Astrophysics III: Stellar and galactic dynamics

Exercises

Problem 1: Derive the *linearised collisionless Boltzmann equation* (5.11 in Binney & Tremaine 1987) from the course:

$$\frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, \Phi_1] = 0 \quad ; \quad \nabla^2 \Phi_{s1} = 4\pi G \int \mathrm{d}^3\mathbf{v}\, f_1 \tag{1}$$

Definitions and hints: The Poisson bracket is defined as

$$[A, B] \equiv \frac{\partial A}{\partial \mathbf{q}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{q}} \tag{2}$$

where A and B are any scalar functions of the phase-space coordinates.

We start from the collisionless Boltzmann equation and from Poisson's equation

$$\frac{\partial f}{\partial t} + [f, H] = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \tag{3}$$

$$\nabla^2 \Phi_s(\mathbf{x}, t) = 4\pi G \int \mathrm{d}^3\mathbf{v}\, f(\mathbf{x}, \mathbf{v}, t) \quad , \tag{4}$$

where $\Phi(\mathbf{x}, t)$ is the total potential, $H = \frac{1}{2}v^2 + \Phi(\mathbf{x}, t)$ is the Hamiltonian, and $\Phi_s(\mathbf{x}, t)$ is the gravitational potential of the stellar system, which may differ from the total potential $\Phi(\mathbf{x}, t)$ if there is an external perturbing potential $\Phi_e(\mathbf{x}, t)$. An isolated stellar system, as far as it is in equilibrium, is described by time-independent DF $f_0(\mathbf{x}, \mathbf{v})$ and potential $\Phi_0(\mathbf{x})$ that are solutions of 3 and 4

$$[f_0, H_0] = 0 \quad ; \quad \nabla^2 \Phi_0 = 4\pi G \int \mathrm{d}^3\mathbf{v}\, f_0 \quad \text{with} \quad H_0 = \frac{1}{2}v^2 + \Phi_0(\mathbf{x}) \tag{5}$$

Now we assume that the equilibrium system is subjected to a weak external potential $\epsilon\Phi_e(\mathbf{x}, t)$, where $|\nabla\Phi_e|$ is of order $|\nabla\Phi_0|$ and $\epsilon \ll 1$. In response to this disturbance, the DF of the stellar system and the potential arising from its stars become

$$f(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{x}, \mathbf{v}) + \epsilon f_1(\mathbf{x}, \mathbf{v}, t) \quad ; \quad \Phi_s(\mathbf{x}, t) = \Phi_0(\mathbf{x}) + \epsilon\Phi_{s1}(\mathbf{x}, t) \tag{6}$$

and the total potential becomes

$$\Phi(\mathbf{x}, t) = \Phi_0(\mathbf{x}, t) + \epsilon\Phi_1(\mathbf{x}, t) \quad \text{with} \quad \Phi_1(\mathbf{x}, t) = \Phi_{s1}(\mathbf{x}, t) + \Phi_e(\mathbf{x}, t) \tag{7}$$

Problem 2: Derive equations (5.23) to (5.26) in Binney & Tremaine 1987 for linearized fluid systems in the course:

$$\frac{\partial \rho_{s1}}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) + \nabla \cdot (\rho_{s1} \mathbf{v}_0) = 0 \tag{8}$$

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_1 + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_0 = -\nabla(h_1 + \Phi_{s1} + \Phi_e) \tag{9}$$

$$\nabla^2 \Phi_{s1} = 4\pi G \rho_{s1} \tag{10}$$

$$h_1 = \frac{p_1}{\rho_0} = \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_{\rho_0} \frac{\rho_{s1}}{\rho_0} = v_s^2 \frac{\rho_{s1}}{\rho_0} \tag{11}$$

knowing that the system is characterized by a density $\rho_s(\mathbf{x}, t)$, a pressure $p(\mathbf{x}, t)$, a velocity $\mathbf{v}(\mathbf{x}, t)$ and a potential $\Phi(\mathbf{x}, t)$, quantities that are linked by the continuity, Euler's and Poisson's equations:

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{v}) = 0 \ , \qquad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho_s}\nabla p - \nabla \Phi \ , \qquad \nabla^2 \Phi_s = 4\pi G \rho_s \ . \tag{12}$$

One has $\Phi = \Phi_s + \epsilon \Phi_e$ and the equation of state is assumed barotropic: $p(\mathbf{x}, t) = p[\rho_s(\mathbf{x}, t)]$. Thus, introducing the specific enthalpy h, Euler's equation becomes

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla(h + \Phi) \quad \text{with} \quad h(\rho_s) \equiv \int_0^{\rho_s} \frac{\mathrm{d}p(\rho)}{\rho}. \tag{13}$$

We further introduce the sound velocity $v_s^2(\mathbf{x}) \equiv \left[\frac{\mathrm{d}p(\rho)}{\mathrm{d}\rho}\right]_{\rho_0(\mathbf{x})}$. The response of the fluid to a weak external potential $\epsilon \Phi_e(\mathbf{x}, t)$ is

$$\begin{aligned} \rho_s(\mathbf{x}, t) &= \rho_0(\mathbf{x}) + \epsilon \rho_{s1}(\mathbf{x}, t) \quad ; \quad h(\mathbf{x}, t) = h_0(\mathbf{x}) + \epsilon h_1(\mathbf{x}, t) \\ \mathbf{v}(\mathbf{x}, t) &= \mathbf{v}_0(\mathbf{x}) + \epsilon \mathbf{v}_1(\mathbf{x}, t) \quad ; \quad \Phi(\mathbf{x}, t) = \Phi_0(\mathbf{x}) + \epsilon \Phi_1(\mathbf{x}, t) \end{aligned} \tag{14}$$

where $\Phi_1 = \Phi_{s1} + \Phi_e$ is the total perturbation in the potential, the sum of the external potential Φ_e and of the Φ_{s1} potential arising from the density perturbation ρ_{s1}.

Problem 3:

Show that the density of a gaseous sphere, which had initial density ρ_0, when compressed by a factor ϵ (i.e., $r_1 = (1 - \epsilon)r$) increases in proportion to $\epsilon \rho_0$. Similarly, show that its pressure increases proportionally to $v_s^2 \epsilon \rho_0$, where v_s is the speed of sound.

Then, using order-of-magnitude estimates, find expressions for the force changes of the pressure and gravity introduced by the contraction, and find the instability criterion expressed by r.