Exercise 10.1. On the plane $\mathbb{R}^2$ with the standard coordinates (x, y) consider the 1-form $\theta = x\,\mathrm{d}y$. Compute the integral of θ along each side of the square $[1, 2] \times [3, 4]$, with each of the two orientations. (There are 8 numbers to compute.)

Solution. The four integrals along the horizontal sides are zero because $\mathrm{d}y \equiv 0$ on any horizontal line.

Along a vertical line given by an equation $x = c$, with $c \in \mathbb{R}$ a constant, the vector field θ coincides with the 1-form $c\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$, which is the differential of the function $h_c(x, y) = cy$. Therefore the integral of θ along a segment of such a vertical line is equal to the variation of the function h_c along this segment.

Along the segment $\{2\} \times [3, 4]$ we have $c = 2$, thus the integral of θ is 2 if we go upwards and -2 if we go downwards. Similarly, along the segment $\{1\} \times [3, 4]$ we have $c = 1$, thus the integral of θ is 1 if we go upwards and -1 if we go downwards. □

Exercise 10.2. Let $\mathcal{B} = (E_i)_i$ and $\widetilde{\mathcal{B}} = (\widetilde{E}_j)_j$ be two bases of a vector space $V \simeq \mathbb{R}^n$, and let $\mathcal{B}^* = (\varepsilon^i)_i$ and $\widetilde{\mathcal{B}}^* = (\widetilde{\varepsilon}^j)_j$ be the respective dual bases. Note that a tensor $T \in \operatorname{Ten}^k V$ can be written as

$$T = \sum_{i_0,\dots,i_{k-1}} T_{i_0,\dots,i_{k-1}}\, \varepsilon^{i_0} \otimes \cdots \otimes \varepsilon^{i_{k-1}} \quad \text{or as} \quad T = \sum_{j_0,\dots,j_{k-1}} \widetilde{T}_{j_0,\dots,j_{k-1}}\, \widetilde{\varepsilon}^{j_0} \otimes \cdots \otimes \widetilde{\varepsilon}^{j_{k-1}}.$$

Find the transformation law that expresses the coefficients $\widetilde{T}_{j_0,\dots,j_{k-1}}$ in terms of the coefficients $T_{i_0,\dots,i_{k-1}}$.

Solution. There exists an invertible $n \times n$ matrix $(a^i_j)_{i,j\in\underline{n}}$ such that $\widetilde{E}_j = \sum_i a^i_j E_i$. For any k-index $J = (j_0, \dots, j_{k-1}) \in \underline{n}^k$ we have

$$\begin{aligned}
\widetilde{T}_J = T(\widetilde{E}_J) = T(\widetilde{E}_{j_0}, \dots, \widetilde{E}_{j_{k-1}}) &= T\left(\sum_{i_0\in\underline{n}} a^{i_0}_{j_0}\, E_{i_0}, \dots, \sum_{i_{k-1}\in\underline{n}} a^{i_{k-1}}_{j_{k-1}}\, E_{i_{k-1}}\right) \\
&= \sum_{I\in\underline{n}^k} a^{i_0}_{j_0} \cdots a^{i_{k-1}}_{j_{k-1}} T(E_{i_0}, \dots, E_{i_{k-1}}) \\
&= \sum_{I\in\underline{n}^k} a^{i_0}_{j_0} \cdots a^{i_{k-1}}_{j_{k-1}}\, T_I.
\end{aligned}$$

□

Exercise 10.3 (Alternating covariant tensors)**.** Let V be a finite-dimensional real vector space.

(a) Let $T \in \operatorname{Ten}^k V$. Suppose that with respect to some basis ε^i of V^*

$$T = \sum_{1\le i_1,\dots,i_k<n} T_{i_1\cdots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}.$$

Show that T is alternating iff for all $\sigma \in S_k$: $T_{i_{\sigma(1)}\cdots i_{\sigma(k)}} = \operatorname{sgn}(\sigma)\, T_{i_1\cdots i_k}$.

Solution. Suppose that T is alternating, then for all $\sigma \in S_n$ we have

$$T_{i_{\sigma(1)},\dots,i_{\sigma(k)}} = T(E_{i_{\sigma(1)}}, \dots, E_{i_{\sigma(k)}}) = \operatorname{sgn}(\sigma)\, T(E_{i_1}, \dots, E_{i_k}) = \operatorname{sgn}(\sigma)\, T_{i_1,\dots,i_k}$$

Conversely if $T_{i_{\sigma(1)},\dots,i_{\sigma(k)}} = \operatorname{sgn}\sigma\, T_{i_1,\dots,i_k}$ for all $\sigma \in S_n$, then in particular

$$T_{\alpha_1,\dots,\alpha_i,\dots,\alpha_j,\dots,\alpha_k} = -T_{\alpha_1,\dots,\alpha_j,\dots,\alpha_i,\dots,\alpha_k}$$

Then for any $X_1, \dots, X_k \in V$ the multi-linearity of T yields (we use the summation convention where repeated indices are summed over)

$$\begin{aligned}
&T(X_1, \dots, X_i, \dots, X_j, \dots, X_k) \\
&= T(X_1^{\alpha_1} E_{\alpha_1}, \dots, X_i^{\alpha_i} E_{\alpha_i}, \dots, X_j^{\alpha_j} E_{\alpha_j}, \dots, X_k^{\alpha_k} E_{\alpha_k}) \\
&= X_1^{\alpha_1} \dots X_k^{\alpha_k} T(E_{\alpha_1}, \dots, E_{\alpha_i}, \dots, E_{\alpha_j}, \dots, E_{\alpha_k}) \\
&= -X_1^{\alpha_1} \dots X_k^{\alpha_k} T(E_{\alpha_1}, \dots, E_{\alpha_j}, \dots, E_{\alpha_i}, \dots, E_{\alpha_k}) \\
&= -T(X_1, \dots, X_j, \dots, X_i, \dots, X_k)
\end{aligned}$$

Hence T is alternating. □

(b) Show that for any covectors $\omega^1, \dots, \omega^k \in V^*$ and vectors $X_1, \dots, X_k \in V$ we have

$$\omega^1 \wedge \dots \wedge \omega^k (X_1, \dots, X_k) = \det(\omega^i(X_j)).$$

Solution. Both sides are multilinear in the ω^i and so the result follows from the one for the basis covectors $\omega^i = \varepsilon^{\ell_i}$, which we have seen in the lecture.

Nevertheless, let us carry out the argument in detail. In the lecture we saw

$$\varepsilon^{\ell_1} \wedge \dots \wedge \varepsilon^{\ell_k}(X_1, \dots, X_k) = \det(\varepsilon^{\ell_r}(X_j))_r^j$$

(on the right hand side we have the determinant of a $k \times k$ matrix ($r, j = 1, \dots k$; think of j as the column index and r as the row index).

Now for arbitrary covectors $\omega^r = \sum_{\ell=1}^n \omega_\ell^r \varepsilon^\ell$ we have (for the first equality we use the multilinearity of the wedge product)

$$\begin{aligned}
\omega^1 \wedge \dots \wedge \omega^k(X_1, \dots, X_k) &= \sum_{\ell_1=1}^n \cdots \sum_{\ell_k=1}^n \omega_{\ell_1}^1 \cdots \omega_{\ell_k}^k \varepsilon^{\ell_1} \wedge \dots \wedge \varepsilon^{\ell_k}(X_1, \dots, X_k) \\
&= \sum_{\ell_1=1}^n \cdots \sum_{\ell_k=1}^n \omega_{\ell_1}^1 \cdots \omega_{\ell_k}^k \det(\varepsilon^{\ell_r}(X_j))_r^j \\
&= \sum_{\ell_1=1}^n \cdots \sum_{\ell_k=1}^n \omega_{\ell_1}^1 \cdots \omega_{\ell_k}^k \det \begin{pmatrix} \varepsilon^{\ell_1}(X_1) & \cdots & \varepsilon^{\ell_1}(X_k) \\ \vdots & & \vdots \\ \varepsilon^{\ell_k}(X_1) & \cdots & \varepsilon^{\ell_k}(X_k) \end{pmatrix} \\
&= \det(\omega^r(X_j))_r^i
\end{aligned}$$

where in the last step we multiplied the r-th line by $\omega_{\ell_r}^r$ and replaced $\sum_{\ell_r=1}^n \omega_{\ell_r}^r \varepsilon^{\ell_r} = \omega^r$.

□

Exercise 10.4 (Some practice with the wedge product)**.** Let V be a finite-dimensional vector space over $\mathbb{R}$.

(i) Show that the covectors $\omega^1, \dots, \omega^k \in V^*$ are linearly dependent if and only if $\omega^1 \wedge \dots \wedge \omega^k = 0$.

Solution. Let $\{\omega^1, \dots, \omega^k\}$ be a linearly dependent set. Then, without loss of generality, we suppose

$$\omega^1 = \sum_{j=2}^k a_j \omega^j$$

where $(a_j) \in \mathbb{R}$. Considering the wedge product, we have

$$\omega^1 \wedge \dots \wedge \omega^k = (\sum_{j=2}^k a_j \omega^j) \wedge \dots \wedge \omega^k = 0$$

where the last inequality follows from the fact that for any 1-covector α, one has $\alpha \wedge \alpha = 0$. Conversely, suppose that $\omega^1, \dots, \omega^k$ are linearly independent and extend it to a basis for V^* and let $\{v_1, \dots, v_n\}$ be the dual basis for V. Then we have

$$\omega^1 \wedge \cdots \wedge \omega^k(v_1, \dots, v_k) = 1$$

hence $\omega^1 \wedge \cdots \wedge \omega^k \neq 0$. □

(ii) Let $\{\omega^1, \dots, \omega^k\}$ and $\{\eta^1, \dots, \eta^k\}$ both be sets of k independent covectors. Show that they span the same subspace if and only if

$$\omega^1 \wedge \cdots \wedge \omega^k = c\,\eta^1 \wedge \cdots \wedge \eta^k$$

for some nonzero real number c.

Solution. "$\Leftarrow$" Assume that $\mathrm{Span}(\omega^1, \dots, \omega^k) \neq \mathrm{Span}(\eta^1, \dots, \eta^k)$, so without loss of generality that $w^1 \notin \mathrm{Span}(\eta^1, \dots, \eta^k)$. Then the covectors $w^1, \eta^1, \dots, \eta^k$ are linearly independent, and thus by (i) we have $w^1 \wedge \eta^1 \wedge \cdots \wedge \eta^k \neq 0$. But by assumption we know that $\omega^1 \wedge \cdots \wedge \omega^k = c\,\eta^1 \wedge \cdots \wedge \eta^k$ for some non-zero constant $c \in \mathbb{R}$. Hence

$$w^1 \wedge \eta^1 \wedge \cdots \wedge \eta^k = \frac{1}{c} w^1 \wedge \omega^1 \wedge \cdots \wedge \omega^k = 0$$

in contradiction with the previous statement.

"$\Rightarrow$" If $\omega^1, \dots, \omega^k$ and $\eta^1, \dots, \eta^k$ span the same subspace then the basis $\omega^1, \dots, \omega^k$ of this subspace can be obtained from $\eta^1, \dots, \eta^k$ by a finite sequence of basis exchange operations $\eta^i \mapsto \eta^i + \lambda\eta^j$ and $\eta^i = \lambda\eta^i$ for a non-zero constant $\lambda \in \mathbb{R}$ and $i \neq j$. But both these operations change the wedge product of the vectors at most by a multiplicative scalar, since

$$\eta^1 \wedge \cdots \wedge \eta^{i-1} \wedge (\eta^i + \lambda\eta^j) \wedge \eta^{i+1} \wedge \cdots \wedge \eta^k = \eta^1 \wedge \cdots \wedge \eta^{i-1} \wedge \eta^i \wedge \eta^{i+1} \wedge \cdots \wedge \eta^k$$

and

$$\eta^1 \wedge \cdots \wedge \eta^{i-1} \wedge (\lambda\eta^i) \wedge \eta^{i+1} \wedge \cdots \wedge \eta^k = \lambda\,\eta^1 \wedge \cdots \wedge \eta^{i-1} \wedge \eta^i \wedge \eta^{i+1} \wedge \cdots \wedge \eta^k$$

by multi-linearity. □

(iii) On the space $V = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, let $(\alpha^0, \dots, \alpha^{n-1},\ \beta^0, \dots, \beta^{n-1})$ be the dual of the standard base. Consider the alternating 2-tensor

$$\omega = \sum_i \alpha^i \wedge \beta^i \in \mathrm{Alt}^2 V.$$

Compute the $2n$-tensor

$$\frac{1}{n!} \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ factors}}.$$

Solution. Define for each $i \in \underline{n} = \{0, \dots, n-1\}$ the alternating 2-tensor $\omega^i = \alpha^i \wedge \beta^i$. Note that $\omega^i \wedge \omega^i = 0$, and that $\omega^i \wedge \omega^j = \omega^j \wedge \omega^i$ since the exterior product of alternating tensors is commutative when at least one of the two factors has even degree.

By the distributive law, we have

$$\underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ factors}} = \sum_{I=(i_0,\dots,i_{n-1}) \in \underline{n}^n} \omega^{i_0} \wedge \cdots \wedge \omega^{i_{n-1}}.$$

However, the wedge product vanishes when I has repeated values, therefore we need only consider the case when I is a permutation. Thus

$$\begin{aligned}
\frac{1}{n!} \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ factors}} &= \frac{1}{n!} \sum_{\sigma \in S_n} \omega^{\sigma(0)} \wedge \cdots \wedge \omega^{\sigma(n-1)} \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} \omega^0 \wedge \cdots \wedge \omega^{n-1} \\
&= \omega^0 \wedge \cdots \wedge \omega^{n-1} \\
&= \alpha^0 \wedge \beta^0 \wedge \cdots \wedge \alpha^{n-1} \wedge \beta^{n-1}.
\end{aligned}$$

□

Exercise 10.5. A tensor $T \in \operatorname{Ten}^k V$ is **symmetric** if it satisfies $T(X_{\sigma(0)}, \ldots, X_{\sigma(k-1)}) = T(X_0, \ldots, X_{k-1})$ for each permutation σ and vectors $X_0, \ldots, X_{k-1} \in V$. Denote $\operatorname{Sym}^k V$ the subspace of $\operatorname{Ten}^k V$ consisting of the symmetric tensors. Show that $\operatorname{Ten}^2 V = \operatorname{Alt}^2 V \oplus \operatorname{Sym}^2 V$ for any real vector space V.

Solution. We can write any tensor $T \in \operatorname{Ten}^2 V$ as a sum $T = \frac{1}{2}(T + \sigma T) + \frac{1}{2}(T - \sigma T)$, where σ is the nontrivial permutation of $\{0, 1\}$. The tensor $\frac{1}{2}(T + \sigma T)$ is symmetric and the tensor $\frac{1}{2}(T - \sigma T)$ is alternating. This shows that $\operatorname{Alt}^2 V + \operatorname{Sym}^2 V = \operatorname{Ten}^2 V$.

We also have to show that $\operatorname{Alt}^2 V \cap \operatorname{Sym}^2 V = \{0\}$. Let $T \in \operatorname{Alt}^2 V \cap \operatorname{Sym}^2 V$. Then for any vectors $X, Y \in V$ we have $T(X, Y) = T(Y, X) = -T(X, Y)$, therefore $T(X, Y) = 0$. This shows that $T = 0$. □