = \sigma^2 \left(k^2 - k_J^2 \right)$$

one k is hidden here

fluid $\qquad k_J^2 = \dfrac{4\pi G \rho_0}{c_s^2}$

$$\frac{k^2}{k_J^2} = 1 + \frac{w^2}{k_J^2 c_s^2}$$

$$w^2 = c_s^2 \left(k^2 - k_J^2 \right)$$

? for $\qquad Im(w) = 0 \qquad$ or $\qquad Im(w) < 0 \qquad$ (a bit more tricky)
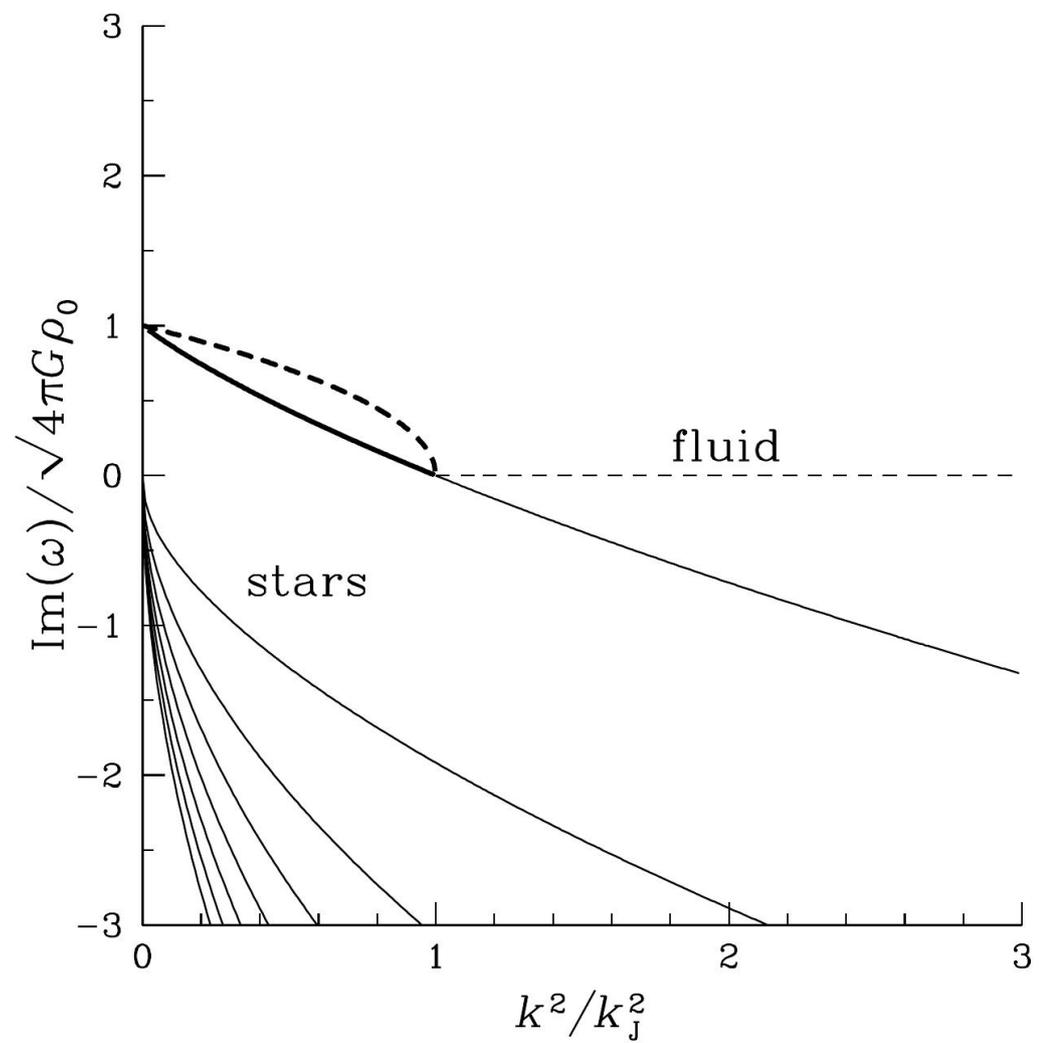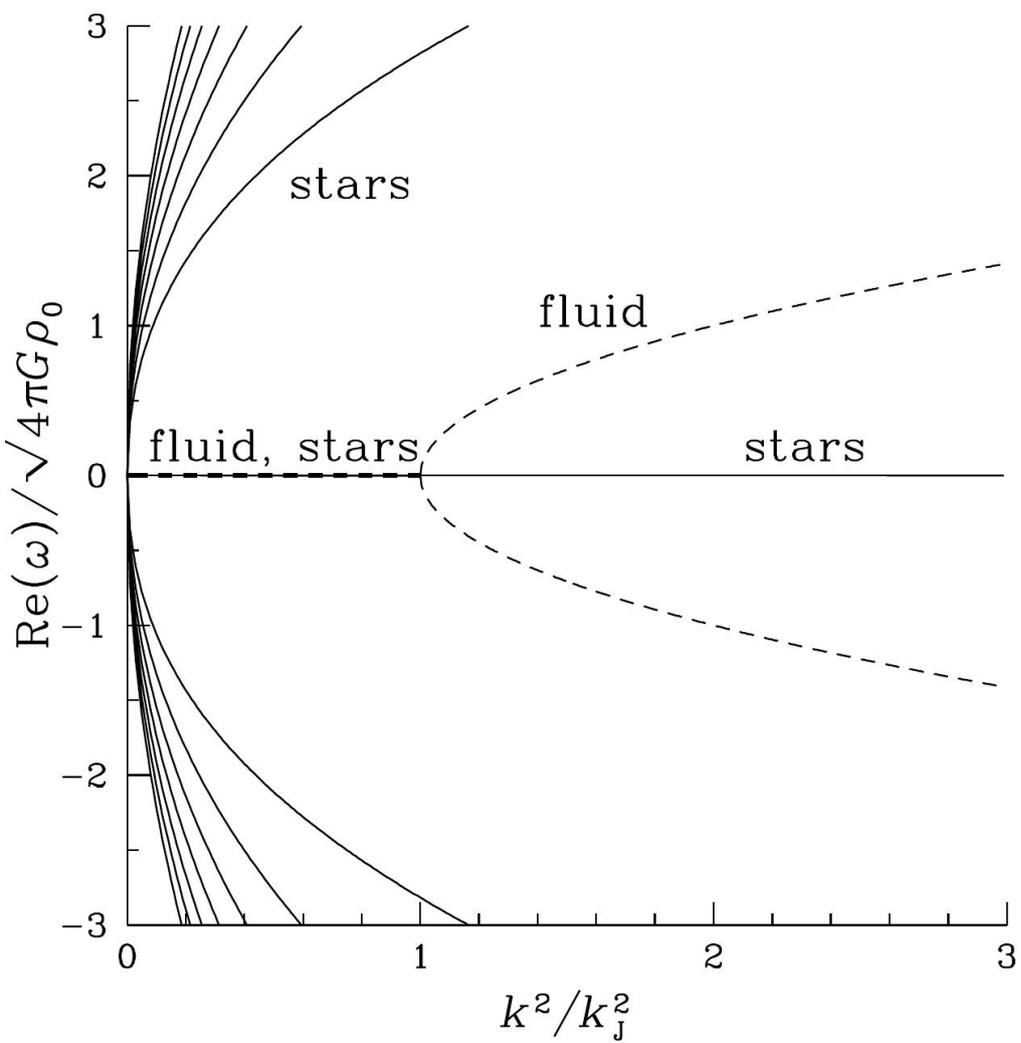
The dispersion relation for fluids
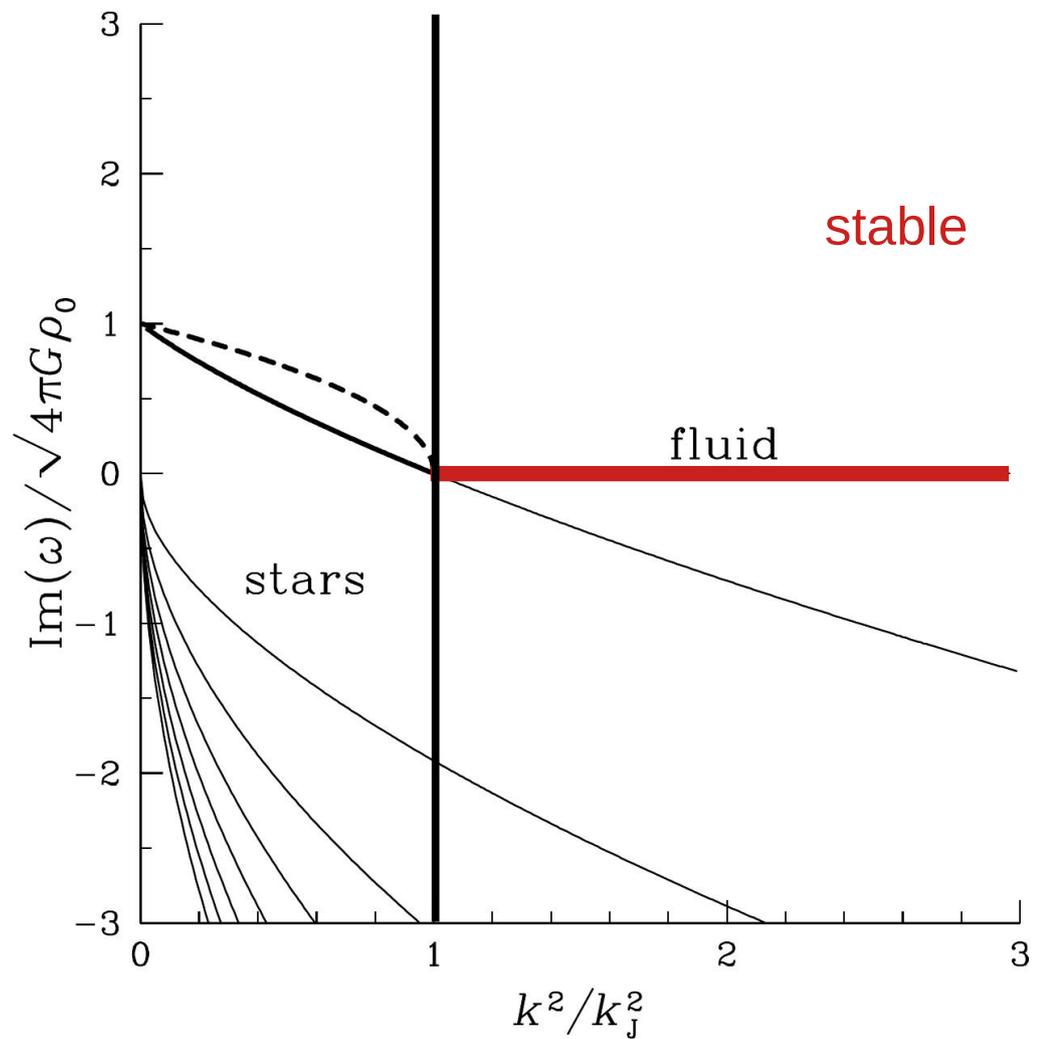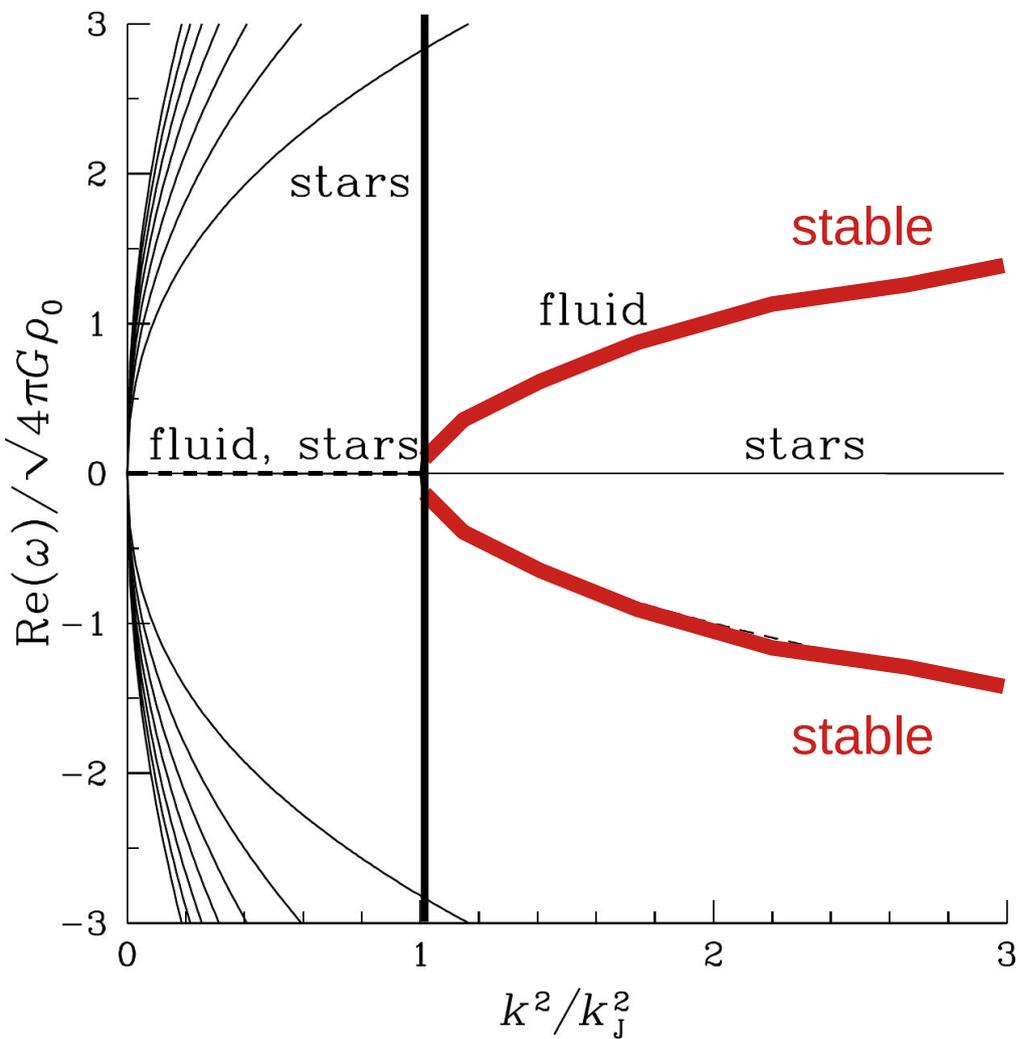
$$\frac{k^2}{k_J^2} = 1 + \frac{\omega^2}{k_J^2 v_s^2}$$

The dispersion relation for fluids

$$\frac{k^2}{k_J^2} = 1 + \frac{\omega^2}{k_J^2 v_s^2}$$

The dispersion relation for stellar systems

$$\frac{k^2}{k_J^2} = 1 + \omega' \cdot \mathcal{Z}(\omega')$$

The dispersion relation for stellar systems

$$\frac{k^2}{k_J^2} = 1 + \omega' \cdot Z(\omega')$$

51

The dispersion relation for stellar systems

$$\frac{k^2}{k_J^2} = 1 + \omega' \cdot z(\omega')$$
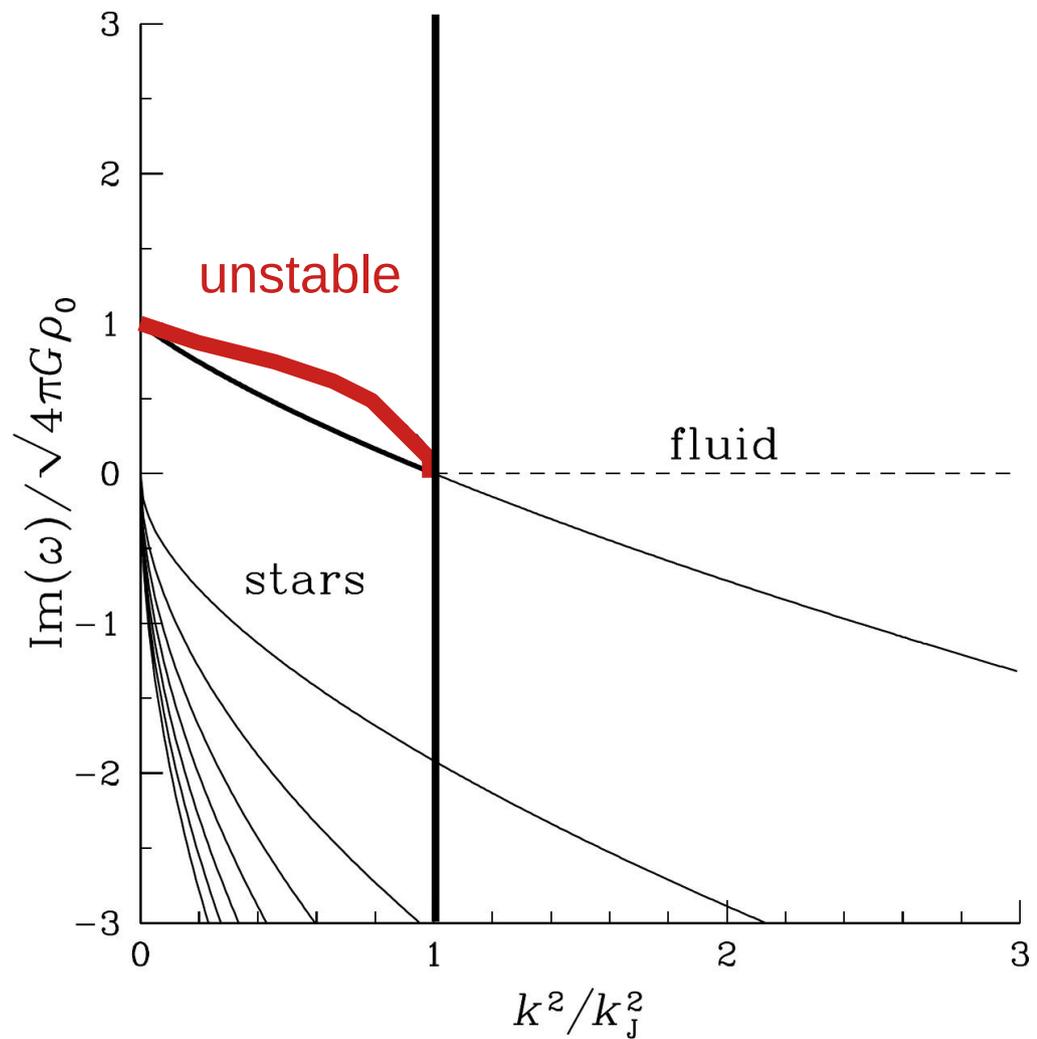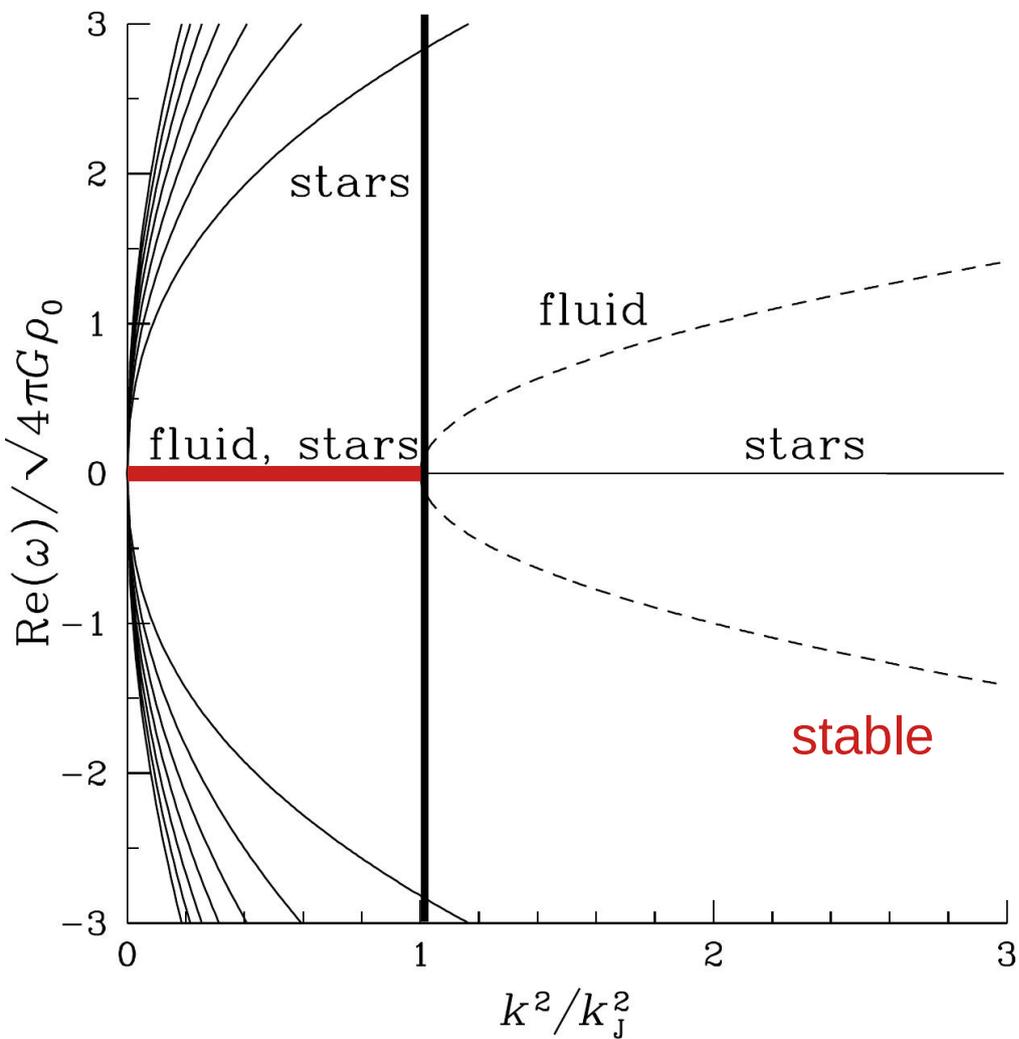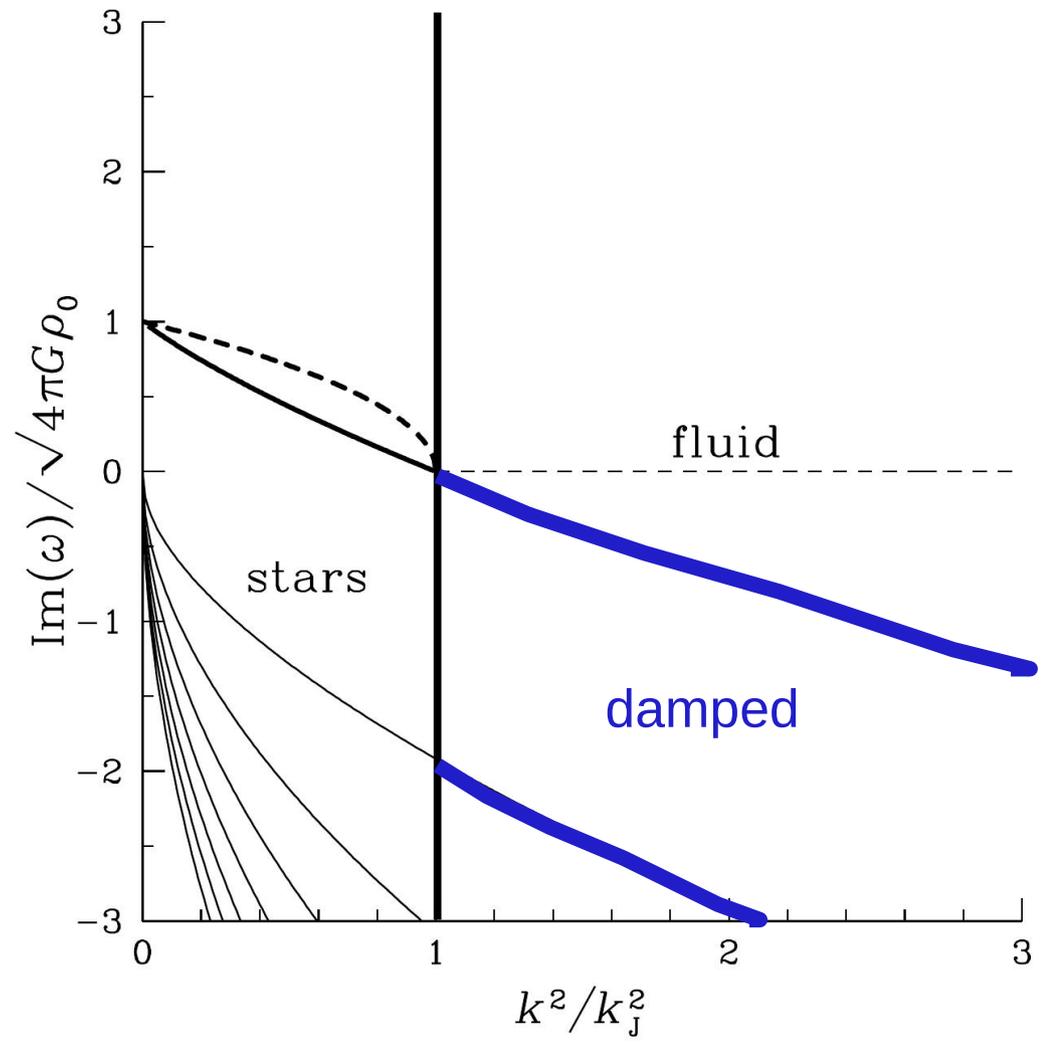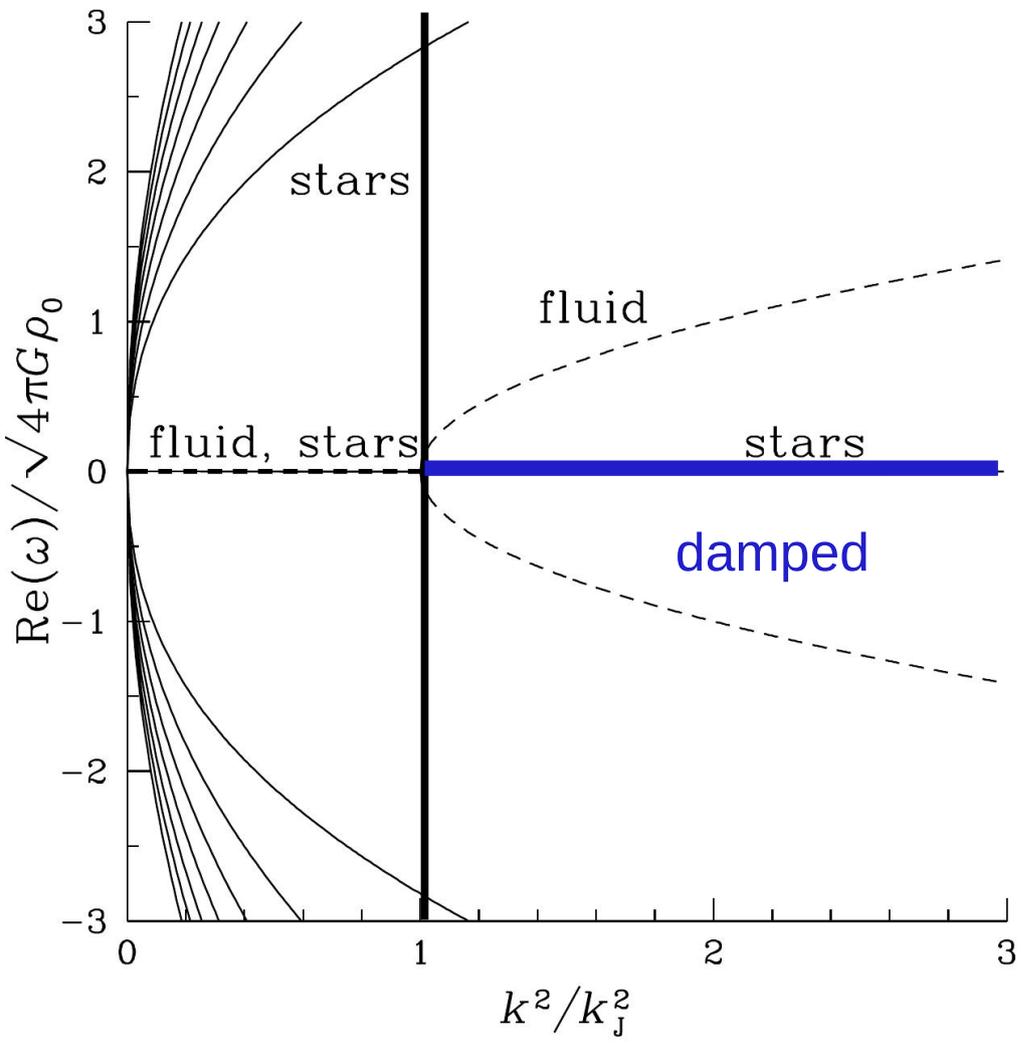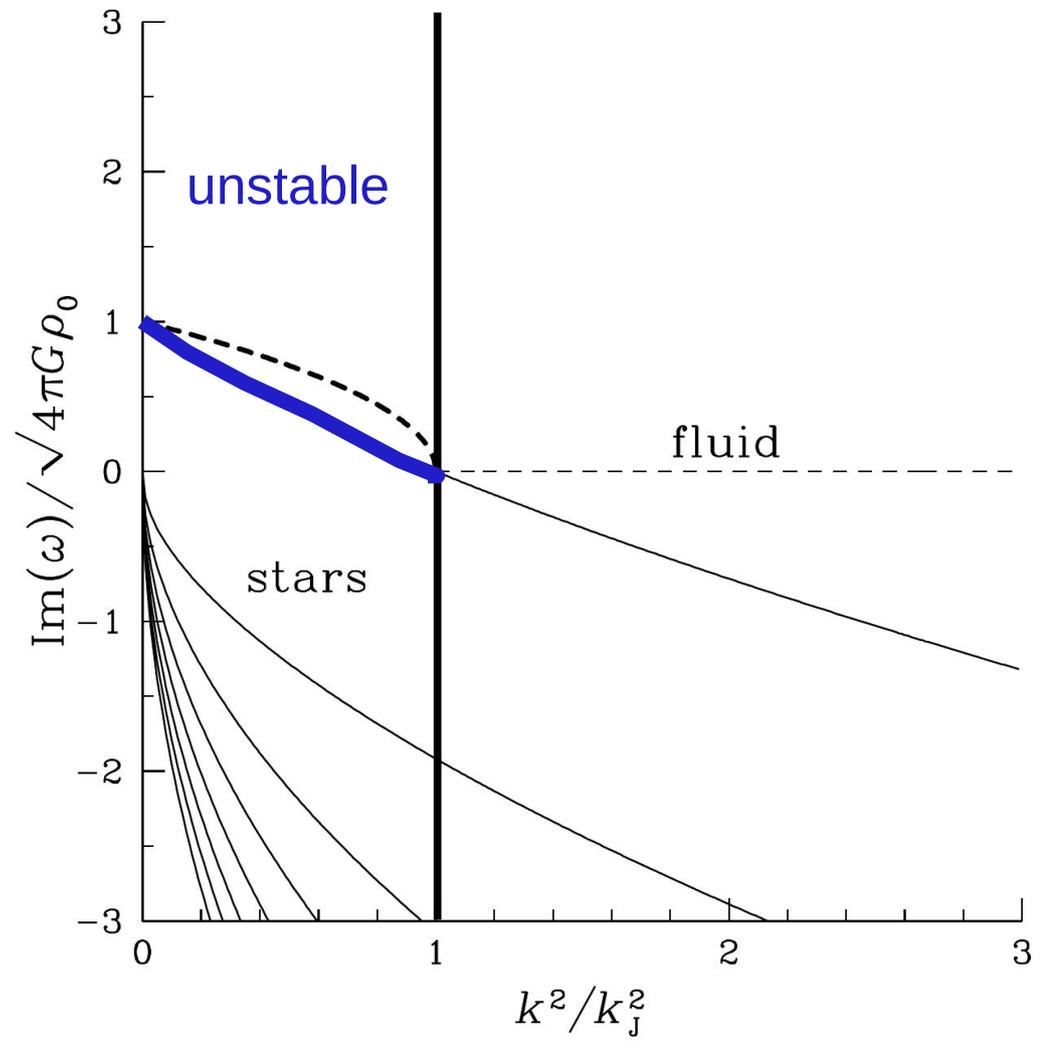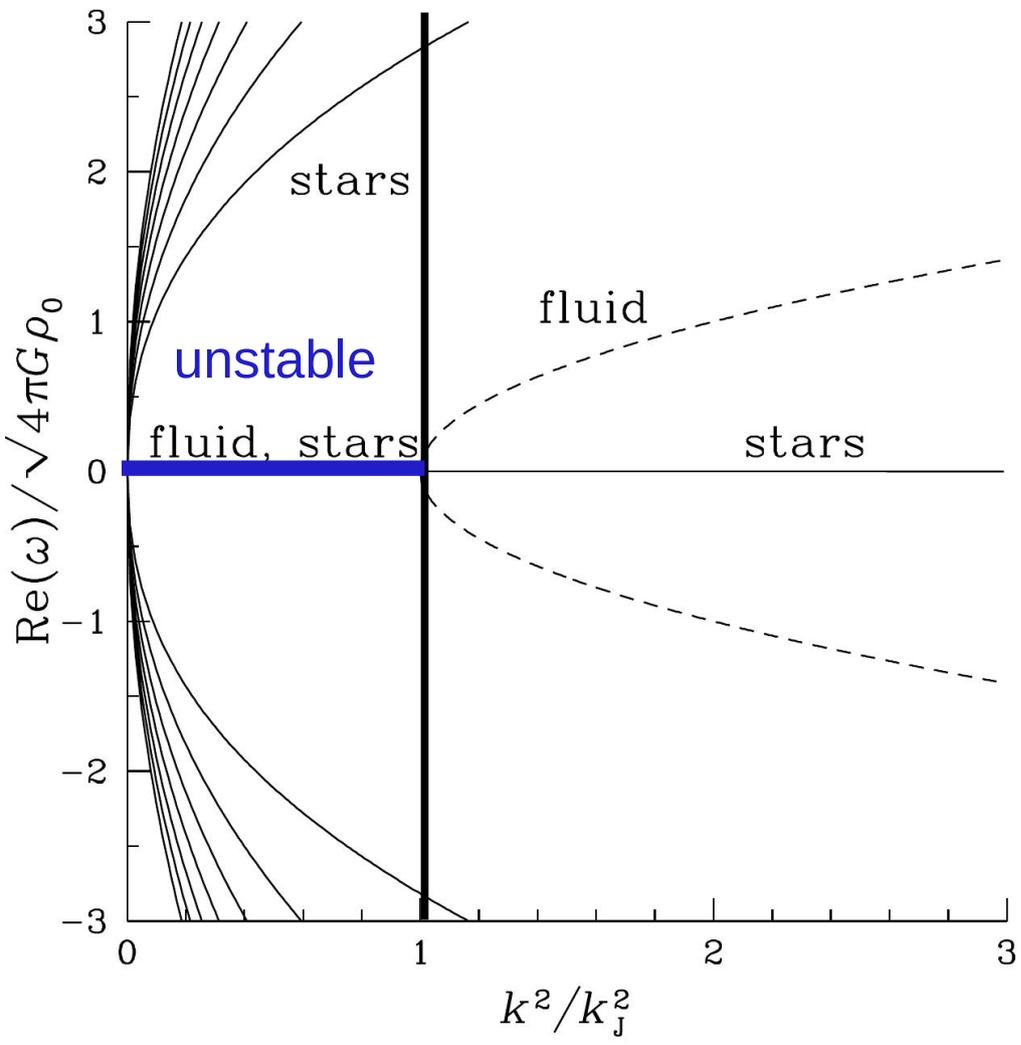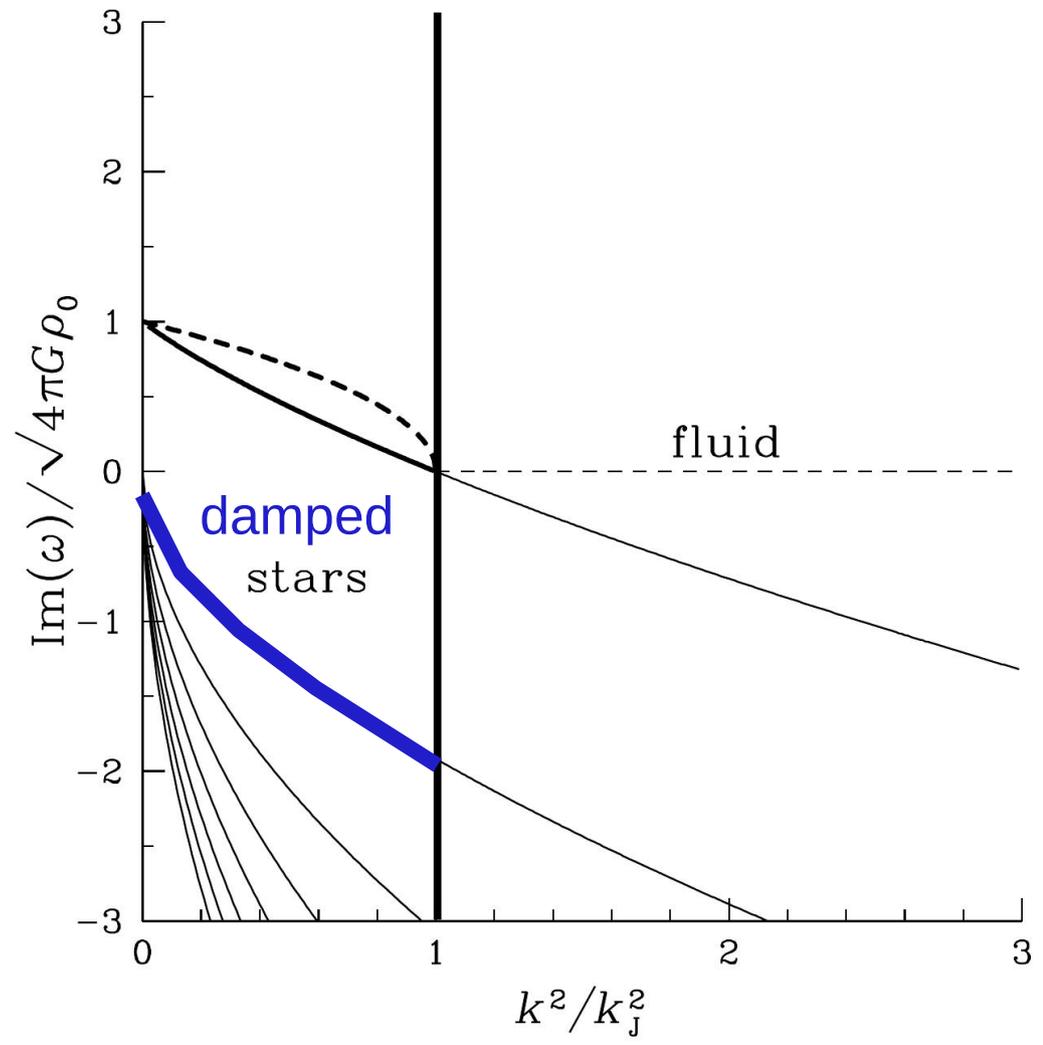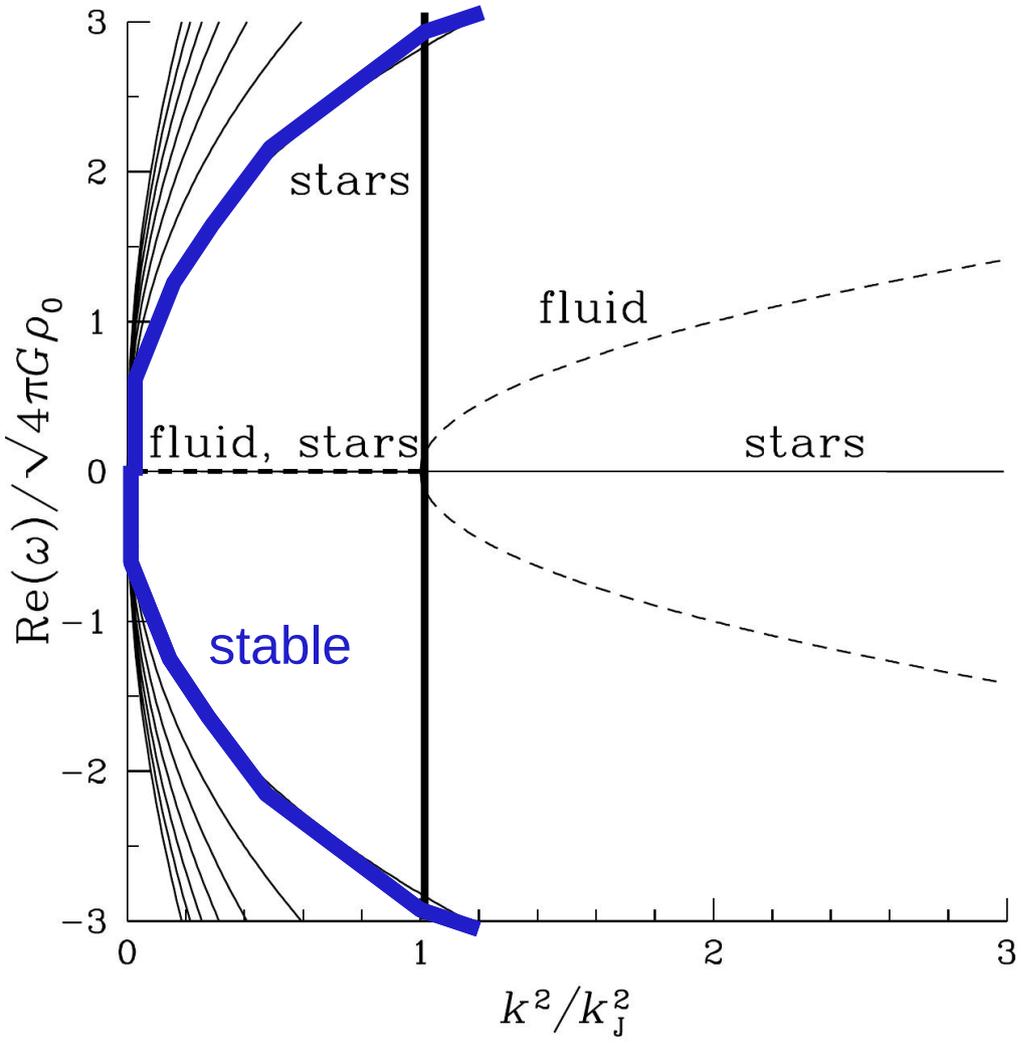
Stability

$$\tilde{\bar{\rho}} \sim e^{-i\omega t}$$

(A) **Unstable** $Im(\omega) > 0, \quad Re(\omega) = 0$

$k^2 < k_J^2$ (same than hydro)

(B) **Stable** $Im(\omega) = 0, \quad Re(\omega) \neq 0$

$\omega = 0, \quad k = k_J$ $\Rightarrow$ **no stable solution**

(C) **Damped** $Im(\omega) < 0$
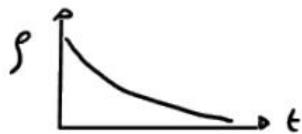
$\longrightarrow$ plenty of damped solutions
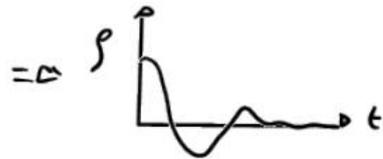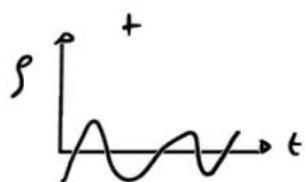
(with or without oscillation)

53

$$\lambda_{\mathrm{J}} = 1.50 \; \lambda = 0.25$$

The Jeans instability in stellar systems

$$\lambda_{\mathrm{J}} = 1.50 \; \lambda = 0.50$$
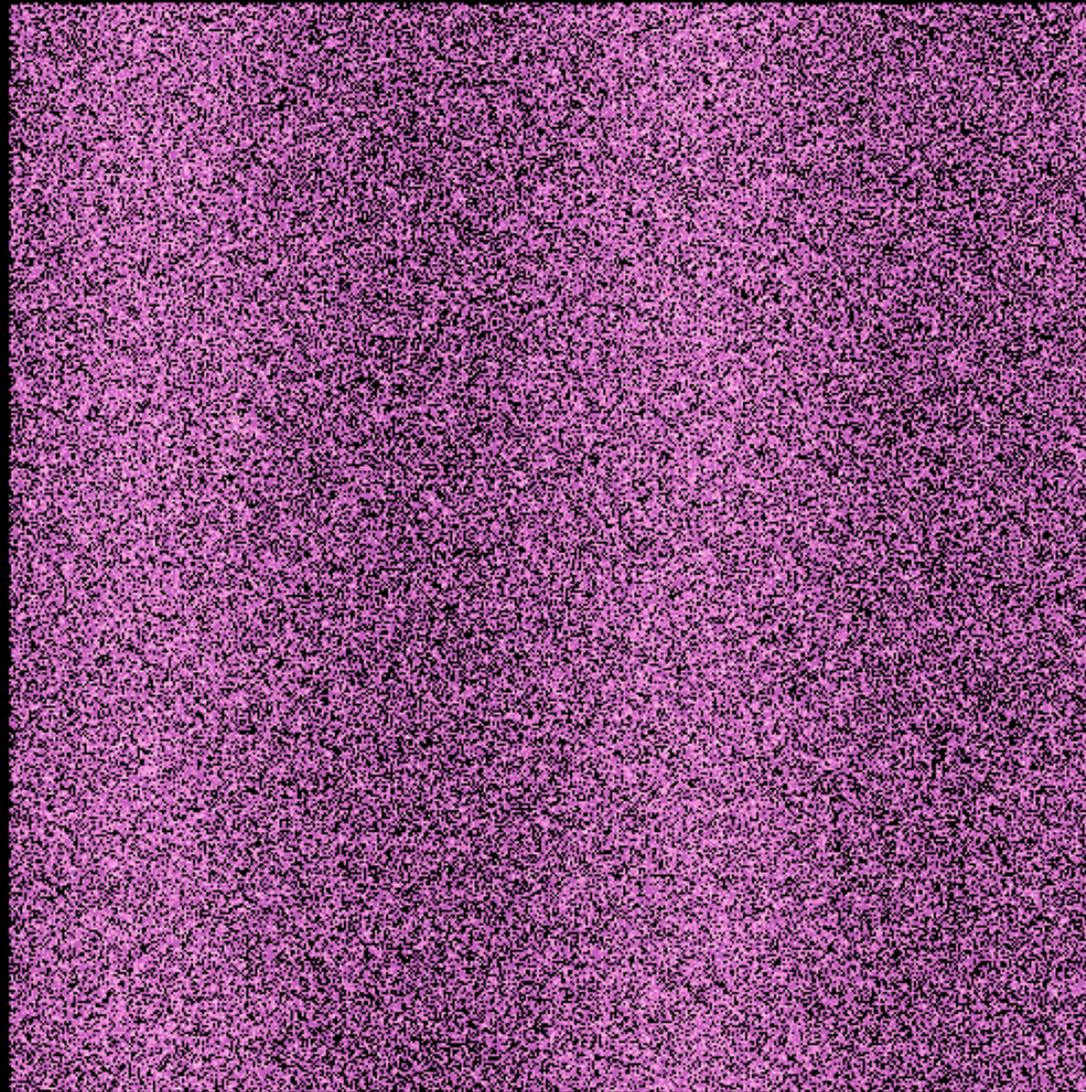
The Jeans instability in stellar systems

$$\lambda_{\mathrm{J}} = 0.25 \; \lambda = 0.25$$
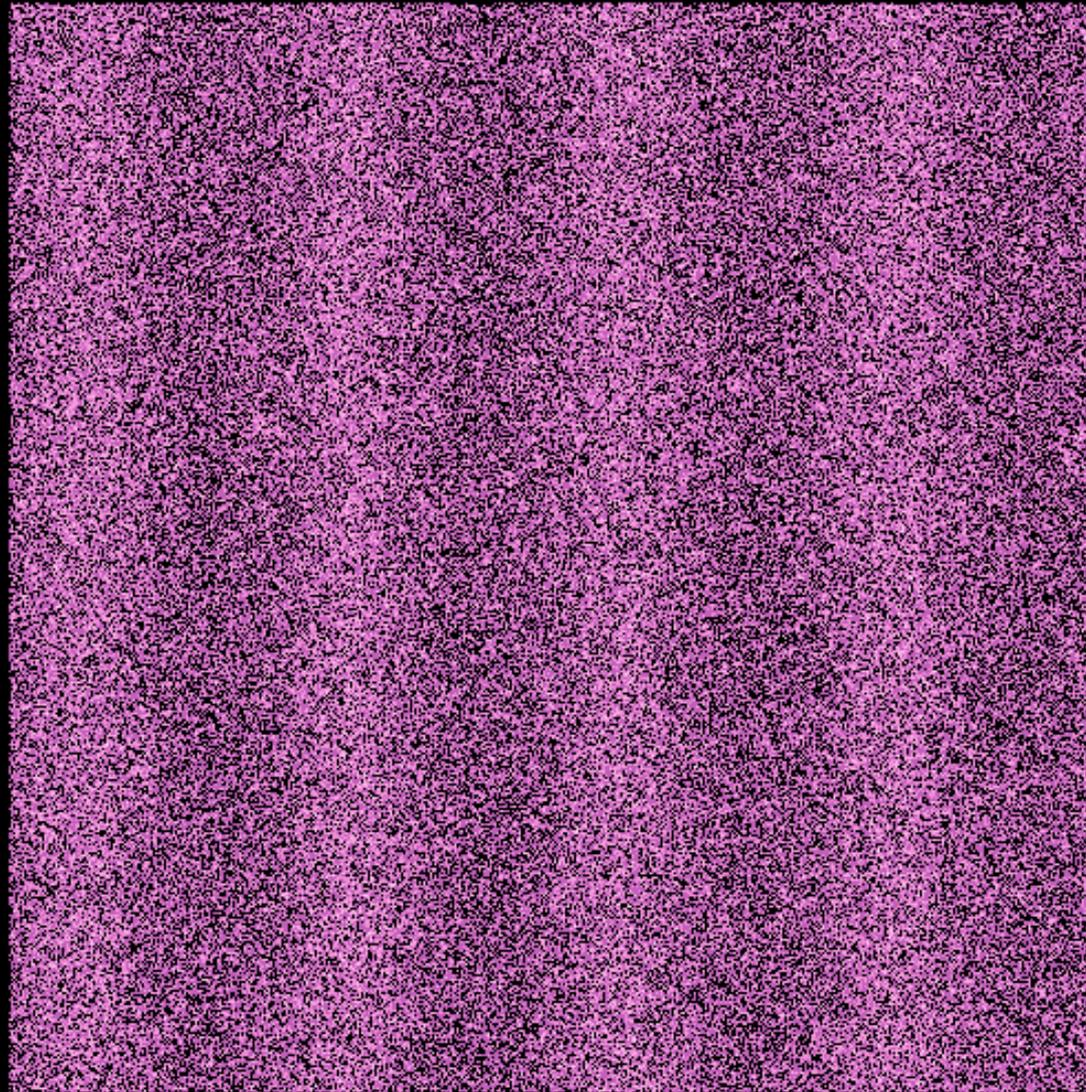
The Jeans instability in stellar systems

$$\lambda_{\mathrm{J}} = 0.25 \; \lambda = 0.50$$
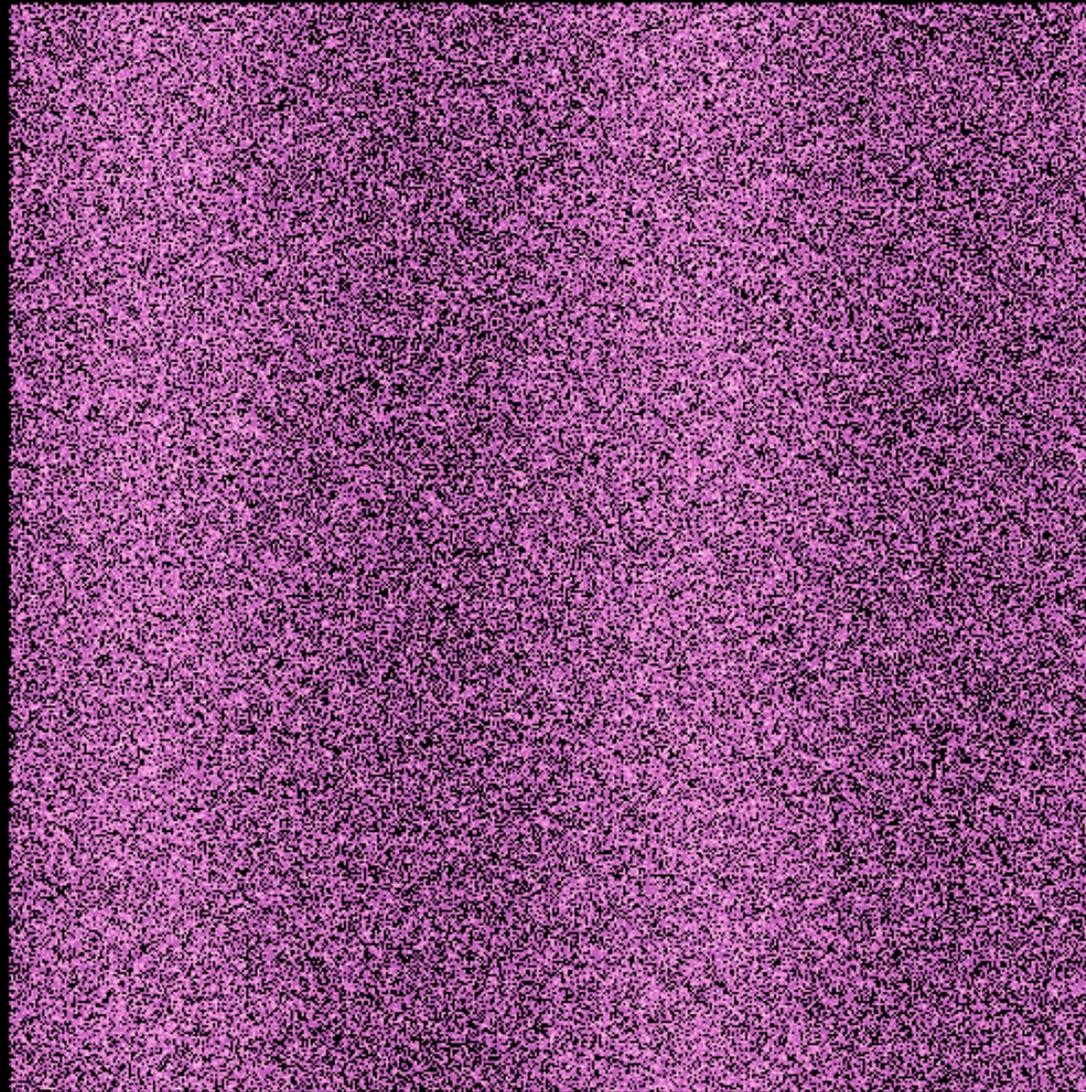
No perturbation : the instability is triggered by the noise due to the discretisation

$$\sigma = 0.1, r_{\mathrm{J}} = 0.005 \qquad \sigma = 0.3, r_{\mathrm{J}} = 0.05 \qquad \sigma = 0.7, r_{\mathrm{J}} = 0.25$$
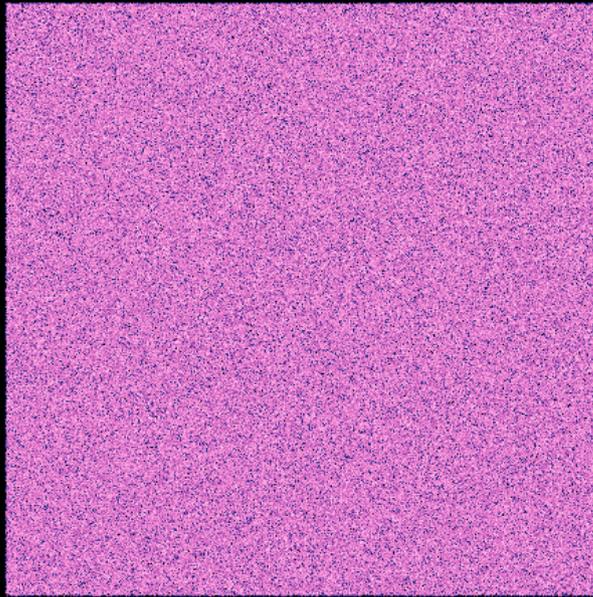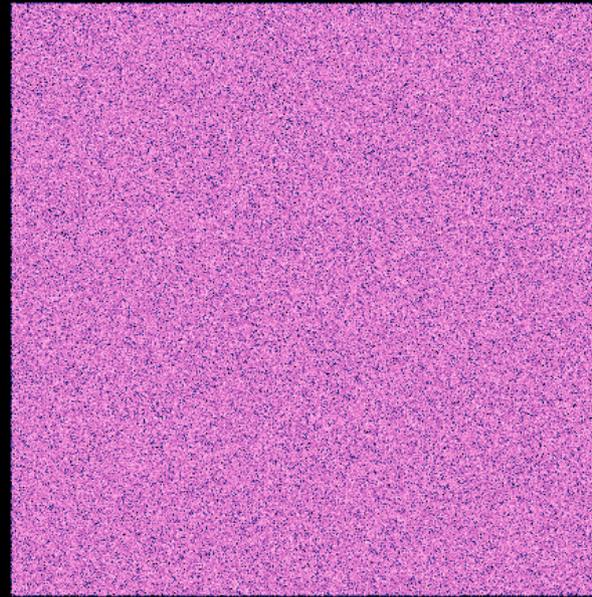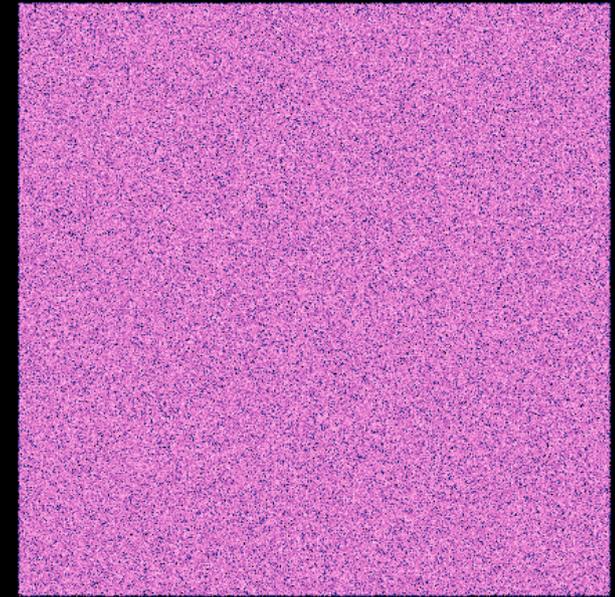
Landau damping

The existence of solutions with $Im(\omega) < 0$ is due to

$$\tilde{\rho}_{s1}(\bar{k}, \omega) = \left(-\frac{u_n G}{k^2} \int \frac{d^3\bar{v}}{\boxed{\bar{k} \cdot \bar{v} - \omega}} \; \bar{k} \cdot \frac{\partial f_o}{\partial \bar{v}} \right) \left(\tilde{\rho}_{s1}(\bar{k}, \omega) + \hat{\rho}_e(\bar{k}, \omega) \right)$$

Wave in 1-D

$$\rho \sim \rho(kx - \omega t)$$


position / velocity of a static point with respect to the wave

$$x = \frac{\omega t}{k} + ck \qquad v = \frac{\omega}{k} \qquad \Rightarrow \boxed{kv - \omega = 0}$$

$\Rightarrow$ the integral diverges for particles moving at the same velocity than the wave

Interpretation : Density wave (perturbation) with a speed $v_w = \frac{\omega}{k}$

① if a particle with $v > v_w$ is trapped by the wave


⟹ energy is <mark>given</mark> to the wave

② if a particle with $v < v_w$ is trapped by the wave

⟹ energy is <mark>taken</mark> from the wave

60

If the velocity distribution function is <u>Maxwellian</u>:

$P_v(x)$

more particles with lower velocities will interact with the wave

$$\boxed{LANDAU \\ DAMPING}$$

V_w wave velocity

$P_v(x)$

same here

V_w wave velocity

61
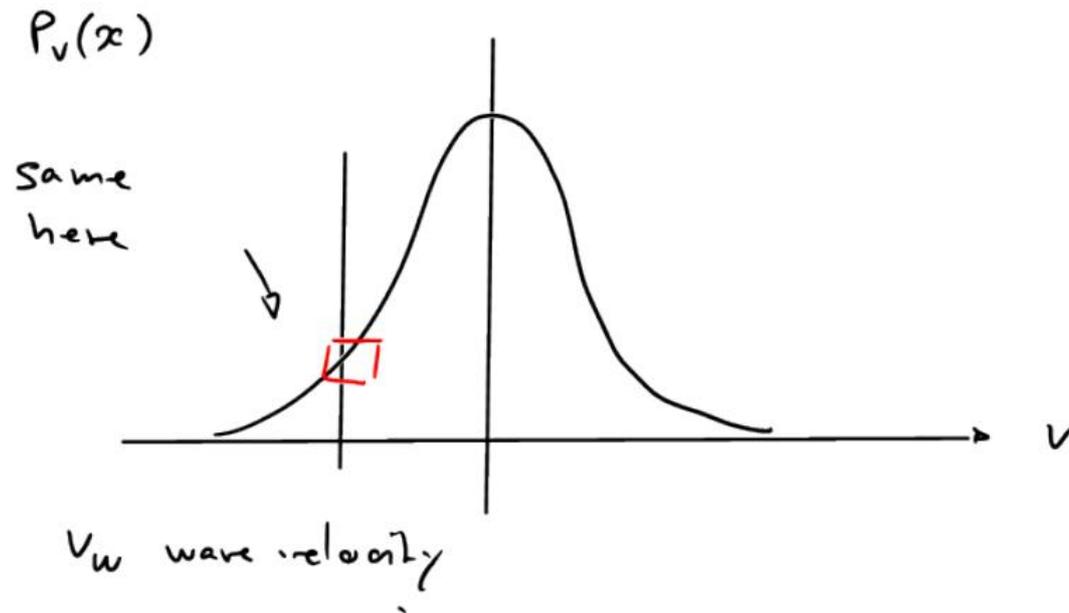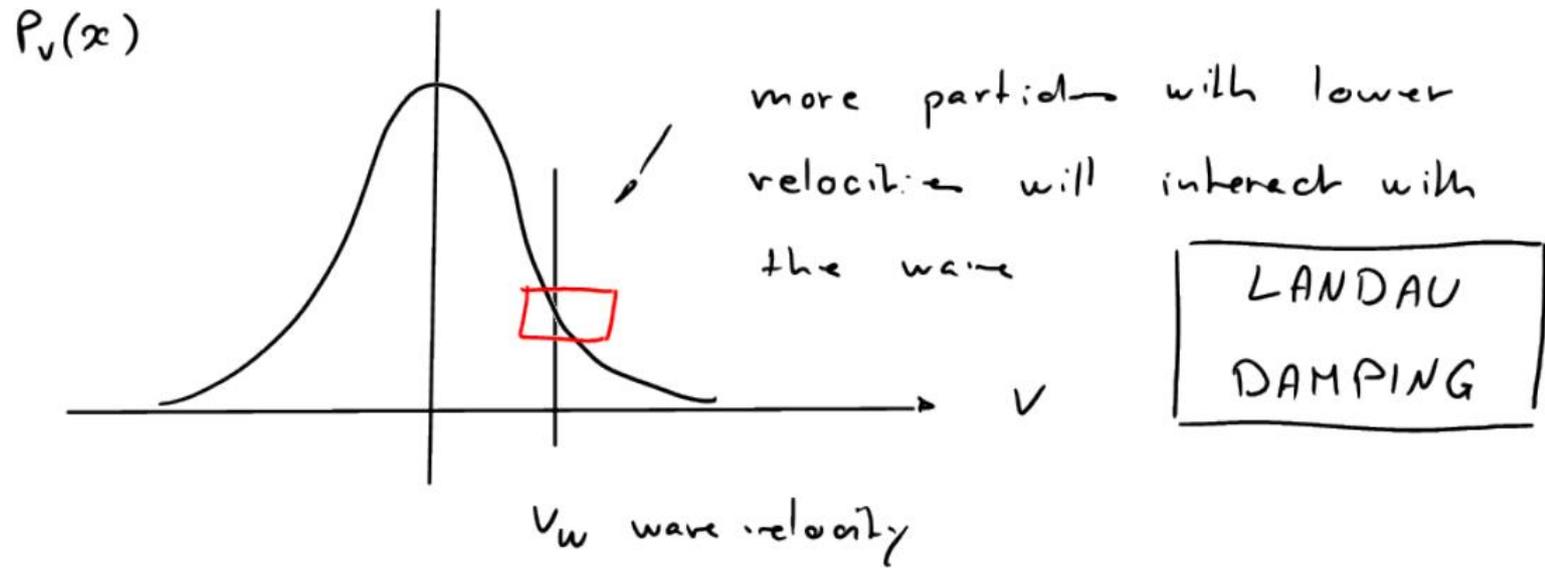
The End

Linear differential equations

$$f(x) : \quad \frac{\partial^n f}{\partial x^n} g_n(x) + \dots + \frac{\partial^2 f}{\partial x^2} g_2(x) + \frac{\partial f}{\partial x} g_1(x) + f(x) g_0(x) = c$$

$g_i(x)$: a continuous function

it f_1, f_2 are solutions $\quad a f_1 + b f_2 \quad$ is a solution

Illustration

- 1-D continuity equation $\quad (f, v)$

$$\frac{\partial}{\partial t} f + f \frac{\partial v}{\partial x} + v \frac{\partial f}{\partial x} = 0$$

$\longrightarrow$ those terms mixes f and v

- 1-D linearized continuity equation $\quad (f_{s_1}, v_1)$

$$\frac{\partial}{\partial t} f_{s_1} + f_0 \frac{\partial v_1}{\partial x} + v_0 \frac{\partial f_{s_1}}{\partial x} = 0$$

$\longrightarrow$ no mixing

63